JIA 119 (1992) 345-364

J.I.A. **119**, II, 345–364

HIV, AIDS AND THE APPROXIMATE CALCULATION OF LIFE INSURANCE FUNCTIONS, ANNUITIES AND NET PREMIUMS

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ABSTRACT

The paper presents an application of the Markov chain as a tool for the calculation of life contingencies functions (e.g. assurance, annuity, net premium, policy value functions) arising from a multi-state model which represents the transmission and development of HIV and AIDS. The transmission model advocated by the Institute of Actuaries AIDS Working Party is modified and simplified and then applied to derive explicit formulae for these standard life contingencies functions. This investigation allows a thorough review of the properties of these functions to be conducted and assists with the calculation of premiums and reserves in the presence of HIV and AIDS.

KEYWORDS

HIV; Insurance Functions; Markov Chain

1. Introduction

At the Twenty-Third International Congress of Actuaries, Hoem (1988) gave an invited lecture entitled 'The Versatility of the Markov chain as a tool in the mathematics of life insurance'. It is proposed here to explore the use of this tool for the calculation of life insurance functions in the presence of HIV and AIDS.

The model put forward by the Institute of Actuaries AIDS Working Party (Daykin *et al*, 1988, 1990) is used as a representation of the transmission and spread of AIDS among male homosexuals.

The paper will thus contain few new ideas. It relies heavily on the early sections of Hoem (1988) and also on an excellent paper by Ramsay (1989) which explored similar themes for models of AIDS transmission, developed for the Society of Actuaries and being used in the United States of America. A full review of such transmission models is provided by Haberman (1990).

2. Institute of Actuaries AIDS Working Party Model

The Institute of Actuaries AIDS Working Party, of which the author is a member, has presented a mathematical model representing the transmission of HIV and the spread of AIDS. The model has been widely published and full details can be found, for example, in Daykin *et al.* (1988).

The Institute of Actuaries Working Party model is similar to many of the other mathematical models described in Haberman (1990). However, because of the emphasis on applying this model to assessing the effect of HIV and AIDS on life insurance underwriting, life insurance premiums and reserving (as well as permanent health insurance and pension provision), the focus of this actuarial model has been different. The model follows on from the terms of reference of the Working Party—these include: "To show the potential impact of HIV on mortality and morbidity and the implications for the use of existing actuarial bases and standard tables for premium rating and reserving." Actuaries require such a model to be age-specific, in order to consider the progress of individuals of a given age and sex through future calendar years, to consider the longer-term trend in transmission and to produce numerical results (although not necessarily by analytical means). Thus, equilibrium models would be of less interest. It is also important for the model to reflect the type of data that would normally be available to an insurance company.

For the above reasons, the Working Party's model is age-specific, and the resulting numerical complexity has meant that elements that depend on detailed assumptions about sexual behaviour have been avoided.

The model belongs to the family of stochastic processes that have been introduced by others, but it addresses male homosexuals only. Each cohort (of a single age) is dealt with independently of other cohorts. It is assumed that infection occurs from a contact between two individuals within a single age group. This assumption is artificial, but, if infections between those of different ages balance out, it may be considered to be a reasonable representaton of reality. The transition intensities between states are allowed to vary with attained age and time. The model allows for immigration of susceptibles and for normal mortality as well as for extra mortality from AIDS.

A further simplification is the assumption that all those males described below as being 'at risk' of infection behave in the same manner at any one time, so that the chance of infection depends on the age of the individual at risk and the particular calendar year, but not on any sub-division according to frequency of sexual contact or frequency of change of sexual partner.

The members of one cohort at age x may be in any one of the distinct states that are indicated in Figure 1. Five of these are live states: 'clear', 'at risk', 'immune', 'positive' and 'sick from AIDS'. There are up to six dead states, which may be kept separate simply to show the live state that someone died in. The dead states include: 'dead from positive', 'dead from sick (other than from AIDS)' and 'dead from AIDS'. It may not be possible to distinguish between the last two categories, but calculated deaths other than from AIDS of those who suffer from AIDS are comparatively trivial.

Those in the clear state are those whose assumed sexual activity is such that they run no risk whatever of becoming infected with HIV. They form the 'normal' pre-AIDS population for comparative purposes. Those in the at risk state are treated as exposing themselves to the risk of acquiring HIV infection by reason of sexual contact with infected people. Those in the immune state are assumed to have acquired HIV infection and to be infectious, but to be wholly immune from becoming sick from AIDS or dying from AIDS.

Those in the positive state are HIV seropositive, but not yet sick from AIDS;

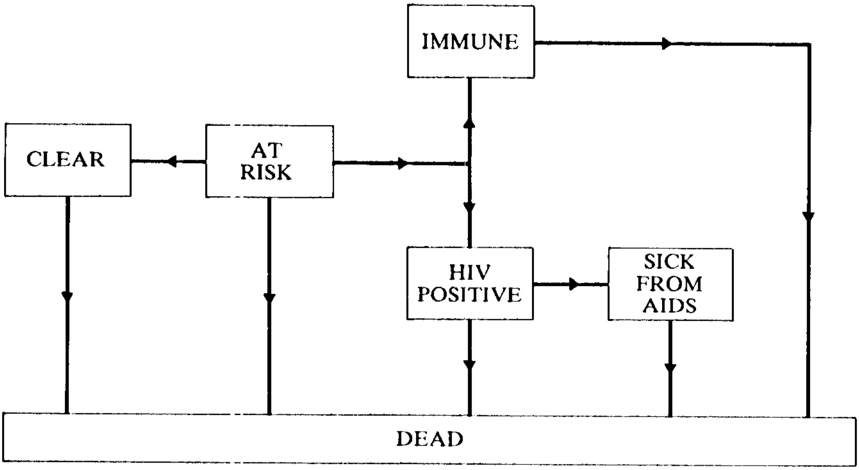

Figure 1. Outline of Institute of Actuaries AIDS Working Party model.

they are infectious and not immune. It is assumed that it is possible to distinguish between those who are HIV seropositive and those who are sick from AIDS. In reality, there are several stages in the transition from HIV infection to death from AIDS. Those who are suffering from AIDS are thought to be highly infectious, but it is possible that their sexual activity may be considerably reduced. The model makes it possible to choose whether those sick from AIDS are treated as contributing to further infections or not.

It is assumed that the current age is part of the status, and that transition intensities can all vary by current age. In addition, since each age cohort (or year-of-birth cohort) is treated separately, each transition intensity can also be varied by calendar year, so that each cohort has its own set of transition intensities.

Duration since entry to the states immune, positive and sick from AIDS are also relevant to the transition intensities. This duration is denoted in each case as z.

Possible transitions are as shown in Figure 1. Those in any of the live states may die, and those who are sick from AIDS may die from AIDS or from causes other than AIDS. Those who are at risk may change their behaviour and become clear, for example, by giving up sexual activity altogether, or by restricting themselves to one equally monogamous partner. There is no representation in the model of transfer from clear to at risk. Those who are at risk may become infected, and at that point are immediately allocated either to the immune state or to the positive state, in proportions that may depend on age (and on calendar year, although it seems unlikely that this would actually exercise any influence).

Those in the positive state may become sick from AIDS, if they do not die first.

Infection is possible from the immunes, positives and sick to the at risk.

The Working Party has proceeded by setting up a complex series of ordinary and partial differential equations for the probabilities of survival in a state and of transition between states, and then solving these equations by numerical methods, given assumptions about the form of the various transition intensities.

3. MODIFIED AIDS MODEL

Given the earlier discussion, it is proposed to modify the Working Party Model and to simplify it.

Firstly, the 'immune' state is removed: as is done in practice by the AIDS Working Party in all of its numerical simulations (see Daykin *et al.* 1988, 1990).

Secondly, the viewpoint is changed from that of the population as a whole to that of an individual male at risk who is considered to progress from state to state over time. Thus, we are concerned, not with the spread of HIV in a population, but with the outcome for a particular individual.

Thirdly, it is assumed that all transition intensities are constants, independent of attained age, duration in current state and secular time. We acknowledge that this assumption contradicts the earlier discussion which explained the importance of these variables, in particular attained age, to an actuarial assessment of the effects of HIV and AIDS on survival prospects. Two arguments can be put forward to support this seemingly 'extreme' assumption:

(a) The magnitude of the AIDS-related transition intensities is such as to outweigh the 'normal' age-related mortality risk. Indeed, many of the AIDS Working Party simulations published do assume intensities that do not vary with respect to age.
(b) The desire to reach some analytical results does require, at least initially, some heroic assumptions. We believe that the results are of value, although perhaps not yet for direct application in pricing and reserving formulae.

It is worth noting that similar assumptions have been made in the context of U.S. actuarial investigations of the impact of AIDS, for example Panjer (1988).

Fourthly, it is assumed that the transition intensity from at risk to seropositive is constant and does not depend on the numbers of persons infected: again a simplifying assumption made to make the resulting mathematical manipulations tractable. As noted by Daykin *et al.* (1990), a constant transition intensity from at risk to seropositive would be consistent with the exponential development of new cases of AIDS in the early stages of the epidemic. To allow for the effect of heterogeneity of risk and behavioural change, it would be reasonable to postulate an intensity that decreases with time as the epidemic develops. However, this assumption is not pursued here on the grounds of mathematical tractability.

It is then possible to calculate numerically standard life insurance, annuity, net premium and policy value functions, as discussed in the subsequent sections of this paper. The investigations reported here would be of value in underpinning

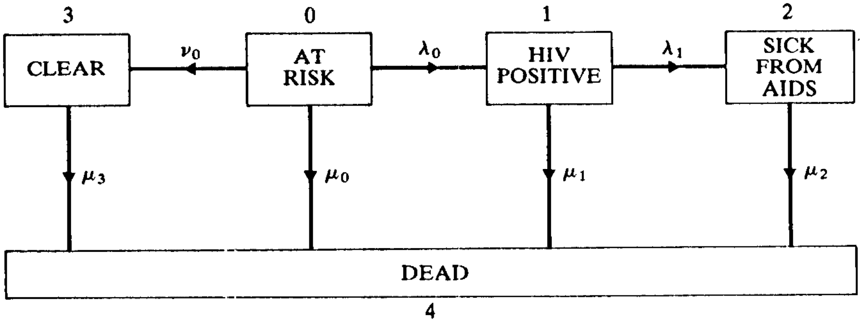

Figure 2. Modified AIDS model.

actuarial decisions on pricing and reserving for life insurance. The impact would be indirect because, for example, pricing would currently need to be based on cash flow considerations (as well as on present values), which allow for realistic assumptions and the cost of capital tied up in the setting aside of reserves calculated on a more stringent basis.

It is clear that the resulting financial functions (e.g. $\bar{A}_x$, $\bar{a}_x$, $\bar{P}_x$, ${}_t\bar{V}_x$ in standard notation) will be complex functions of the underlying transition intensities in the multi-state model. The sensitivity of the actuarial functions to changes in these transition intensities can be explored numerically using computer-based calculations. However, it would be helpful to have some clear picture of the sensitivities from an analytical route—in the same way that actuarial students learn, for example, about $\partial\bar{A}_x/\partial x$ or $\partial\bar{A}_x/\partial i$ (Neill, 1977, Chapter 6). The explicit formulae obtained will enable such partial differentials to be derived.

Figure 2 depicts the modified version of the model, with states that, for convenience, have been reduced in number to 5, and with the corresponding transition intensities.

It is proposed to use a continuous time Markov process to represent the transitions. Thus, a person in state 0 is subject to a constant force of progression out of state 0 into state 1, out of state 0 into state 3 and to a constant force of mortality out of state 0 into state 4. For a person in state 1, the possible transitions are to states 2 and 4. For a person in state 2 or 3, the transitions are to state 4 only. Once a life leaves a state it cannot return to that state. Clearly, the result is a Markov process, similar to the combined multi and single decrement models discussed in Neill (1977, Chapter 9), in respect of marriage and mortality.

The constant transition intensities are as depicted in Figure 2, viz.:

λ_i = force of progression from state i to state $(i+1)$ for $i=0, 1$

μ_i = force of mortality while in state i for $i=0, 1, 2, 3$

ν_0 = force of progression from state 0 to state 3.

Since the forces of progression remain constant while an individual is in any state, a 'memoryless' property exists. This means that the length of time already spent in the current state has no effect on the future length of time that the person will remain in this state. This permits us to speak in terms of the future time spent in a state without having to condition on the amount of time already spent in the state.

4. DERIVATION OF ACTUARIAL FUNCTIONS

4.1 *Transition Probabilities*

As Ramsay (1989) points out, there are a number of ways of setting up equations for the required probabilities, actuarial functions and so on. Our general approach is to use the Chapman Kolmogorov backward system of difference-differential equations (Grimmett & Stirzaker, 1982, Chapter 6). Because the transition intensities are assumed to be constant, we obtain simple recursive solutions to these equations.

Assume that insurance is issued to a life in state i at the time of issue, i.e. at $t=0$. Denote this life by $[i]$.

Let $p_{ij}(t)$ be the probability that $[i]$ will be alive t years from now and be in state j. The backward system of equations is derived by considering the interval $(0, t+h]$ as being made up of the subintervals $(0, h]$ and $(h, t+h]$ and then letting $h \to 0$. The complete set of equations for the $p_{ij}(t)$ is thus:

$$p_{00}(t+h) = (1-\alpha_0 h)\, p_{00}(t) + o(h)$$

$$p_{01}(t+h) = \lambda_0 h\, p_{11}(t) + (1-\alpha_0 h)\, p_{01}(t) + o(h)$$

$$p_{02}(t+h) = \lambda_0 h\, p_{12}(t) + (1-\alpha_0 h)\, p_{02}(t) + o(h)$$

$$p_{03}(t+h) = v_0 h\, p_{33}(t) + (1-\alpha_0 h)\, p_{03}(t) + o(h)$$

$$p_{11}(t+h) = (1-\alpha_1 h)\, p_{11}(t) + o(h)$$

$$p_{12}(t+h) = \lambda_1 h\, p_{22}(t) + (1-\alpha_1 h)\, p_{12}(t) + o(h)$$

$$p_{22}(t+h) = (1-\mu_2 h)\, p_{22}(t) + o(h)$$

$$p_{33}(t+h) = (1-\mu_3 h)\, p_{33}(t) + o(h)$$

where for convenience we have introduced $\alpha_0 = v_0 + \mu_0 + \lambda_0$ and $\alpha_1 = \lambda_1 + \mu_1$.

These lead, as $h \to 0$, to a set of difference-differential equations as follows:

$$\frac{d}{dt} p_{00}(t) = -\alpha_0\, p_{00}(t)$$

$$\frac{d}{dt} p_{01}(t) = \lambda_0\, p_{11}(t) - \alpha_0\, p_{01}(t)$$

$$\left.\begin{aligned}
\frac{d}{dt}p_{02}(t) &= \lambda_0\, p_{12}(t) - \alpha_0\, p_{02}(t) \\
\frac{d}{dt}p_{03}(t) &= \nu_0\, p_{33}(t) - \alpha_0\, p_{03}(t) \\
\frac{d}{dt}p_{11}(t) &= -\alpha_1\, p_{11}(t) \\
\frac{d}{dt}p_{12}(t) &= \lambda_1\, p_{22}(t) - \alpha_1\, p_{12}(t) \\
\frac{d}{dt}p_{22}(t) &= -\mu_2\, p_{22}(t) \\
\frac{d}{dt}p_{33}(t) &= -\mu_3\, p_{33}(t).
\end{aligned}\right\} \tag{1}$$

Since $p_{ij}(0)=\delta_{ij}$, the Kronecker delta with $\delta_{ij}=1$ if $i=j$
$=0$ otherwise

we have that
$$\left.\begin{aligned}
p_{ii}(t) &= e^{-\alpha_i t} \quad \text{for } i = 0, 1 \\
p_{ii}(t) &= e^{-\mu_i t} \quad \text{for } i = 2, 3
\end{aligned}\right\} \tag{2}$$

Given the solutions for $p_{ii}(t)$, it is possible by successive substitution to solve each of equations (1) in turn.

For example:

$$\begin{aligned}
p_{01}(t) &= e^{-\alpha_0 t}\lambda_0 \int_0^t p_{11}(s)e^{\alpha_0 s}ds \\
&= \frac{\lambda_0}{\alpha_0 - \alpha_1}(e^{-\alpha_1 t} - e^{-\alpha_0 t}) \text{ on integration} \\
&= \Pi_0^{(1)}e^{-\alpha_0 t} + \Pi_1^{(1)}e^{-\alpha_1 t} \qquad (3)
\end{aligned}$$

where: $$-\Pi_0^{(1)} = \Pi_1^{(1)} = \frac{\lambda_0}{\alpha_0 - \alpha_1}.$$

Similarly, it can be shown that:

$$\begin{aligned}
p_{12}(t) &= \lambda_1\, e^{-\alpha_1 t}\int_0^t p_{22}(s)e^{\alpha_1 s}ds \\
&= \Pi_1^{(2)}e^{-\alpha_1 t} + \Pi_2^{(2)}e^{-\mu_2 t} \qquad (4)
\end{aligned}$$

where: $$-\Pi_1^{(2)} = \Pi_2^{(2)} = \frac{\lambda_1}{\alpha_1 - \mu_2}.$$

$$p_{02}(t) = \lambda_0 e^{-\alpha_0 t} \int_0^t p_{12}(s) e^{\alpha_0 s} ds$$

$$= \Pi_0^{(3)} e^{-\alpha_0 t} + \Pi_1^{(3)} e^{-\alpha_1 t} + \Pi_2^{(3)} e^{-\mu_2 t} \qquad (5)$$

where:

$$\Pi_0^{(3)} = \frac{-\lambda_0 \lambda_1}{(\alpha_0 - \alpha_1)(\alpha_0 - \mu_2)}$$

$$\Pi_1^{(3)} = \frac{\lambda_0 \lambda_1}{(\alpha_1 - \mu_2)(\mu_0 - \alpha_1)}$$

$$\Pi_2^{(3)} = \frac{-\lambda_0 \lambda_1}{(\alpha_0 - \mu_2)(\alpha_1 - \mu_2)}.$$

$$p_{03}(t) = v_0 e^{-\alpha_0 t} \int_0^t p_{33}(s) e^{\alpha_0 s} ds$$

$$= \Pi_0^{(4)} e^{-\alpha_0 t} + \Pi_1^{(4)} e^{-\mu_3 t} \qquad (6)$$

where: $$-\Pi_0^{(4)} = \Pi_1^{(4)} = \frac{v_0}{\alpha_0 - \mu_3}.$$

4.2 *Probabilities of Dying and Life Expectancy*

Let $q_{ij}(t)$ be the probability that $[i]$ will die in state j within the next t years. Then:

$$q_{ij}(t) = \int_0^t p_{ij}(s)\, \mu_j\, ds,$$

enabling these terms to be evaluated. Let $q_i(t)$ be the probability that $[i]$ will die within the next t years, then:

$$q_i(t) = \sum_{j \varepsilon S_i} q_{ij}(t)$$

where S_i denotes the appropriate set of j for a given i, for example $S_0 = (0, 1, 2, 3)$, but $S_2 = (2)$. $q_i(t)$ can be determined either from the above summation or by writing down the Chapman–Kolmogorov backward equations directly for $q_i(t)$. Let $T_i(d)$ be the future lifetime until death for a life currently in state i.

Then let $\mathring{e}_i$ be the life expectancy for a life now in state i.

Then $\mathring{e}_i = \mathrm{E}(T_i(d))$.

A recursive procedure for determining $\mathring{e}_i$ can be set up by introducing D_i for the event that the life progresses to the next live state and $\bar{D}_i$ for the complement of this event.

Then $\mathring{e}_i = \mathrm{E}(T_i(d)) = \mathrm{E}(T_i(d)|D_i)\ \mathrm{pr}(D_i) + \mathrm{E}(T_i(d)|\bar{D}_i)\ \mathrm{pr}(\bar{D}_i)$.

The following expressions are derived in Appendix I.

$$\mathring{e}_0 = \frac{1 + \lambda_0 \mathring{e}_1 + v_0 \mathring{e}_3}{\alpha_0}$$

$$\left.\begin{aligned}\mathring{e}_1 &= \frac{1+\lambda_1\mathring{e}_2}{\alpha_1}\\ \mathring{e}_2 &= \frac{1}{\mu_2}\\ \mathring{e}_3 &= \frac{1}{\mu_3}\\ \mathring{e}_4 &= 0\end{aligned}\right\} \qquad (7)$$

4.3 *Assurances, Annuities and Premiums*

With subscript i referring to $[i]$ at $t=0$, we now define the following financial functions where the standard actuarial notation has been adapted to fit with the current context:

$A_i(t)$ = net single premium for a t year continuous temporary insurance of sum insured 1 unit.

$E_i(t)$ = net single premium for a t year pure endowment of 1 unit.

$a_i(t)$ = present value of a t year continuous life annuity of 1 per annum.

$A_i(t)$ is to be thought of as the expected value of a function of a random variable.

Let:

$$\begin{aligned} G_i(t) &= v^{T_i(d)} && \text{if } T_i(d) \leqslant t \\ &= 0 && \text{if } T_i(d) > t. \end{aligned}$$

Then $A_i(t) = \mathrm{E}(G_i(t))$. Similarly $E_i(t)$ and $a_i(t)$ can be referred to as expected values of functions of random variables.

It should be noted that with $A_i(t)$, $E_i(t)$, $a_i(t)$ being regarded as expectations of underlying random variables, it would be of interest to derive expressions for the corresponding variances. Thus, just as $A_i(t) = \mathrm{E}(G_i(t))$, it can be shown that:

$$\mathrm{Var}(G_i(t)) = A_i^{(2)}(t) - (A_i(t))^2$$

where $A_i^{(2)}(t)$ is calculated as for $A_i(t)$, but at twice the force of interest.

Similarly with $a_i(t) = \mathrm{E}(H_i(t))$:

$$\mathrm{Var}(H_i(t)) = \frac{2}{\delta}\,(a_i(t) - a_i^{(2)}(t)) - (a_i(t))^2.$$

The details would follow the derivations in Forfar & Waters (1986). It would be possible to extend this discussion to incorporate a stochastic interest rate model—for example, independent and identically distributed random variables as in Frees (1990).

Then, using an approach again based on the Chapman–Kolmogorov equations we have that:

$$A_0(t+h) = v^h[\mu_0 h + A_1(t)\,\lambda_0 h + A_3(t)\,v_0 h + (1-\alpha_0 h)A_0(t)] + o(h)$$

$$A_1(t+h) = v^h[\mu_1 h + A_2(t)\,\lambda_1 h + (1-\alpha_1 h)\,A_1(t)] + o(h)$$

$$A_2(t+h) = v^h[\mu_2 h + (1-\mu_2 h)A_2(t)] + o(h)$$

$$A_3(t+h) = v^h[\mu_3 h + (1-\mu_3 h)A_3(t)] + o(h).$$

Writing v^h as $(1-\delta\cdot h+o(h))$ it is then possible as $h\to 0$ to rewrite these equations as a system of difference—differential equations, viz.:

$$\left.\begin{aligned}
\frac{d}{dt}A_0(t) &= \mu_0 + \lambda_0 A_1(t) + v_0 A_3(t) - (\delta+\alpha_0)A_0(t)\\
\frac{d}{dt}A_1(t) &= \mu_1 + \lambda_1 A_2(t) - (\delta+\alpha_1)A_1(t)\\
\frac{d}{dt}A_2(t) &= \mu_2 - (\delta+\mu_2)A_2(t)\\
\frac{d}{dt}A_3(t) &= \mu_3 - (\delta+\mu_3)A_3(t).
\end{aligned}\right\} \tag{8}$$

The last 2 of these equations are similar in format to the corresponding standard formula for dA_x/dx; see Neill (1977) equation (6.1.5).

With boundary conditions $A_i(0)=0$, we can solve as follows:

$$\left.\begin{aligned}
A_2(t) &= \frac{\mu_2}{\mu_2+\delta}\,[1-\exp(-(\mu_2+\delta)t)]\\
A_3(t) &= \frac{\mu_3}{\mu_3+\delta}\,[1-\exp(-(\mu_3+\delta)t)]
\end{aligned}\right\} \tag{9}$$

$$\begin{aligned}
A_1(t) &= e^{-(\alpha_1+\delta)t}\int_0^t (\mu_1 + \lambda_1 A_2(s))\,e^{(\alpha_1+\delta)s}ds\\
&= A_{10} - A_{11}\,e^{-(\alpha_1+\delta)t} - A_{12}\,e^{-(\mu_2+\delta)t}
\end{aligned} \tag{10}$$

where A_{10}, A_{11} and A_{12} are constants with the following expressions in terms of the basic parameters:

$$A_{10} = \frac{\mu_1}{\alpha_1+\delta} + \frac{\lambda_1\mu_2}{(\mu_2+\delta)(\alpha_1+\delta)} = (A_{11}+A_{12})$$

$$A_{11} = \frac{\mu_1}{\alpha_1+\delta} - \frac{\lambda_1\mu_2}{(\alpha_1+\delta)(\alpha_1-\mu_2)}$$

$$A_{12} = \frac{\lambda_1\mu_2}{(\mu_2+\delta)(\alpha_1-\mu_2)}.$$

Similarly:

$$A_0(t) = e^{-(\alpha_0+\delta)t}\int_0^t (\mu_0 + \lambda_0 A_1(s) + \nu_0 A_3(s))e^{(\alpha_0+\delta)s}ds$$

$$= A_{00} + A_{01}e^{-(\alpha_0+\delta)t} + A_{02}e^{-(\alpha_1+\delta)t} + A_{03}e^{-(\mu_2+\delta)t} + A_{04}e^{-(\mu_3+\delta)t} \qquad (11)$$

where A_{00}, A_{01}, A_{02}, A_{03} and A_{04} are constants as follows:

$$A_{00} = \frac{1}{\alpha_0+\delta}(\mu_0 + \lambda_0 A_{10} + \nu_0 A_{30}) \qquad \text{where } A_{30} = \frac{\mu_3}{\mu_3+\delta}$$

$$A_{01} = -(A_{00} + A_{02} + A_{03} + A_{04})$$

$$A_{02} = -\frac{\lambda_0 A_{11}}{\alpha_0 - \alpha_1}$$

$$A_{03} = -\frac{\lambda_0 A_{12}}{\alpha_0 - \mu_2}$$

$$A_{04} = \frac{-\nu_0\mu_3}{(\alpha_0-\mu_3)(\mu_3+\delta)}.$$

The backward equations for $E_i(t)$ are derived similarly.

Thus for $i=0$:

$$E_0(t+h) = v^h(\lambda_0 E_1(t)h + \nu_0 E_3(t)h + (1-\alpha_0 h)E_0(t)) + o(h).$$

Also:

$$E_1(t+h) = v^h(\mu_1 E_2(t)h + E_1(t)(1-\alpha_1 h)) + o(h)$$

$$E_2(t+h) = -v^h(\delta+\mu_2)E_2(t) + o(h)$$

$$E_3(t+h) = -v^h(\delta+\mu_3)E_3(t) + o(h).$$

As before, writing v^h as $(1-\delta\cdot h+o(h))$ and letting $h\to 0$ leads to:

$$\left.\begin{aligned}
\frac{d}{dt}E_0(t) &= \lambda_0 E_1(t) + \nu_0 E_3(t) - (\delta+\alpha_0)E_0(t)\\
\frac{d}{dt}E_1(t) &= \lambda_1 E_2(t) - (\delta+\alpha_1)E_1(t)\\
\frac{d}{dt}E_2(t) &= -(\delta+\mu_2)E_2(t)\\
\frac{d}{dt}E_3(t) &= -(\delta+\mu_3)E_3(t)
\end{aligned}\right\} \qquad (12)$$

with boundary conditions $E_i(0)=1$.

Hence $E_2(t) = e^{-(\delta+\mu_2)t}$ and $E_3 = e^{-(\delta+\mu_3)t}$.

The explicit solutions for $E_0(t)$ and $E_1(t)$ are:

$$E_0(t) = e^{-(\delta+\alpha_0)t}\left[1 + \int_0^t (\lambda_0 E_1(s) + v_0 E_3(s))e^{(\delta+\alpha_0)s}ds\right]$$

$$E_1(t) = e^{-(\delta+\alpha_1)t}\left[1 + \int_0^t \lambda_1 E_2(s)e^{(\delta+\alpha_1)s}ds\right]$$

which can be evaluated as before.

For $a_i(t)$ we can proceed as for A_i and E_i. Thus, for $i=0$ the backward equations lead to:

$$a_0(t+h) = h + v^h(\lambda_0 a_1(t)h + v_0 a_3(t)h + (1-\alpha_0 h)a_0(t)) + o(h).$$

Thus a set of difference-differential equations can be derived by letting $h \to 0$. viz.:

$$\left.\begin{aligned}
\frac{d}{dt}a_0(t) &= 1 + \lambda_0 a_1(t) + v_0 a_3(t) - (\delta+\alpha_0)a_0(t) \\
\frac{d}{dt}a_1(t) &= 1 + \lambda_1 a_2(t) - (\delta+\alpha_1)a_1(t) \\
\frac{d}{dt}a_2(t) &= 1 - (\delta+\mu_2)a_2(t) \\
\frac{d}{dt}a_3(t) &= 1 - (\delta+\mu_3)a_3(t)
\end{aligned}\right\} \qquad (13)$$

with boundary conditions $a_i(0)=0$.

The reader should compare these expressions with the corresponding standard life table results from Neill (1977) e.g. $d\bar{a}_x/dx = -1 + (\mu_x + \delta)\bar{a}_x$, equation (6.1.4). Thus:

$$a_2(t) = \frac{1}{\delta+\mu_2}(1 - e^{-(\delta+\mu_2)t})$$

$$a_3(t) = \frac{1}{\delta+\mu_3}(1 - e^{-(\delta+\mu_3)t})$$

$$a_0(t) = e^{-(\delta+\alpha_0)t}\int_0^t (1 + \lambda_0 a_1(s) + v_0 a_3(s))e^{(\delta+\alpha_0)s}ds$$

$$a_1(t) = e^{-(\delta+\alpha_1)t}\int_0^t (1 + \lambda_1 a_2(s))e^{(\delta+\alpha_1)s}ds.$$

These last two integrals can be evaluated as before.

The reader can check that premium conversion relationships of the form $1 = A_i(t) + E_i(t) + \delta\alpha_i(t)$ also hold in these circumstances.

Corresponding results for whole life insurances can be obtained by letting $t \rightarrow \infty$.

Given the full formulae for functions like $A_i(t)$, $E_i(t)$, $a_i(t)$, it is straightforward to derive expressions for their partial differentials with respect to key underlying parameters. For example, it is straightforward to show that:

$$\frac{\partial A_2(t)}{\partial \mu_2} > 0 \qquad \text{and} \qquad \frac{\partial A_3(t)}{\partial \mu_3} > 0.$$

Appendix II lists for completeness the full expressions for $A_i(t)$ for $i=0, 1, 2, 3$.

Given the development so far, it is clear that with estimates of the parameter values ν_0, λ_i, μ_i we could calculate net premiums for various simple life insurance contracts, e.g. for an n-year endowment assurance on a fully continuous basis the net premium would be:

$$P_i(n) = \frac{A_i(n) + E_i(n)}{a_i(n)}$$

and for a term insurance policy:

$$\tilde{P}_i(n) = \frac{A_i(n)}{a_i(n)}.$$

4.4 *Numerical Examples*

As an illustration, we give in this section some numerical values for $A_0(t)$ and $A_1(t)$ for different combinations of some of the key parameters.

Following Daykin *et al.* (1990), we choose $\mu_2 = 0{\cdot}35$ and take $\mu_0 = \mu_1 = \mu_3 = 0{\cdot}001$ throughout (being approximately equivalent to the value for μ_x for a male aged 30–34 according to English Life Tables No. 14).

Then Table 1 gives values for the single premiums for a t-year temporary insurance issued to a life HIV positive at outset for different values of λ_1 and δ. Tables 2 and 3 consider a life at risk at outset: Table 2 explores values for the single premiums at different levels of ν_0 and λ_0 (for fixed λ_1 and δ), while Table 3 considers the case of different levels of λ_1 and δ (for fixed ν_0 and λ_0). It might be useful to compare these single premiums with those that would be obtained from conventional life contingencies for a life anticipated to experience a constant $\mu_x = 0{\cdot}001$, viz.:

$$\bar{A}^{1}_{x:\overline{t}|} = \frac{0{\cdot}001}{0{\cdot}001 + \delta}\,[1 - e^{-(0{\cdot}001 + \delta)t}].$$

	t years						
	1	2	3	4	5	10	15
$\delta = 0{\cdot}01$	0·00099	0·00198	0·00295	0·00391	0·00486	0·00947	0·01382
$\delta = 0{\cdot}05$	0·00097	0·00190	0·00278	0·00362	0·00441	0·00783	0·01048

Table 1. *Sample temporary insurance single premiums for life HIV positive at inception (state 1)*

$$\mu_0=\mu_1=\mu_3=0{\cdot}001 \quad \mu_2=0{\cdot}35$$

Period t	$\delta=0{\cdot}01$				$\delta=0{\cdot}05$			
(yrs)	$\lambda_1=0{\cdot}01$	0·05	0·10	0·15	$\lambda_1=0{\cdot}01$	0·05	0·10	0·15
1	0·0025	0·0086	0·016	0·023	0·0025	0·0084	0·016	0·022
2	0·0075	0·029	0·054	0·077	0·0071	0·027	0·051	0·073
3	0·014	0·056	0·1034	0·1461	0·013	0·052	0·096	0·1357
4	0·022	0·087	0·1593	0·2218	0·020	0·079	0·1446	0·2015
5	0·030	0·1208	0·2173	0·2978	0·026	0·1072	0·1930	0·2650
10	0·074	0·2875	0·4753	0·6032	0·059	0·2311	0·3854	0·4935
15	0·1172	0·4232	0·6454	0·7683	0·086	0·3139	0·4896	0·5950
20	0·1563	0·5253	0·7467	0·8468	0·105	0·3649	0·5405	0·6345

Table 2. *Sample temporary insurance single premiums for life at risk at inception (state 0)*

$$\mu_0=\mu_1=\mu_3=0{\cdot}001 \quad \text{and} \quad \mu_2=0{\cdot}35$$
$$\lambda_1=0{\cdot}01 \quad \delta=0{\cdot}01$$

Period t	$v_0=0$			$v_0=0{\cdot}025$			$v_0=0{\cdot}05$		
(yrs)	$\lambda_0=0{\cdot}05$	0·10	0·15	$\lambda_0=0{\cdot}05$	0·10	0·15	$\lambda_0=0{\cdot}05$	0·10	0·15
1	0·0010	0·0011	0·0011	0·0010	0·0011	0·0011	0·0010	0·0010	0·0011
2	0·0022	0·0023	0·0025	0·0022	0·0023	0·0025	0·0022	0·0023	0·0025
3	0·0035	0·0041	0·0046	0·0035	0·0040	0·0045	0·0035	0·0040	0·0045
4	0·0052	0·0063	0·0073	0·0051	0·0062	0·0072	0·0051	0·0062	0·0071
5	0·0071	0·0090	0·0107	0·0070	0·0089	0·0105	0·0069	0·0087	0·0103
10	0·0209	0·0297	0·0365	0·0202	0·0285	0·0350	0·0196	0·0275	0·0337
15	0·0407	0·0585	0·0707	0·0382	0·0549	0·0665	0·0362	0·0517	0·0629
20	0·0640	0·0907	0·1070	0·0587	0·0834	0·0992	0·0543	0·0773	0·0926

Table 3. *Sample temporary insurance single premiums for life at risk at inception (state 0)*

$$\mu_0=\mu_1=\mu_3=0{\cdot}001 \quad \text{and} \quad \mu_2=0{\cdot}35$$
$$\lambda_0=0{\cdot}10 \quad v_0=0{\cdot}05$$

Period t	$\lambda_1=0{\cdot}01$				$\delta=0{\cdot}05$		
(yrs)	$\delta=0{\cdot}03$	0·05	0·07	0·09	$\lambda_1=0{\cdot}03$	0·05	0·10
1	0·0010	0·0010	0·0010	0·0010	0·0011	0·0012	0·0015
2	0·0023	0·0022	0·0022	0·0022	0·0029	0·0036	0·0051
3	0·0039	0·0038	0·0036	0·0035	0·0057	0·0075	0·0119
4	0·0059	0·0056	0·0054	0·0051	0·0095	0·0132	0·0217
5	0·0082	0·0078	0·0073	0·0069	0·0142	0·0203	0·0342
10	0·0243	0·0216	0·0192	0·0171	0·0467	0·0692	0·1155
15	0·0432	0·0363	0·0307	0·0261	0·0813	0·1191	0·1894
20	0·0612	0·0490	0·0396	0·0324	0·1098	0·1576	0·2391

4.5 *Reserves*

Consider the prospective net loss random variable L_i at the time of issue of an n-year continuous term insurance policy to an individual in state i at time 0.

Then:
$$L_i = v^{T_i(d)} - \tilde{P}_i(n)\bar{a}_{\overline{T_i(d)}|} \qquad \text{if } T_i(d) \leqslant n$$
$$= -\tilde{P}_i(n)\bar{a}_{\overline{n}|} \qquad \text{if } T_i(d) > n$$

where $\tilde{P}_i(n)$ is the corresponding net premium, $A_i(n)/a_i(n)$ in our notation as above.

A corresponding formula can be set down for $L_{ij}(t)$, the prospective net loss at time t, given that the life is in state j at time $t(<n)$ i.e.:

$$L_{ij}(t) = v^{T_j(d)} - \tilde{P}_i(n)\bar{a}_{\overline{T_j(d)}|} \qquad \text{if } T_j(d) \leqslant n - t$$
$$= -\tilde{P}_i(n)\bar{a}_{\overline{n-t}|} \qquad \text{if } T_j(d) > n - t.$$

Then we introduce the net premium reserve at time t, given that the life is in stage j at time t, $V_{ij}(t)$, as:

$$V_{ij}(t) = E(L_{ij}(t)).$$

If the status of the life at time t is unknown, then we denote the net premium reserve at time t as $V_i(t)$ where:

$$V_i(t) = \sum_{j \text{ feasible}} \text{pr}[(i) \text{ is in state } j | i \text{ alive at time } t] V_{ij}(t)$$

i.e.
$$V_i(t) = \sum \frac{p_{ij}(t)}{1 - q_i(t)} V_{ij}(t)$$

by the laws of conditional probability.

Difference-differential equations for $V_{ij}(t)$ can be derived using the methodology exploited earlier. The solutions could lead to expressions for $V_i(t)$ from the above. The details are presented in Appendix III.

5. CONCLUDING REMARKS

The simple approximations presented here enable explicit formulae to be derived for the present values of assurance and annuity functions, as well as net premiums and policy values, for lives in the 'high risk' groups. Where appropriate, variances can also be estimated. It is hoped that these approximations will assist the life insurance actuary in gaining a 'feel' for the effect of HIV and AIDS on the values of such actuarial functions.

The presentation can be made more realistic, and hence more complex, in a number of different ways: for example by including normal mortality rates which are functions of attained age, by including a transition from the 'clear' state back to the 'at risk' state (from '3' to '0'), by allowing the transition intensities in the model to vary with age and duration as in the full AIDS Working Party Model.

It should be noted that the case $v_0 = 0$ leads to equations which are equivalent to those derived by Ramsay (1989).

REFERENCES

DAYKIN, C. D., CLARK, P. N. S., EVES, M. J., HABERMAN, S., LE GRYS, D. J., LOCKYER, J., MICHAELSON R. W. & WILKIE, A. D. (1988). The Impact of HIV Infection and AIDS on Insurance in the United Kingdom. *J.I.A.*, **115**, 727–837.

DAYKIN, C. D., CLARK, P. N. S., EVES, M. J., HABERMAN, S., LE GRYS, D. J., LOCKYER, J., MICHAELSON, R. W. & WILKIE, A. D. (1990). Projecting the Spread of AIDS in the United Kingdom: A Sensitivity Analysis. *J.I.A.*, **117**, 95–133.

FORFAR, D. O. & WATERS, H. R. (1986). A Stochastic Approach to Life Contingencies. Institute of Actuaries Education Service Special Note, Oxford.

FREES, E. W. (1990). Stochastic Life Contingencies with Solvency Considerations. *Trans. Soc. Act.*, **42**, 91–130 (with discussion).

GRIMMETT, G. R. & STIRZAKER, D. R. (1982). *Probability and Random Processes*. Oxford University Press, Oxford.

HABERMAN, S. (1990). Actuarial Review of Models Describing and Predicting the Spread of HIV Infection and AIDS. *J.I.A.*, **117**, 319–405.

HOEM, J. M. (1988). Versatility of Markov Chain as a Tool in the Mathematics of Life Insurance. Introductory Lecture to Subject 3. *Transactions of 23rd International Congress of Actuaries.* **Volume R.** 171–202.

NEILL, A. (1977). *Life Contingencies*. Heinemann, London.

PANJER, H. H. (1988). AIDS: Survival Analysis of Persons Testing HIV Positive. *Trans. Soc. Act.*, **40**, 517–530 (with discussion).

RAMSAY, C. M. (1989). AIDS and the Calculation of Life Insurance Functions. *Trans. Soc. Act.*, **41**, 393–422 (with discussion).

APPENDIX I

DERIVATION OF LIFE EXPECTANCY FORMULAE

Let T_i be the future time spent in state i before leaving alive to enter the next state in the progression. For $i=0$, this refers to either state 1 or 3. For $i=1$, this refers to state 2.

Let T_i' be the future time spent in state i before immediate transition to state 4 (i.e. death).

Let $T_i^{(d)}$ be the future lifetime until death for a life currently in state i.

Let D_i represent the event that the life in state i progresses alive to the next state in the progression.

Let $\bar{D}_i$ represent the event that the life dies in state i.

Then:

$$\mathring{e}_0 = \mathrm{E}(T_0^{(d)}|D_0)\Pr(D_0) + \mathrm{E}(T_0^{(d)}|\bar{D}_0)\Pr(\bar{D}_0).$$

D_0 can be subdivided further according to whether the life progresses to state 1 or state 3, i.e. $D_0 = D_{01} \bigcup D_{03}$.

Then:

$$\mathring{e}_0 = \mathrm{E}(T_0 + T_1^{(d)}|D_{01})\Pr(D_{01}) + \mathrm{E}(T_0 + T_3^{(d)}|D_{03})\Pr(D_{03}) + \mathrm{E}(T_0'|\bar{D}_0)\Pr(\bar{D}_0)$$

$$= \left(\frac{1}{\alpha_0} + \mathring{e}_1\right)\frac{\lambda_0}{\alpha_0} + \left(\frac{1}{\alpha_0} + \mathring{e}_3\right)\frac{\nu_0}{\alpha_0} + \frac{1}{\alpha_0}\left(\frac{\mu_0}{\alpha_0}\right)$$

$$= \frac{1}{\alpha_0} + \frac{\lambda_0}{\alpha_0}\mathring{e}_1 + \frac{\nu_0}{\alpha_0}\mathring{e}_3.$$

$$\mathring{e}_1 = \mathrm{E}(T_1 + T_2^{(d)}|D_1)\Pr(D_1) + \mathrm{E}(T_1'|\bar{D}_1)\Pr(\bar{D}_1)$$

$$= \left(\frac{1}{\alpha_1} + \mathring{e}_2\right)\frac{\lambda_1}{\alpha_1} + \frac{1}{\alpha_1}\frac{\mu_1}{\alpha_1}$$

$$= \frac{1}{\alpha_1} + \frac{\lambda_1}{\alpha_1}\mathring{e}_2$$

$$\mathring{e}_2 = \frac{1}{\mu_2}$$

$$\mathring{e}_3 = \frac{1}{\mu_3}$$

$$\mathring{e}_4 = 0.$$

Hence:

$$\mathring{e}_1 = \frac{1}{\alpha_1} + \frac{\lambda_1}{\alpha_1}\frac{1}{\mu_2}$$

$$= \frac{1}{\alpha_1}\left(1 + \frac{\lambda_1}{\mu_2}\right)$$

$$\mathring{e}_0 = \frac{1}{\alpha_0} + \frac{\lambda_0}{\alpha_0}\left(\frac{1}{\alpha_1} + \frac{\lambda_1}{\alpha_1}\frac{1}{\mu_2}\right) + \frac{v_0}{\alpha_0}\frac{1}{\mu_3}$$

$$= \frac{1}{\alpha_0}\left(1 + \frac{\lambda_0}{\alpha_1}\left(1 + \frac{\lambda_1}{\mu_2}\right) + \frac{v_0}{\mu_3}\right).$$

APPENDIX II

NET SINGLE PREMIUMS FOR A CONTINUOUS TEMPORARY INSURANCE

The following are the full expressions for $A_i(t)$ for $i=0, 1, 2, 3$, as referred to in Section 4.3.

$$A_0(t) = A_{00}(1 - e^{-(\alpha_0+\delta)t}) + A_{02}(e^{-(\alpha_1+\delta)t} - e^{-(\alpha_0+\delta)t}) + A_{03}(e^{-(\mu_2+\delta)t} - e^{-(\alpha_0+\delta)t}) + A_{04}(e^{-(\mu_3+\delta)t} - e^{-(\alpha_0+\delta)t})$$

where:
$$A_{00} = \frac{1}{(\alpha_0+\delta)}\left(\mu_0 + \frac{\lambda_0\mu_1}{\alpha_1+\delta} + \frac{\lambda_0\lambda_1\mu_2}{(\mu_2+\delta)(\alpha_1+\delta)} + \frac{\nu_0\mu_3}{\mu_3+\delta}\right)$$

$$A_{02} = \frac{\lambda_0}{(\alpha_0-\alpha_1)(\alpha_1+\delta)}\left(-\mu_1 + \frac{\lambda_1\mu_2}{\alpha_1-\mu_2}\right)$$

$$A_{03} = \frac{-\lambda_0\lambda_1\mu_2}{(\alpha_0-\mu_2)(\mu_2+\delta)(\alpha_1-\mu_2)}$$

$$A_{04} = \frac{-\nu_0\mu_3}{(\alpha_0-\mu_3)(\mu_3+\delta)}.$$

$$A_1(t) = A_{11}(1 - e^{-(\alpha_1+\delta)t}) + A_{12}(1 - e^{-(\mu_2+\delta)t})$$

where:
$$A_{11} = \frac{1}{(\alpha_1+\delta)}\left(\mu_1 - \frac{\lambda_1\mu_2}{(\alpha_1-\mu_2)}\right)$$

$$A_{12} = \frac{\lambda_1\mu_2}{(\mu_2+\delta)(\alpha_1-\mu_2)}.$$

$$A_2(t) = \frac{\mu_2}{(\mu_2+\delta)}(1 - e^{-(\mu_2+\delta)t})$$

and
$$A_3(t) = \frac{\mu_3}{(\mu_3+\delta)}(1 - e^{-(\mu_3+\delta)t})$$

where: $\alpha_0 = \mu_0 + \nu_0 + \lambda_0$ and $\alpha_1 = \mu_1 + \lambda_1$.

APPENDIX III

DERIVATION OF EXPRESSIONS FOR THE NET PREMIUM RESERVES

This gives the derivation of expressions for $V_{ij}(t)$ as referred to at the end of Section 4.5.

$$\frac{d}{dt} V_{00}(t) = \tilde{P}_0(n) + (\delta + \alpha_0)V_{00}(t) - \lambda_0 V_{01}(t) - v_0 V_{03}(t) - \mu_0$$

$$= \tilde{P}_0(n) + \delta_1 V_{00}(t) - \mu_0(1 - V_{00}(t))$$

$$- \lambda_o(V_{01}(t) - V_{00}(t)) - v_0(V_{03}(t)) - V_{00}(t))$$

where the terms on the right hand side can be interpreted successively as the premium and interest income, the net amount at risk if death occurs in state 0 and the net amounts at risk for progressing to states 1 or 3.

The corresponding equation in conventional life contingencies would be:

$$\frac{d}{dt}\, {}_t\bar{V}^1_{x:\overline{n}|} = \bar{P}^1_{x:\overline{n}|} + \delta\, {}_t\bar{V}^1_{x:\overline{n}|} - \mu_{x+t}(1 - {}_t\bar{V}^1_{x:\overline{n}|})$$

which is attributed widely to Thiele (Hoem, 1988).

Correspondingly we may derive the following:

$$\frac{d}{dt} V_{01}(t) = \tilde{P}_0(n) + (\delta + \alpha_0)V_{01}(t) - \lambda_1 V_{02}(t) - \mu_1$$

$$\frac{d}{dt} V_{02}(t) = \tilde{P}_0(n) + (\delta + \alpha_0)V_{02}(t) - \mu_2$$

$$\frac{d}{dt} V_{03}(t) = \tilde{P}_0(n) + (\delta + \alpha_0)V_{03}(t) - \mu_3$$

$$\frac{d}{dt} V_{11}(t) = \tilde{P}_1(n) + (\delta + \alpha_1)V_{11}(t) - \lambda_1 V_{12}(t) - \mu_1$$

$$\frac{d}{dt} V_{12}(t) = \tilde{P}_1(n) + (\delta + \alpha_1)V_{12}(t) - \mu_2$$

$$\frac{d}{dt} V_{22}(t) = \tilde{P}_2(n) + (\delta + \mu_2)V_{22}(t) - \mu_2$$

$$\frac{d}{dt} V_{33}(t) = \tilde{P}_3(n) + (\delta + \mu_3)V_{33}(t) - \mu_3.$$