

THE IMPACT OF PROPORTIONAL MORTALITY  
PROFIT DISTRIBUTION ON SOLIDARITY

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# **The impact of proportional mortality profit distribution on solidarity**

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## **Summary**

For a given period, a portfolio of individual life contracts to which the same amount at risk applies, will be considered. The portfolio consists of two homogeneous subgroups mutually different with respect to the mortality rate. At the end of the period, a fixed proportion of the mortality result realized by the insurer will be shared equally among the survivors.

At the beginning of the period considered, all individuals pay the same average risk premium, while the insurer's aim is to achieve equivalence on the level of the entire portfolio. In this paper it will be investigated how the mentioned proportion affects the absolute and relative subsidizing solidarity.

## The impact of proportional mortality profit distribution on solidarity

### 1. Introduction

Unobserved heterogeneity in a portfolio of contracts, which are identical with respect to observable risk characteristics, is an obstacle for the insurer to use a tarification system where, on an individual level, premiums and risks match one another as closely as possible. A remedy in non-life as well in group life insurance can be to update the premiums by means of experience rating. In individual life insurance, however, this is not possible as individuals only die once and contracts directly after the moment of dying are removed from the portfolio.

In this paper an individual life insurance system based on distribution of the aggregate mortality result, realized in a period, is introduced. Starting point of our analyses is a life portfolio consisting of individual contracts with the same amount of risk, which can be either positive or negative, for a certain unit period. To each individual contract, the same average risk premium, to be paid at the beginning of the period applies. (This risk premium may also be negative, implying that the company pays its absolute value to each insured.) A fixed proportion of the mortality result, the latter obtained by subtracting the product of the number of deaths times the amount at risk from the total amount of risk premiums paid, will be equally distributed among the survivors at the end of the period. The mortality result may be positive or negative. In the former case the survivors receive a(n) (extra) benefit while in the latter case they are charged a(n) (extra) premium. The insurer's aim is to achieve equivalence on the level of the group. The major topic of this paper is to examine whether, and in which respect, this will contribute to a system being more close to one meeting equivalence on an individual level as described in the first paragraph.

The system of proportional mortality distribution will be explained in Section 2. Then, in Section 3, the average risk premium, satisfying equivalence on a group level, will be derived. An overview of the several possible final states, one of which will be entered by any individual, together with an overview of the state-dependent transfers, the latter defined as the individual's loss due to the insurance contract, will be given. In order to be able to construct reasonable measures of solidarity, a vector of risk premiums satisfying equivalence on an individual level, in the remainder of this paper called individual risk premiums, will be derived. Then each of the transfers will be split in: a) the loss applying in case of equivalence on an individual level, and b) the insurer's expected loss due to the insurance contract, irrespective of the final state being equal to the difference between the average risk premium and the individual risk premium. Similar to Posthuma (1985) and Spreeuw (1996), these will be named *ex post transfers* and *ex ante transfers*, respectively.

In Section 4, two solidarity measures will be defined and considered: a) the *Absolute Subsidizing Solidarity* (ASS), the average-over-all-individuals of the squared

ex ante transfers, and b) the *Relative Subsidizing Solidarity* (RSS), equal to the ratio of ASS to the average-over-all-individuals of the expectations of the squared total transfers. The impact of the proportion of profit distribution on both measures will be analyzed. The conclusions drawn will be accompanied by a numerical example. Section 5 concludes this paper.

## 2. Basic assumptions

We consider a portfolio with  $n$  contracts for a certain period. To each contract, the same amount at risk, denoted by  $\hat{R}$ , applies. This amount is paid out at the end of the given period to the insured's heirs in case of death. The amount can also be negative, which means that the absolute value of  $\hat{R}$  is transferred to the insurer by the insured.

The portfolio consists of two risk classes indexed by 1 and 2, where risk class  $i$  contains  $n_i$  individuals,  $n_i > 1$ , with mortality rate  $\theta_i$ ,  $i \in \{1, 2\}$ . Hence  $n_1 + n_2 = n$ . All contracts pay the same average risk premium, denoted by  $\Pi r$ . The interest rate for the entire period is indicated by  $i$ . At the end of the period, a predetermined proportion of the mortality result, whether positive or negative, will be equally distributed among the survivors. If nobody survives, no distribution will take place. In case  $k$  individuals die, the mortality profit, denoted by  $\hat{MP}(k)$ , is equal to

$$\hat{MP}(k) = n \Pi r (1 + i) - k \hat{R}. \quad (1)$$

The proportion of the mortality profit allocated to those who survived the period is indicated by  $\rho$ . So in that case the company has to pay the following amount to each survivor (if the amount is negative it involves a benefit transferred from the survivor to the company):

$$\frac{\hat{MP}(k)}{n-k} = \rho \left[ \frac{n}{n-k} \Pi r (1 + i) - \frac{k}{n-k} \hat{R} \right] \quad (2)$$

In the remainder of this paper we will use the quantities  $MP(k)$  and  $R$  instead of  $\hat{MP}(k)$  and  $\hat{R}$ , respectively, where

$$MP(k) = \frac{\hat{MP}(k)}{1+i}; \quad R = \frac{\hat{R}}{1+i}. \quad (3)$$

## 3. Average and individual risk premiums; ex ante and ex post transfers

In this section, first, in Subsection 3a, the average risk premium, which is the same for each individual and which corresponds to the principle of equivalence on the level of the group, will be computed. Afterwards, in Subsection 3b, the vector of individual risk premiums, satisfying the principle of equivalence on an individual level, will be derived. The section will be concluded by giving an overview of the final states, to be entered by the individuals at the end of the period, together with

the respective entry probabilities, as well as the corresponding transfers. Each of these transfers, defined as the individual's loss due to the insurance contract, will be separated into two parts, namely the transfer applying in case of equivalence on an individual level (the ex post transfer) and the difference between the average risk premium and individual risk premium (or ex ante transfer).

### 3a. Average risk premium

Denoting the probability of  $k$  deaths by  $\text{Pr}(k)$ , we have that the expected loss incurred by the insurer, denoted by  $EL$ , is equal to

$$\begin{aligned}
 EL &= (1 - \rho) \sum_{k=0}^{n-1} [\text{Pr}(k) (kR - n\Pi r)] + \text{Pr}(n) n(R - \Pi r) \\
 &= R \left[ (1 - \rho) \sum_{k=0}^n \text{Pr}(k) k + \rho \text{Pr}(n) n \right] - n\Pi r (1 - \rho (1 - \text{Pr}(n))) \\
 &= R \left[ (1 - \rho) (n_1 \theta_1 + n_2 \theta_2) + \rho n \theta_1^{n_1} \theta_2^{n_2} \right] - n\Pi r \left[ 1 - \rho (1 - \theta_1^{n_1} \theta_2^{n_2}) \right]. \quad (4)
 \end{aligned}$$

The second expression of above equation shows that the absolute value of the mortality result is always reduced by  $100\rho\%$ , unless all individuals die.

It is assumed that the company aims to achieve equivalence on a group level (meaning  $EL = 0$ ) and hence the following average risk premium results:

$$\begin{aligned}
 \Pi r(\theta_1, \theta_2; \rho) &= R \frac{(1 - \rho) (n_1 \theta_1 + n_2 \theta_2) + \rho n \theta_1^{n_1} \theta_2^{n_2}}{n \left[ 1 - \rho (1 - \theta_1^{n_1} \theta_2^{n_2}) \right]} \\
 &= R \left[ 1 - \frac{(1 - \rho) (n_1 (1 - \theta_1) + n_2 (1 - \theta_2))}{n \left[ 1 - \rho (1 - \theta_1^{n_1} \theta_2^{n_2}) \right]} \right] \quad (5)
 \end{aligned}$$

Since

$$\frac{\partial \Pi r(\theta_1, \theta_2; \rho)}{\partial \rho} = \frac{R}{n} \frac{(\theta_1^{n_1} \theta_2^{n_2}) (n_1 (1 - \theta_1) + n_2 (1 - \theta_2))}{\left[ 1 - \rho (1 - \theta_1^{n_1} \theta_2^{n_2}) \right]^2} \quad (6)$$

we have that  $\Pi r(\theta_1, \theta_2; \rho)$  is monotone in  $\rho$ . Furthermore:

$$\Pi r(\theta_1, \theta_2; 0) = R \frac{n_1 \theta_1 + n_2 \theta_2}{n}, \quad (7)$$

and

$$\Pi r(\theta_1, \theta_2, 1) = R. \quad (8)$$

Hence, by assuming that  $0 < \rho < 1$ , it is ensured that the signs of the amount at risk and the risk premium are the same since  $\Pi r(\theta_1, \theta_2; \rho)$  is monotonously increasing in  $\rho$ .

**Remark 1 (Interpretation of equations (7) and (8))**

Note that the average risk premium displayed in equation (7) involves the classical case of no distribution of the mortality result, while (8) shows that, for  $\rho = 1$ , there is no insurance at all. ■

In case  $k$  individuals die, the insurer's mortality result, involving a loss if positive and a profit if negative, defined as  $MR(k)$ , is equal to

$$MR(k) = R(1 - \rho)^{I_{(k < n)}(k)} \frac{(1 - \rho)(k - n_1 \theta_1 - n_2 \theta_2) + \rho(k - n) \theta_1^{n_1} \theta_2^{n_2}}{1 - \rho(1 - \theta_1^{n_1} \theta_2^{n_2})}, \quad (9)$$

where  $I_{(k < n)}(k)$  is an indicator function, being zero if  $k = n$  and one in other cases.

Hitherto, we have restricted ourselves to the mortality result on an aggregate level. In order to obtain a degree of solidarity however, one has to compare the given situation with the one of equivalence on an individual level. In the latter case, the risk premiums to be paid by the respective individuals are such that for each individual the loss incurred by the insurer has expectation zero. These quantities, in the remainder of this paper defined as *individual risk premiums*, will be derived in the next subsection.

3b. Final states; equivalence on an individual level; individual risk premiums

At the end of the period considered, each individual will be in one of  $n + 1$  different final states, which will be defined below:

<u>State</u>	<u>Meaning</u>
A	Dead
B. $k$	Alive, together with $n - k$ others ( $k \in \{0, \dots, n - 1\}$ ).

In the remainder of this paper the probability of  $k$  deaths within a portfolio equal to the given one, except that one member of risk class  $i$  has been left out, will be denoted by  $\Pr_i(k)$ .

If the insurer was able to monitor for each individual the risk class to which they belong, and individuals of type  $i$  paid risk premium  $\Pi r_i^*$ , the transfer for an individual of risk class 1, reflecting the loss incurred by this individual due to the insurance contract, at the same time being the profit made by the insurer for the given contract, would for the several states be equal to the expressions given in the right column of the following table:

Table 1		
<i>Possible final states and corresponding entering probabilities and transfers for an individual belonging to risk class 1.</i>		
State	Probability	Transfer
A	$\theta_1$	$\Pi r_1^* - R$
B.k	$(1 - \theta_1) \Pr_1(k)$	$\left[ 1 - \frac{\rho n_1}{n-k} \right] \Pi r_1^* - \frac{\rho n_2}{n-k} \Pi r_2^* + \frac{\rho k}{n-k} R$

For an individual belonging to risk class 2, a similar result follows by interchanging the indices  $_1$  and  $_2$ .

Calculating the individual risk premiums, the risk premiums in case of equivalence on an individual level, is not as simple as it is in the ordinary case, since the insured's transfer depends on what happens to the other individuals within the portfolio. The individual risk premiums need to be calculated simultaneously, as will be done below.

The couple of premiums satisfying the principle of equivalence on an individual level is a solution of the matrix equality:

$$A \Pi r^* = R b, \quad (10)$$

with

$$A = \begin{pmatrix} 1 - \rho n_1(1 - \theta_1) \sum_{k=0}^{n-1} \frac{\Pr_1(k)}{n-k} & -\rho n_2(1 - \theta_1) \sum_{k=0}^{n-1} \frac{\Pr_1(k)}{n-k} \\ -\rho n_1(1 - \theta_2) \sum_{k=0}^{n-1} \frac{\Pr_2(k)}{n-k} & 1 - \rho n_2(1 - \theta_2) \sum_{k=0}^{n-1} \frac{\Pr_2(k)}{n-k} \end{pmatrix},$$

$$\Pi r^* = \begin{bmatrix} \Pi r_1^* \\ \Pi r_2^* \end{bmatrix}, b = \begin{bmatrix} \theta_1 - \rho(1 - \theta_1) \sum_{k=0}^{n-1} \frac{k \Pr_1(k)}{n-k} \\ \theta_2 - \rho(1 - \theta_2) \sum_{k=0}^{n-1} \frac{k \Pr_2(k)}{n-k} \end{bmatrix} \quad (11)$$

Taking into account that

$$n_1(1 - \theta_1) \sum_{k=0}^{n-1} \frac{\Pr_1(k)}{n-k} + n_2(1 - \theta_2) \sum_{k=0}^{n-1} \frac{\Pr_2(k)}{n-k} = 1 - \theta_1^{n_1} \theta_2^{n_2}, \quad (12)$$

the following solution is obtained by using Cramer's rule:

$$\begin{bmatrix} \Pi r_1^* \\ \Pi r_2^* \end{bmatrix} = \frac{R}{1 - \rho(1 - \theta_1^{n_1} \theta_2^{n_2})} \times \begin{bmatrix} (1 - \rho) \theta_1 + \rho \left[ \theta_1^{n_1} \theta_2^{n_2} + (1 - \rho) n_2 \left[ \prod_{i=1}^2 (1 - \theta_i) \right] \sum_{k=0}^{n-1} \frac{\Pr_2(k) - \Pr_1(k)}{n-k} \right] \\ (1 - \rho) \theta_2 + \rho \left[ \theta_1^{n_1} \theta_2^{n_2} - (1 - \rho) n_1 \left[ \prod_{i=1}^2 (1 - \theta_i) \right] \sum_{k=0}^{n-1} \frac{\Pr_2(k) - \Pr_1(k)}{n-k} \right] \end{bmatrix}. \quad (13)$$

In the remainder of this paper, this solution will be denoted by

$$\Pi r = \begin{bmatrix} \Pi r_1 \\ \Pi r_2 \end{bmatrix}. \quad (14)$$

Noticing that

$$n_1 \Pi r_1 + n_2 \Pi r_2 = n \Pi r,$$

the values of the transfers shown in Table 1, appear to be as follows in case of equivalence on an individual level:



Table 2		
<i>Transfers for an individual of risk class <math>i</math>, <math>i \in \{1, 2\}</math>, in case of equivalence on an individual level.</i>		
State	Probability	Transfer
$A$	$\theta_i$	$\Pi r_i - R$
$B.k$	$(1 - \theta_i) \Pr_i(k)$	$\Pi r_i - \rho \left[ \frac{n \Pi r - k R}{n - k} \right]$

### 3c. Ex ante and ex post transfers

The situation of equivalence on a group level and an average risk premium will now be reconsidered. In the given case the transfers are as given below:

Table 3		
<i>Transfers for an individual of risk class <math>i</math>, <math>i \in \{1, 2\}</math>, in case of equivalence on a group level and payment of an average risk premium.</i>		
State	Probability	Transfer
$A$	$\theta_i$	$\Pi r - R$
$B.k$	$(1 - \theta_i) \Pr_i(k)$	$\Pi r - \rho \left[ \frac{n \Pi r - k R}{n - k} \right]$

The transfers displayed in Table 2 are the losses suffered by the individual in case of equivalence on an individual level and will, similar to Posthuma (1985), be called *ex post transfers*. By subtracting these values from the corresponding ones in Table 3, it appears that the remaining part is independent of the final state, for an individual of type  $i$ ,  $i \in \{1, 2\}$ , being equal to:

$$\Pi r - \Pi r_i. \quad (15)$$

This quantity represents the expected loss incurred by the individual, due to the insurance contract, and will, again according to Posthuma (1985), be called *ex ante transfer*.

The several ex ante transfers will in the remainder of this paper be the base of two measures of solidarity to be considered in the next two sections.

## 4. Absolute and Relative Subsidizing Solidarity

### 4a Absolute Subsidizing Solidarity

The Absolute Subsidizing Solidarity is defined as the average over all individuals of the square of the ex ante transfers:

$$ASS = \sum_{i=1}^2 \frac{n_i}{n} \left( \Pi r(\theta_1, \theta_2; \rho) - \Pi r_i \right)^2. \quad (16)$$

#### Remark 1 (Terminology)

The quantity just defined has essentially the same meaning as the "Risk Solidarity" considered in De Wit & Van Eeghen (1984). Similar to Willekes & Van den Hoogen (1998), we however prefer the appellation "Subsidizing Solidarity", since in our view the former name does not indicate the type of solidarity accurately enough. ■

By substituting the average and individual risk premiums, derived in formulas (5) and (14), respectively, and by using the following equality

$$\sum_{\ell=0}^{n-1} \frac{\Pr_2(\ell) - \Pr_1(\ell)}{n - \ell} = (\theta_1 - \theta_2) \sum_{\ell=0}^{n-2} \frac{\Pr_{12}(\ell)}{(n - \ell)(n - \ell - 1)}, \quad (17)$$

with  $\Pr_{12}(\ell)$  denoting the probability of  $\ell$  deaths within a portfolio equal to the given one, except that one member of both risk classes has been left out, the right hand side of (16) appears, after some rewriting, to be equal to

$$\begin{aligned} ASS &= \frac{n_1 n_2}{n^2} \left[ \frac{R(1 - \rho)(\theta_1 - \theta_2)}{1 - \rho \left( 1 - \theta_1 \frac{n_1}{n} - \theta_2 \frac{n_2}{n} \right)} \right]^2 \\ &\times \left[ 1 + \rho n \prod_{i=1}^2 (1 - \theta_i) \sum_{\ell=0}^{n-2} \frac{\Pr_{12}(\ell)}{(n - \ell)(n - \ell - 1)} \right]^2. \end{aligned} \quad (18)$$

It follows immediately that ASS is equal to zero for

$$\rho = - \frac{1}{n \prod_{i=1}^2 (1 - \theta_i) \sum_{\ell=0}^{n-2} \frac{\Pr_{12}(\ell)}{(n - \ell)(n - \ell - 1)}}. \quad (19)$$

So it is possible to arrange a contract lacking any subsidizing solidarity but, contrary to the case  $\rho = 1$ , still containing a probabilistic element. However, the solution of

$\rho$  given in (19), has a negative sign and an absolute value greater than one, so it is doubtful whether such a contract can be regarded as an insurance agreement since the original intention of such an agreement, that is, covering the amount at risk, is dominated by the element of profit distribution. Therefore we will now concentrate on values of  $\rho$  lying in the interval  $[0, 1]$ , for which ASS adopts an extremal value. By taking the derivative, we get the following result:

$$\frac{\partial \text{ASS}}{\partial \rho} = \frac{2n_1 n_2}{n^2} \frac{R^2 (1 - \rho) (\theta_1 - \theta_2)^2}{\left[ 1 - \rho \left( 1 - \theta_1^{n_1} \theta_2^{n_2} \right) \right]^3} \times \left[ 1 + \rho n \prod_{i=1}^2 (1 - \theta_i) \sum_{\ell=0}^{n-2} \frac{\text{Pr}_{12}(\ell)}{(n - \ell)(n - \ell - 1)} \right] C(\rho), \quad (20)$$

with

$$C(\rho) = -\theta_1^{n_1} \theta_2^{n_2} \left[ n \prod_{i=1}^2 (1 - \theta_i) \sum_{\ell=1}^2 \frac{\text{Pr}_{12}(\ell)}{(n - \ell)(n - \ell - 1)} \rho^2 + 1 \right] + n \left[ \prod_{i=1}^2 (1 - \theta_i) \right] \left[ \sum_{\ell=0}^{n-2} \frac{\text{Pr}_{12}(\ell)}{(n - \ell)(n - \ell - 1)} \right] (1 - \rho)^2, \quad (21)$$

being a continuous quadratic function of  $\rho$ . We have

$$C(1) < 0, \quad (22)$$

Furthermore:

$$C(0) = -\theta_1^{n_1} \theta_2^{n_2} + n \left[ \prod_{i=1}^2 (1 - \theta_i) \right] \left[ \sum_{\ell=0}^{n-2} \frac{\text{Pr}_{12}(\ell)}{(n - \ell)(n - \ell - 1)} \right]. \quad (23)$$

If  $\theta_i < 0.5$ ,  $i \in \{1, 2\}$ , which can be considered a reasonable assumption in many cases, then the following inequality holds:

$$C(0) > \theta_1^{n_1} \theta_2^{n_2} \left[ \frac{1}{2} n - 1 \right] > 0. \quad (24)$$

Hence under the conditions just described above, there is a value of  $\rho$  being an element of the interval  $[0, 1]$  for which the ASS is maximal. This value, being a solution of the equality:

$$C(\rho) = 0, \quad (25)$$

is equal to:

$$\rho = \frac{1 - \theta_1^{n_1} \theta_2^{n_2} \left[ 1 + \frac{1 - \theta_1^{n_1} \theta_2^{n_2}}{n \left[ \prod_{i=1}^2 (1 - \theta_i) \right] \sum_{\ell=0}^{n-2} \frac{\text{Pr}_{12}(\ell)}{(n-\ell)(n-\ell-1)}} \right]}{1 - \theta_1^{n_1} \theta_2^{n_2}}. \quad (26)$$

Denoting this solution by  $\rho_{\text{abs}}$ , we have that for  $\rho \in \langle \rho_{\text{abs}}, 1 \rangle$ , the subsidizing solidarity ASS decreases, being zero for  $\rho = 1$ .

For not very large values of  $\theta_1$  and  $\theta_2$  and/or not very small numbers  $n_1$  and  $n_2$ ,  $\rho_{\text{abs}}$  tends to be very close to 1, as the following example shows. So the conclusion is that by mortality result distribution in a "conventional" way, i.e. for  $\rho \in \langle \rho_{\text{abs}}, 1 \rangle$ , the Absolute Subsidizing Solidarity is only increased, compared with the situation of no distribution at all.

### Example 1

For

$$n_1 = n_2 = 5; \theta_1 = 0.36; \theta_2 = 0.001; R = 100, \quad (27)$$

the right hand side of (19), being the value of  $\rho$  for which ASS vanishes is equal to  $-9.643$ . Restricting ourselves to  $\rho < 1$ , the Absolute Subsidizing Solidarity is maximal for:

$$\rho_{\text{max}} = 1 - 8 \times 10^{-9}. \quad (28)$$

The following table shows ASS for  $\rho$  varying from  $-1$  to  $\rho_{\text{max}}$ .

Table 4	
<i>The Absolute Subsidizing Solidarity as a function of <math>\rho</math>.</i>	
$\rho$	ASS
-1.0	258.842
-0.7	277.123
-0.4	296.027
-0.2	308.976
-0.1	315.555
0.0	322.203
0.1	328.920
0.2	335.706
0.4	349.487
0.6	363.545
0.8	377.880
0.9	385.152
$\rho_{\max}$	392.492

#### 4b Relative Subsidizing Solidarity

The Absolute Subsidizing Solidarity may not be an ideal measure for solidarity because not only the ex ante transfers, but also the ex post transfers, depend on  $\rho$ . Therefore another quantity, expressing ASS as a proportion of the average variance of the total transfers, the latter having been exhibited in Table 3, will now be introduced.

The variable TS (short for *Total Solidarity*) is defined as the average over all individuals of the expected value of the squared total transfers:

$$TS = \sum_{i=1}^2 \frac{n_i}{n_1 + n_2} \times \left[ \theta_i (\Pi r - R)^2 + (1 - \theta_i) \sum_{k=0}^{n-1} \Pr_i(k) \left[ \Pi r - \rho \left( \frac{n \Pi r - k R}{n - k} \right) \right]^2 \right]. \quad (29)$$

By substituting the average risk premium, derived in (5), we get

$$TS = \left[ \frac{R(1 - \rho)}{1 - \rho(1 - \theta_1^{n_1} \theta_2^{n_2})} \right]^2 \frac{\sum_{i=1}^2 n_i (1 - \theta_i)}{n} (a_{TS} \rho^2 + c_{TS}), \quad (30)$$

with

$$a_{TS} = \left[ \sum_{i=1}^2 n_i (1 - \theta_i) \right] \left[ \sum_{i=1}^2 \left[ n_i (1 - \theta_i) \sum_{k=0}^{n-1} \frac{\Pr_i(k)}{(n - k)^2} \right] - (1 - \theta_1^{n_1} \theta_2^{n_2})^2 \right], \quad (31)$$

$$c_{TS} = \frac{\sum_{i=1}^2 n_i \theta_i}{n}. \quad (32)$$

Introducing the following expectations and variances:

$$E[f(K_i)] = \sum_{k=0}^{n-1} f(k) \Pr_i(k), \quad i \in \{1, 2\}, \quad (33)$$

and

$$\text{Var}[f(K_i)] = \sum_{k=0}^{n-1} [f(k) - E[f(K_i)]]^2 \Pr_i(k), \quad i \in \{1, 2\}, \quad (34)$$

for any real valued function  $f(\cdot)$ , the random variable  $K_i$  denoting the conditional number of deaths, given that a member of risk class  $i$  survived, we have that  $a_{TS}$  can be rewritten as follows:

$$a_{TS} = \left[ \sum_{i=1}^2 n_i (1 - \theta_i) \right] \left[ \sum_{i=1}^2 n_i (1 - \theta_i) \text{Var} \left[ \frac{1}{n - K_i} \right] \right] + \left[ \prod_{i=1}^2 n_i (1 - \theta_i) \right] \left[ E \left[ \frac{1}{n - K_2} \right] - E \left[ \frac{1}{n - K_1} \right] \right]^2 \quad (35)$$

## Remark 2 (Correlation between individual transfers)

Note that the Total Solidarity is not equal to the variance of the insurer's loss divided by the number of individuals, since the individual transfers are not uncorrelated. ■

The Relative Subsidizing Solidarity, denoted by RSS, will now be specified as the ratio of the Absolute Subsidizing Solidarity to the Total Solidarity:

$$\text{RSS} = \frac{\text{ASS}}{\text{TS}}. \quad (36)$$

The derivative of RSS to  $\rho$  is

$$\begin{aligned} & \frac{\partial \text{RSS}}{\partial \rho} \\ = & \frac{2 n_1 n_2 \left( \theta_1 - \theta_2 + \rho n \left[ \prod_{i=1}^2 (1 - \theta_i) \right] \left[ E \left[ \frac{1}{n - K_2} \right] - E \left[ \frac{1}{n - K_1} \right] \right] \right)}{n \sum_{i=1}^2 n_i (1 - \theta_i)} \\ & \times \frac{\text{NUM}}{\text{DENOM}^2}. \end{aligned} \quad (37)$$

In the above expression, NUM is equal to

$$\text{NUM} = a_{\text{RSS}} \rho + b_{\text{RSS}}, \quad (38)$$

with

$$a_{\text{RSS}} = -(\theta_1 - \theta_2) a_{\text{TS}}, \quad (39)$$

and

$$b_{\text{RSS}} = \left[ \sum_{i=1}^2 n_i \theta_i \right] \left[ \prod_{i=1}^2 (1 - \theta_i) \right] \left[ E \left[ \frac{1}{n - K_2} \right] - E \left[ \frac{1}{n - K_1} \right] \right], \quad (40)$$

$a_{\text{TS}}$  having been defined in (35), while DENOM appears to be:

$$\text{DENOM} = a_{\text{TS}} \rho^2 + c_{\text{TS}}, \quad (41)$$

being strictly positive.

So we have that the Relative Subsidizing Solidarity is maximal for

$$\rho = - \frac{b_{\text{RSS}}}{a_{\text{RSS}}} = \frac{b_{\text{RSS}}}{(\theta_1 - \theta_2) a_{\text{TS}}}. \quad (42)$$

For smaller values, RSS increases with increasing  $\rho$ , while for larger ones, bearing in mind that the following values for  $\rho$  are prohibited:

$$\rho \neq 1, \rho \neq \frac{1}{1 - \theta_1^{n_2} \theta_2^{n_1}}, \quad (43)$$

RSS monotonously decreases if  $\rho$  increases.

Hence, in order to achieve a low RSS,  $\rho$  must exceed the value given in (42). The following example shows that this value can be quite high:

### Example 2

The values taken for  $n_1$ ,  $n_2$ ,  $\theta_1$  and  $\theta_2$  are the same as in Example 1, while  $R$  does not play any role regarding the Relative Subsidizing Solidarity. The right hand side of (42), is equal to 0.99662. The next table shows RSS for  $\rho$  varying between  $-1$  and this value:



Table 5	
<i>The Relative Subsidizing Solidarity as a function of <math>\rho</math>.</i>	
$\rho$	RSS
-1.0	0.15850
-0.7	0.17826
-0.4	0.19685
-0.2	0.20802
-0.1	0.21311
0.0	0.21782
0.1	0.22213
0.2	0.22601
0.4	0.23240
0.6	0.23690
0.8	0.23951
0.9	0.24014
0.99662	0.24033



## 5. Conclusions

In this paper, a system of proportional mortality profit distribution has been introduced. This system involves distributing a fixed proportion of the insurer's mortality result for a certain heterogeneous group and a certain period among those who survived the given period. It has been investigated how this proportion affects solidarity, for which two measures, the Absolute and the Relative Subsidizing Solidarity, have been defined. This paper shows that, at least for the former measure, the proportion only has an increasing impact on solidarity as long as it lies between zero and one. It has also been shown that it is possible to construct a contract without subsidizing solidarity, but such a policy would not be a very conventional one.

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