Incorporating model uncertainty into mortality forecasts

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Background

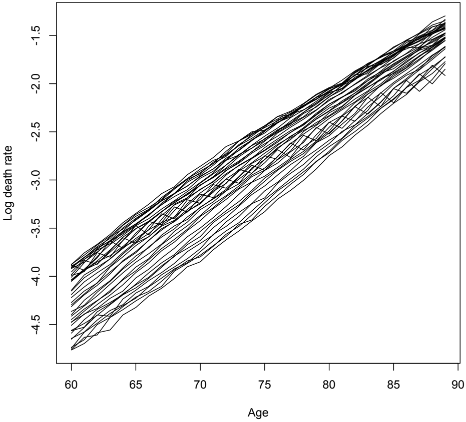
Mortality projections are essential for the actuarial profession for the quantification and management of longevity risk.
Existing methods typically fail to coherently account for all sources of uncertainty. Addressing this would allow

- improved pricing and assessment of longevity risk transactions
- greater understanding of and confidence in quantification of the tail of longevity risk
- improved management of longevity risk, ultimately to the benefit of consumers

Data and notation

$\{y_{xt}\}$ – England/Wales male deaths in year t aged x last birthday
$\{e_{xt}\}$ – corresponding average population in year t

Here, we focus on pensioner ages so x represents $60, 61, \ldots, 89$
Observed time window is for years $1971, \ldots, 2009 \equiv T$



A standard model

The Poisson Lee-Carter model (with random-walk period effects)

$$y_{xt} \sim \mathrm{Poisson}(e_{xt}\mu_{xt})$$

$$\log \mu_{xt} = \alpha_x + \beta_x \kappa_t$$

$$\kappa_t = \eta + \kappa_{t-1} + \epsilon_t$$

$$\epsilon_t \sim N(0, \sigma^2)$$

We will use this model as a reference to discuss some key concepts.

Sources of uncertainty

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1. Natural uncertainty
2. Parameter uncertainty
3. Model uncertainty
    a. rate model

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1. Natural uncertainty
2. Parameter uncertainty
3. Model uncertainty
    a. rate model
    b. projection model

Alternative models (rate)

Following Dowd et al (2010), we focus on the following 6 models

$$
\begin{aligned}
\text{M1:}\quad \log \mu_{xt} &= \beta_x^{(1)} + \beta_x^{(2)}\kappa_t && \text{(Lee-Carter)}\\
\text{M2:}\quad \log \mu_{xt} &= \beta_x^{(1)} + \beta_x^{(2)}\kappa_t + \beta_x^{(3)}\gamma_{t-x} && \text{(Lee-Carter + cohort)}\\
\text{M3:}\quad \log \mu_{xt} &= \beta_x^{(1)} + \kappa_t + \gamma_{t-x} && \text{(Age period cohort)}\\
\text{M5:}\quad \mathrm{logit}\, q_{xt} &= \kappa_t^{(1)} + \kappa_t^{(2)}(x-\bar{x}) && \text{(Linear in age)}\\
\text{M6:}\quad \mathrm{logit}\, q_{xt} &= \kappa_t^{(1)} + \kappa_t^{(2)}(x-\bar{x}) + \gamma_{t-x} && \text{(Linear + cohort)}\\
\text{M7:}\quad \mathrm{logit}\, q_{xt} &= \kappa_t^{(1)} + \kappa_t^{(2)}(x-\bar{x}) && \text{(Quadratic + cohort)}\\
&\quad +\kappa_t^{(3)}(x-\bar{x})^2 + \gamma_{t-x}
\end{aligned}
$$

Models must be stochastic, and we have restricted to models of fixed dimensionality (excluding semiparametric models).

Alternative models (projection)

Many possible time-series models for period effects, κ_t or cohort effects γ_c, typically based on the ARIMA family, including

- ARIMA(0,1,0) – random walk (with drift)

$$\gamma_c = \eta + \gamma_{c-1} + N(0, \sigma^2)$$

- ARIMA(1,1,0)

$$\gamma_c = \eta + \gamma_{c-1} + \phi(\gamma_{c-1} - \gamma_{c-2} - \eta) + N(0, \sigma^2)$$

- ...

Here, we assume ARIMA(0,1,0) for κ_t and restrict model uncertainty for γ_c to ARIMA(0,1,0) v. ARIMA(1,1,0).

Bayesian statistical inference

All uncertainty is quantified through probability distributions, including

1. Natural uncertainty
2. Parameter uncertainty
3. Model uncertainty

For a given model m with associated likelihood $p_m(y|\theta)$, our *prior* probability distribution $p_m(\theta)$ for the model parameters θ is updated to a *posterior* probability distribution through Bayes's theorem

$$p_m(\theta|y) \propto p_m(y|\theta)p_m(\theta)$$

Bayesian coherence

The posterior probability distribution $p_m(\theta)$ simultaneously represents uncertainty about *all* model parameters, *e.g.* for the Lee-Carter model above,

$$\theta = \left(\{\alpha_x\}, \{\beta_x\}, \{\kappa_t\}, \eta, \sigma^2\right)$$

Hence rate model and projection model are fully integrated, through (here)

$$p(y|\theta) = p\left(y|\{\alpha_x\}, \{\beta_x\}, \{\kappa_t\}\right) p\left(\{\kappa_t\}|\eta, \sigma^2\right)$$

Posterior $p(\theta|y)$ is computed and summarised (marginalised) using Markov chain Monte Carlo (MCMC).

e.g. Czado et al (2005), Girosi and King (2008)

Bayesian predictive inference

The Bayesian approach is particularly natural for projection, as all posterior uncertainty is naturally integrated into a *posterior predictive probability distribution*, *e.g.* for $y_f = \{y_{xt}, t = T+1, \ldots\}$

$$p_m(y_f|y) = E_{\text{post}}\left[p_m(y_f|\theta)\right]$$

Projections are typically based on means or medians of $p_m(y_f|y)$, with associated intervals derived from its quantiles.

Bayesian inference under model uncertainty

1. Natural uncertainty
2. Parameter uncertainty
3. Model uncertainty

Our prior probability distribution $p(m)$, representing our prior beliefs concerning (3), is similarly updated as

$$p(m|y) \propto p(y|m)p(m)$$

where $p(y|m)$ is the *marginal likelihood*

$$p(y|m) = E_{\text{prior}}\left[p_m(y|\theta)\right]$$

It is typical to assume prior neutrality between models, so models are compared using $p(y|m)$ or $\log p(y|m)$

Computed marginal likelihoods (log scale)

	ARIMA(0,1,0)	ARIMA(1,1,0)
M1:	-10101.2	
M3:	-7886.4	-7888.3
M5:	-11803.1	
M6:	-7360.4	-7364.1
M7:	-7347.8	-7352.9

Computational problems with M2, so this model omitted.

Projection under model uncertainty

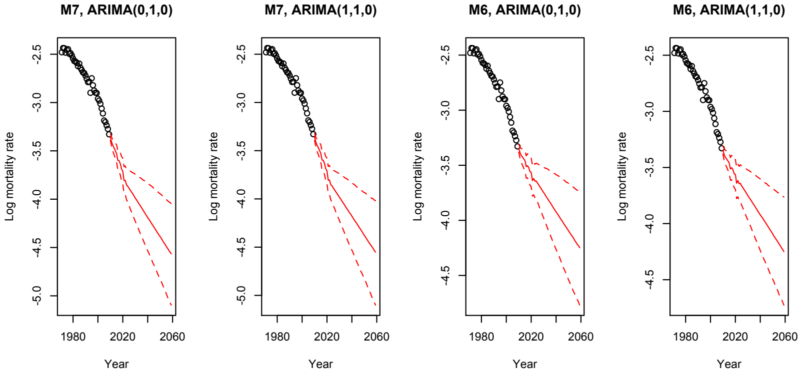
Under model uncertainty, the posterior predictive probability distribution incorporates model uncertainty, as

$$p(y_f|y) = \sum p(m|y) E_{\mathrm{post}} \left[p_m(y_f|\theta) \right]$$

which is a mixture of the individual model projections *weighted by their posterior probabilities*.

The individual model projections are computed using MCMC.
The weights $p(m|y)$ can be awkward to compute – we use the *bridge sampler* of Meng and Wong (1996).

Forecast mortality for age 75 (median and 90% intervals)



$$\text{M6:} \quad \operatorname{logit} q_{xt} = \kappa_t^{(1)} + \kappa_t^{(2)}(x - \bar{x}) + \gamma_{t-x}$$

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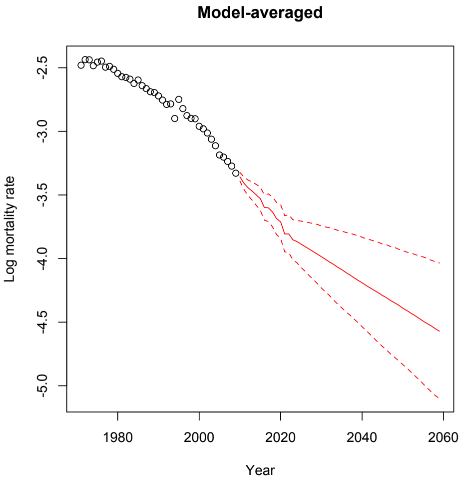
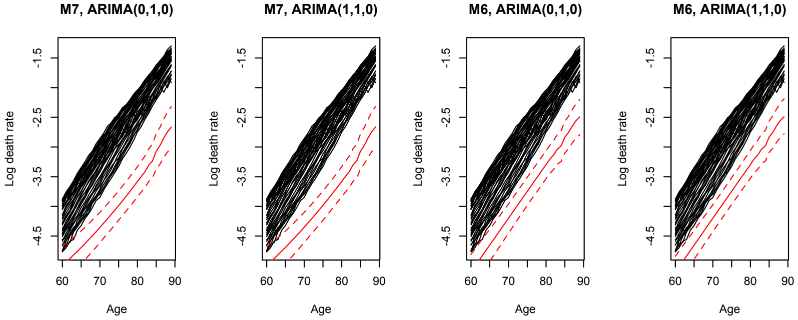


Forecast mortality for 2030 (median and 90% intervals)



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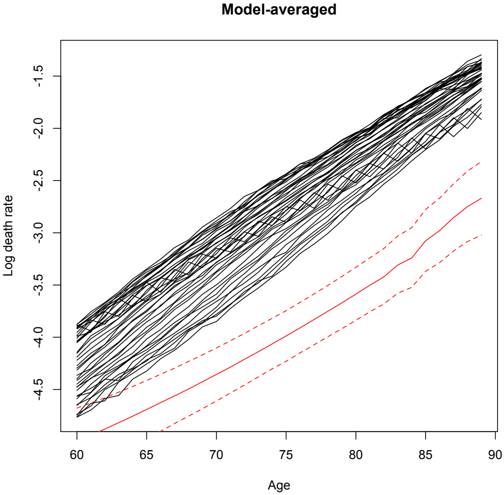

Summary and next steps

We have established a framework for fully integrating uncertainty into mortality forecasts, including model uncertainty.

Further research directions:

- Accounting for overdispersion should bring more models into play in the integrated forecast. Lack of fit (as illustrated by Dowd et al, 2010) accentuates differences in posterior probability.
- Expanding the range of models used:
  - Other rate models (semiparametric?)
  - Other projection models (straightforward)
- Integrating prior information (*e.g.* concerning smoothness)