

Insurance as a Capital Project

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Summary

Actuarial techniques coupled with the "big picture" knowledge base of actuaries make the actuary extremely well placed to add value when it comes to appraising capital projects.

Ordinarily when the words "insurance" and "capital projects" are used together, the writer is referring to the purchase of an insurance policy to transfer or share some of the capital project risk with an insurer.

The objective of this paper is to consider whether added insight can be gained into the likely future performance of an insurance company by treating it as a capital project.

Insurance companies are well suited to investigation using capital project techniques because both are based on:

- i) a structure;
- ii) a series of cash flows between the elements of the structure; and
- iii) an association with the achievement of a financial goal.

Capital project techniques typically involve forecasting the future cash flows arising from the given financial structure. The actuary needs to develop sophisticated financial modelling techniques including the use of a number of suitable and consistent financial assumptions. All risks must be identified, modelled and managed.

This paper considers two common structures for insurance suppliers and considers the risks and rewards of those structures. This paper then considers the steps involved in a capital project approach to investigating an insurance company.

Insurance as a Capital Project

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1 Introduction

The objective of this paper is to consider whether it is possible for an insurer to gain financial advantage by treating his business as being one or more capital projects. Will an insurer better understand and better mix the business across different classes and in so doing increase the expected profitability for a given level of risk.

Section 2 defines what we mean by a capital project. For now it is useful to consider the capital project as having two stages namely, choosing a structure and analysing and estimating the cash flows arising from the structure over the future life of the project.

The primary objective of this paper is to promote the notion that it is well worth considering a range of different structures for any capital projects including insurance companies. Structuring is basically considering what participants should have which legal entities and should arrange themselves in which fashion so as to best achieve the various goals of the project for the parties involved. Even before analysing the cash flows, the process of structuring will add to the understanding of the issues involved in the structure. By way of example we have considered a structure for a typical insurance company and for Lloyds of London in Section 5 below.

The cash flows of the structure must be comprehensively modelled. Tools suitable for insurance companies will include risk theory techniques and more recently risk based capital techniques. These tools are well known to insurance actuaries. This paper promotes another useful tool for analysing the cash flows of an insurance company namely a utility theory based approach for choosing the optimal relative size of each insurance portfolio. A discussion of this technique is contained in Appendix A.

This paper considers an insurance company to be a set of capital projects, one for each class of business being underwritten.

2 Definition

The following definition was adopted for the paper entitled "Capital Projects" published by the Institute of Actuaries and presented on 28/11/94.

"A capital project may be defined in a wide sense as meaning any scheme which involves the investment of resources at the outset, in return for the expectation of a net benefit at a later stage. In this paper however we shall use the term in a narrower sense to include only those projects where the investment has significant physical, social, or organisational consequences and is not merely to secure a transfer of ownership of an existing asset."

For our purpose we prefer to adopt a definition which places greater emphasis on the existence a particular asset or group of assets. Our definition follows. It has been based on a definition contained in the "Handbook of Australian Corporate Finance", R Bruce et al., Butterworths, 1989.

A capital project is a series of cash flows associated with an asset or a group of assets.

Unlike the normal borrowing and investment sequence, the key difference between a capital project and other investments is that the project lenders/investors will look primarily to the cash flows and assets of the project itself for repayment and collateral rather than to the project's sponsors. It is the economics of the project rather than the financial strength of its sponsors on which project finance evaluation is initially based. It is this, coupled with commitments from the sponsors and third parties, that provide the basis for extensive borrowing.

The above definition encapsulates the following key distinguishing features of a capital project.

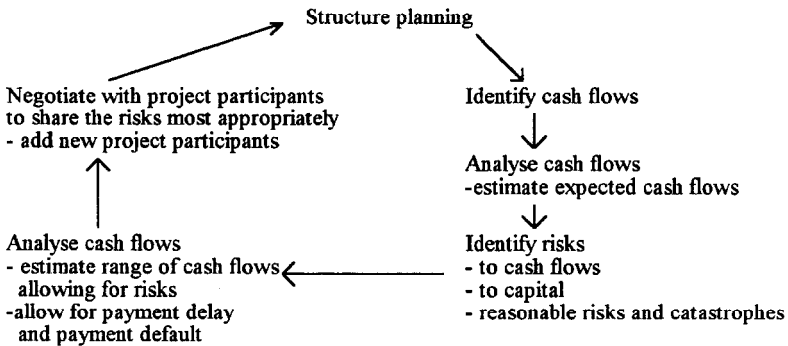
- 1 the project is established as a separate financial entity and relies on its own cash flows to support the debt financing and returns on equity invested. For example, a typical debt to equity ratio is 2:1
- 2 the sponsor's guarantees to the lender do not as a general rule cover all of the risks involved.
- 3 Firm commitments by various third parties (eg. suppliers) and the project sponsors make up significant components of the credit support.
- 4 The debt of the project entity is often completely separate (at least for balance sheet purposes) from the sponsor companies' direct obligations.
- 5 The lender's security is usually only in the project's assets.

An actuary is well placed to assist in:

- 1 the determination and analysis of the cash flows; and
- 2 recognition and measurement of risk. Insurance is just one way of transferring and/or sharing risk.

3 Elements of capital projects

Capital projects commence with one party desiring to achieve a given physical and/or financial goal. The party then typically considers the types of project participants that might be necessary to achieve this goal, approaches these other parties and commences to develop a suitable structure. The proposed structure is analysed, the risks are identified and estimated and the project participants decide which risks, funding, project income etc.. they are prepared to accept. Participants may come and go and the structure continues to be refined until all participants are happy with their share of the project. The following diagram highlights the revolving nature of the process.



The analysis of the cash flows, amount and timing uncertainties will often be quite complex. The evaluation techniques will typically include:

- i) building an asset/liability model which can be used to test the sensitivity of the results to various foreseeable changes in the assumptions eg. future economic conditions, payment default by a party, tax changes etc...
- ii) building utility models of the various project participants to help them to decide the most appropriate distribution of the project risks and rewards.

In this paper we examine another useful tool for analysing the cash flows of an insurance company being a utility theory based approach for choosing the optimal relative size of each insurance portfolio. A discussion of this technique is contained in Appendix A.

The process of structuring and restructuring helps the participants to better understand the likely risks and rewards from their participation. The process also helps to minimise leakage from the structure from tax and other expenses and to make the structure as robust as a possible to unforeseen future changes in legislation, supply and demand etc..

4 Insurance in capital projects

We are concerned with treating an insurance company as a capital project as distinct from the traditional relationship between insurers and capital projects. Traditionally, insurance has been used in capital projects to transfer risk from the project in return for a fee.

For example, in the past, insurance has been used in the following ways.

- i) commercial property puts - The insurer underwrote that the value of a property would exceed a minimum agreed value in three years time. To our knowledge, commercial property puts are no longer available from insurers although residual value guarantees can still be found on other assets including motor cars, commercial vehicles and aeroplanes.
- ii) standard insurance business including asset damage, pecuniary loss, liability etc..
- iii) force majeure risk eg. incl. earthquakes, floods etc..
- iv) political and regulatory risk expropriation and nationalisation
- v) pollution risks

Insurance tends to be designed to suit each capital project and is often both difficult to place and apparently expensive. The price of the insurance reflects:

- i) the costs associated with negotiating and tailoring the insurance;
- ii) the risks are often difficult to pool;
- iii) the lack of competition amongst other insurers;
- iv) the often catastrophic nature of the risk with a very small chance of loss makes the pricing very difficult and uncertain; and
- v) insurers are aware of the relatively large benefits being shared by the other project participants.

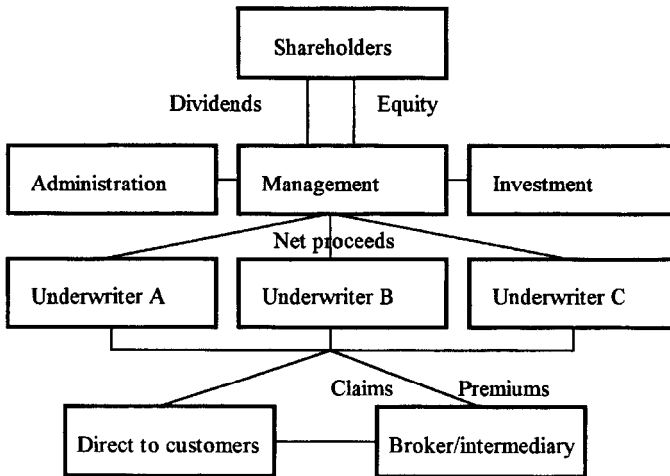
5 Insurance as a capital project

Projects and products are developed to suit a particular set of circumstances and goals. For example the leasing structures evolved because the lessee was not the most efficient owner of the asset.

Turning to insurance companies, in insurance there is a famous concept, often declared in the past, that investing in an insurance portfolio can double your returns. Members agents at Lloyds are often cited as promoters of such philosophy. What the concept fails to do is to recognise that additional return is accompanied by additional risk, and that an investor's risk/return preference needs to be taken into account when formulating the investment decision. The classic "money for old rope" syndrome often

hides a considerable remote unquantifiable risk that needs careful management. Catastrophes and mortgage guarantees are two such examples.

Consider the following financial structure diagram for a typical general insurance company.



The management basis of the structure follows.

- 1 The underwriter accepts or rejects a specific risk. In making the assessment he takes into account the quality of the individual risk (risk vs. reward), and how the risk fits in with his portfolio objectives (aggregate risk vs. reward).

For example, although US catastrophe business may have historically given the best return, the return needs to be weighed against the risk and managed by limiting the exposure.

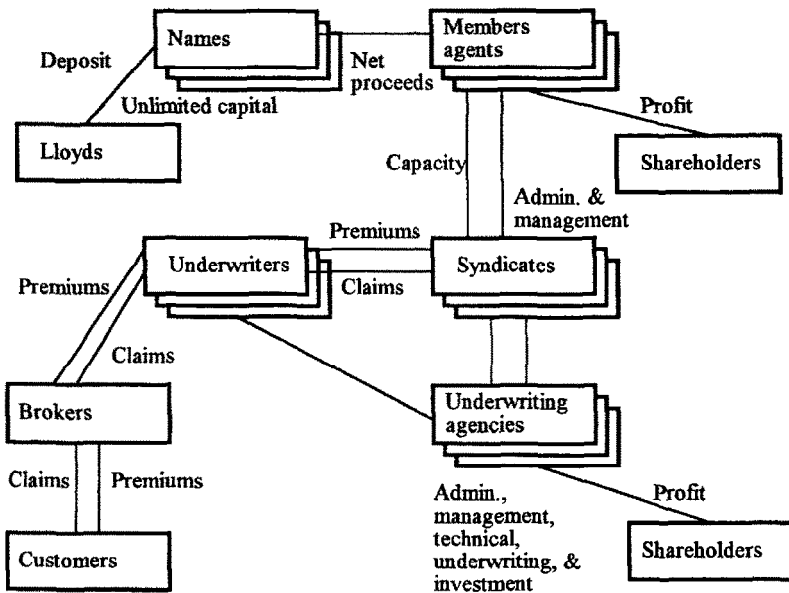
- 2 Management views each class as a separate risk/reward project. This may include allowance for investment returns and the risk of achieving those returns. Mismatching assets and liabilities by term and currency will be a further consideration. It then allocates capital between portfolios by placing restrictions on the underwriter as to the quantity and type of risks that the company will underwrite.

In practice this is often done in an ad-hoc manner with little regard to the risk/reward profile, with some deterministic profit forecast and with scant regard to the risks being run or if so then only in aggregate with respect to specific catastrophes.

- 3 The shareholder views each insurance company as an investment and considers the risk/reward of the investment relative to the market of possible investments.

- 4 The policyholder will be interested in the security of the insurance contract and hence is interested in a low risk insurance operation.

It is clear that the managers, the underwriters, the shareholders, and the policyholders may all have different and conflicting objectives in the structure. A program or strategy is needed to manage all of these issues. This strategy is to be found in the approach employed for capital projects. Before proceeding to an example of the capital project strategy we give below the financial structure diagram for Lloyds of London.



The flow diagram immediately reveals a number of key issues at Lloyds.

First, the structure is too complex and too rigid. Names are not free to buy and sell their position in Lloyds as shareholders in an insurance company can do. There exists too many parties in the structure which adds to delays, inefficiencies and unnecessary costs.

Second, the yearly raising of capital produces an unnecessary emphasis on short term results and corresponding lack of long term continuity and suitable management.

Last, and perhaps worst, the payment structure of the members agents and underwriting agencies are not equitable to the Names as effectively large bonuses are

paid in the times of underwriting profits which are not recouped in the times of underwriting losses. This is an inappropriate structure for all cyclical businesses.

It is clearly necessary to change the structure of Lloyds to make it a more efficient generator of insurance. At least the following changes are already happening at Lloyds.

- i) "Names" with limited liability known as "Corporate Capital".
- ii) A simplified structure should eventually emerge as the changes reduce and remove the need for certain party participants.
- iii) Names to be able to buy and sell participation in syndicates more readily eg. corporate capital and the new auction system for participation.
- iv) Corporate capital effectively opens up the equity raising capability of Lloyds and provides it with longer term funds into the future.

Eventually we believe that Lloyds will become a group of insurance companies regulated as such by the DTI. This approach will give Lloyds a more stream-lined structure, longer term funds, and it will remove the need for members agents and realign the payment structure of underwriting agencies. Removing the need for agents and changing the role of underwriting agencies is necessary because they have in the past received too great a bonus in good underwriting cycles and not then shared the loss in the poor years.

We believe that the change will take place by the introduction of the new Lloyds running in tandem with the old Lloyds. The old Lloyds will then gradually disappear as Names die and as corporate capital becomes available in greater amount. This prolonged process of evolution is necessary because Lloyds still needs the Names for capital and some Names may not wish to convert to limited liability because the conversion can be expected to impact on the Name's gearing and perhaps on the Name's tax position.

An example of a capital project approach

This example follows the system used by USF&G and outlined in a paper by Correnti & Sweeney entitled "Asset-Liability Management and Asset Allocation for Property and Casualty Companies - The Final Frontier". This paper was presented to the 4th International Congress on Insurance Solvency and Finance held at Wharton.

The process addressed in that paper gives a five step programme to determine an efficient frontier in much the same way that the Capital Assets Pricing Model (CAPM) identifies an efficient investment frontier. The steps follow.

- 1 Make an economic evaluation of the balance sheet. This should consider the continuing nature of the business.
- 2 Make an evaluation of the capital markets using simulation models possibly similar to the Wilkie models.

- 3 Optimise the profit by using a multi-time period non-linear optimisation model. This develops efficient frontiers in much the same way as CAPM. In the method used by USF&G they use semi-variance as a measure of risk and the non-linear optimisation takes the place of the linear constrained model.
- 4 Test for sensitivity the key asset, liability and capital market factors.
- 5 Establish appropriate performance measurement indices to give a benchmark against which to compare the results of the decision.

Step 1 - The balance sheet evaluation

Assets

The asset side of the balance sheet of an insurance company consists primarily of marketable securities and moneys owed by brokers, reinsurers etc. To make life simple it is often easy to proxy the asset classes by a suitable stock or index. These need to take account of the effective duration and yield curve of the assets.

Liabilities

The liability side is more difficult and perhaps the most difficult part of the modelling process. This is not surprising.

No single duration model may be appropriate for a designated class of business. The liquidation duration differs from the ongoing duration and the latter depends on the extent of the volume of business anticipated (or allocated) in the future. The balance sheet value should be a "discounted" value. Clearly there are a number of key sensitivities (risks) and namely the impact of the renewal assumption on the duration and the sensitivity of the discount rates used. This can be seen below for a typical Personal Lines account.

Liability duration	Term
Liquidation duration	1.5 years
Including renewals only	4.4 years
Including renewals and new business for 3 years and then renewals	5.1 years
Including renewals and indefinite new business	10.8 years

This clearly has significant impact on liability cash flows, duration, asset allocation and involves (unmeasured) risk. The above example duration represents expected payments on existing reserves and expected payments on new and renewal business and will differ from company to company as well as between lines.

The selection of the appropriate discount rate will also impact on the analysis. A choice must be made to use one average rate say representing the average duration gilt or

different rates representing a function of the yield curve. A possible impact on mean duration is given below.

Insurance class	Duration based on a 4% pa. discount rate	Duration based on a 7% pa. discount rate	Duration based on a 10% pa. discount rate
Employers liability	9.0	6.9	5.5
Fire	1.2	1.1	1.0

Balance sheet analysis - Capital

Capital must be allocated to each line of business. This should take account of the risk that the portfolio carries. In the USF&G model, Risk Based Capital is used. This is a proxy. It may be necessary to run a number of scenarios (simulations) to assess the capital allocation that management feels comfortable but not over-comfortable with; this is supposed to be a market rated system with expensive capital. It is also necessary to split the business lines into appropriate management segments. Profit centres may be appropriate. Within each profit centre the process may be further continued to form a cascade of business analysis. The fewer the number of lines, the easier the analysis is to manage, although sensitivity to the separation into lines needs to be understood.

Step 2 - Simulation of capital markets and allocation of assets

A wealth of actuarial literature may be found on this process. The model should allow for the duration of the assets, the convexity of the yield curve, the volatility in market values and investment returns, and potential default loss risk.

Step 3 - Optimisation

The original Markowitz model is a single period/variance model. There has been some recent debate on this model and modern methodology is now tending to multi-period/semi-variance models and using non-linear optimisation techniques. In the model used in this example there are three key assumptions as follows:

- i) asset returns follow a lognormal distribution;
- ii) to allow for multi-period simulation, the year to year returns are independent;
and
- iii) equilibrium assumptions remain constant (constant return and variability).

These assumptions are capable of being challenged. This is not the issue. There can be many different and valid models. The aim of the process is to maximise the final surplus with respect to the standard deviation of the surplus.

The process is to conduct a number of simulations for a particular level of risk. The levels of risk are then changed and further simulations undertaken producing a series of points which form the outline of the efficient frontier. This optimisation can be based on return on assets, surplus or some other appropriate index. A particular profile can then be selected for the purpose of managing the risks.

Within this process there may be constraints to handle riskier assets, income constraints (eg. solvency) and the mismatching of asset and liability durations.

Step 4 - Sensitivity

It is vital to test each assumption both independently and in tandem with other assumptions. This helps give initial benchmarks on how increasing one line and decreasing another may impact on the return and corresponding asset allocation. The above methodology enables not only existing business to be analysed but also to review the likely future impact of renewals, new business growth, and modification of business segments.

Step 5 - Performance measurement

Undertaking the above work is of little value unless it improves the performance of the operation by at least the cost of the analysis. There may be additional benefits which may not be quantifiable in the accounting sense eg. better information.

Further no model is perfect and we need to measure "model errors" as well as "parameter errors". Also it may be impossible to obtain the ideal portfolio because of regulatory or other business constraints. For example more than a certain percentage of assets in a specific class may be regarded as unsuitable by management/regulators. It is not possible to "jump" in and out of certain lines of business and in any case there is a loss of information if a class is excluded for a period.

Other asset/liability models exist with similar objectives. Most are content with modelling to give management additional information. Models of the type discussed above seem to differ by being more than just a management tool but also a tool to create an improved business philosophy. This philosophy is so similar to capital project analysis that insurance itself should be viewed as being a series of capital projects with reinsurance acting as the risk transfer party typically occupied by insurance in a traditional capital project.

6 Conclusion

As we have seen, a capital project may be considered as being a series of cash flows associated with some entity(ies) and goal(s). The entities and goals are not directly relevant to the analysis excepting in so far as they determine the type and nature of the cash flows and the risks or uncertainties associated with the cash flows. From the analysis viewpoint it all boils down to cash flows; dates; and amounts.

At the analytical level what is important is:

- i) the determination (possibly in a stochastic manner) and analysis of the cash flows; and
- ii) recognition, measurement and valuation of the risks.

The analysis outlined in Section 5 above clearly places insurance itself as a series of capital projects with specific constraints and the analytic tools used in the process are valid for capital projects generally. Actuaries in Australia were originally involved in capital projects concerned with optimising debt structures for building projects within debt/equity, taxation, and legislative constraints. This is similar to optimising the portfolios of an insurance company.

The way this may be undertaken is a structured cascade. An individual risk needs a relatively large amount of capital, a portfolio of risks of the same type will almost certainly require relatively much less capital to support the aggregate (by the law of large numbers assuming some degree of independence) and the aggregate of portfolios (the insurance company) will need less capital again because of the independence and even anticorrelation of losses.

The aim of the business manager is to maximise the profit or, in other words, optimise the structure within the constraints to give the best likely profit for a given risk level. The process described in Section 5 meets all of these criteria and therefore viewing an insurance company as a series of capital projects is not only a useful idea, it is essential to fully understand the processes and the risks and to manage the problems.

We therefore suggest that in the future actuaries should use and develop the capital project's tools and concepts to help them to manage insurance companies. It is also an excellent forum for developing ideas and techniques that may be used in a non-insurance context.

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There are two interesting papers published in the Casualty Actuarial Society Forum for Summer 1994. The first is "Risk and Uncertainty- A Fallacy of Large Numbers" by Paul Samuelson. The second is "Portfolio of Risky Projects" by John M Cozzelino.

In the first paper Samuelson demonstrates the fallacy that there is safety in large numbers - that is that actuarial risks must allegedly cancel out in the sense relevant for investment decisions. I quote from the paper.

"First when an insurance company doubles the number of ships it insures, it does also double the range of possible losses or gains. (This does not deny it reduces the probability of its losses). If at the same time that it doubles its pool of risks, it also doubles the number of its owners (shares), it has indeed left the full maximum possible loss per owner unchanged, but, and this is the germ of truth in the expression "there is safety in numbers", the insurance company has now succeeded in reducing the probability of each loss, the gain to each new owner becomes more certain."

Samuelson's argument uses utility theory to draw the necessary conclusions.

The second paper which is based on utility theory is of particular interest to insurance concerns, and continues the ideas laid down by Samuelson.

x represents total wealth

$U(x)$ represents the decision maker's utility assigned to that wealth.

\tilde{z} represents the profit from some business
(correspondingly z is a random variable with probability
distribution $f(z)$)

Expected utility of z is

$$E\{U(x+\tilde{z})\} = \int U(x+z)f(z)dz$$

Thus uncertainty becomes certainty through integration.

Let $\pi_a(x, \tilde{z})$ be the value at which the decision maker, owning x plus the lottery z would be indifferent to buying or selling.

$$U(x + \pi_a(x, \tilde{z})) = E\{U(x + \tilde{z})\}$$

$U(x)$ is monatomic increasing and thus has a unique inverse function.

Therefore $\pi_a(x, \tilde{z})$ is uniquely determined.

$$\pi(x, \tilde{z}) = E\{\tilde{z}\} - \pi_a(x, \tilde{z})$$

$\pi(x, \tilde{z})$ is the risk premium

For a venture having a small variance and mean $E(\tilde{z})$ then

$$\pi(x, \tilde{z}) = \frac{1}{2}r(x + E(\tilde{z}))\sigma_z^2 + 0(\sigma_z^3)$$

The last term represents a set of terms of order higher than σ_z^2 .

$$r(x) = -U''(x)/U'(x)$$

where primes denote derivatives.

This function is called the local risk aversion because it represents (twice) the risk premium per unit of variance for risks with small variance. It also represents all the risk preference information implied.

Risk Sharing (or Syndication)

Under this we have an α , a share, which itself is a decision variable.

$\alpha = 0$ implies no participation and $\pi_a(x, 0) = 0$

If the utility function is concave then $E\{U(x + \alpha\tilde{z})\}$ is also concave and hence if there is an α^* satisfying

$$\frac{d}{d\alpha} E\{U(x + \alpha\tilde{z})\} = 0$$

then it is a unique maximum. Furthermore, for small variances, if $E(\tilde{z}) > 0$ then $\alpha^* > 0$.

Thus, in theory, every project with positive expected profit can be made desirable if the appropriate financial institute and arrangement can meet.

Constant Risk Aversion

If $r(x) = \text{constant}$, then it can be shown that the utility function has exponential form. This assumption, while not true in the global sense appears reasonable over a limited range.

Let us consider n possible projects.

The i^{th} project has investment cost C_i and uncertain return \tilde{z}_i to be received at the end of the period. This return may be negative.

$f_i(z_i)$ = probability density function of the i^{th} projects return, \tilde{z}_i .

Taking shares we have $\alpha_i C_i$ and $\alpha_i z_i$

Let $A_i(\alpha_i)$ represent the certainty equivalent of the uncertain return $\alpha_i \tilde{z}_i$ based upon the exponential utility function with local risk aversion parameter r .

$$\text{Then } A_i(\alpha_i) = -\frac{1}{r} \ln E(e^{-r\alpha_i \tilde{z}_i})$$

This is the "risk adjusted return."

The risk adjusted net value of the profit of share α_i in project i is $A_i(\alpha_i) - \alpha_i C_i$.

The object is to maximise the risk adjusted value of the portfolio

$$\begin{aligned}\tilde{z} &= \sum_{i=1}^n \alpha_i (\tilde{z}_i - C_i) \\ A(\alpha_1, \alpha_2, \dots, \alpha_n) &= \sum_{i=1}^n [A_i(\alpha_i) - \alpha_i C_i]\end{aligned}$$

This needs maximising subject to $0 \leq \alpha_i \leq 1$ for all i .

If there are no portfolio constraints then every project can be decided upon independently.

Then there are n independent subproblems $\max[A_i(\alpha_i) - \alpha_i C_i]$ subject to $0 \leq \alpha_i \leq 1$.

If the expected profit is positive, then there is a unique solution and the optimum share $\alpha_i^* > 0$. This is at a stationary point if it is in the unit interval or if $\alpha_i^* = 1$.

$$\alpha_i^*(C_i) = \text{initial cost}$$

If $C_i \geq E(\tilde{z}_i)$ then $\alpha_i^*(C_i) = 0$.

For $C_i < E(\tilde{z}_i)$ and such that $\alpha_i^* < 1$, $\alpha_i^*(C_i)$ is the stationary point of the function

$$A_i(\alpha_i) - \alpha_i C_i$$

Thus it is the solution of $A_i'(\alpha_i) = C_i$

Differentiating with respect to C_i

$$A_i''(\alpha_i) d\alpha_i^*(C_i)/dC_i = 1$$

$A_i''(\alpha_i)$ is negative at the optimum. Thus $\alpha_i^*(C_i)$ is a decreasing function of its argument at points where $0 < \alpha_i^*(C_i) < 1$. As C_i approaches $E(\tilde{z}_i)$, the projects expected profit approaches zero and the optimum share $\alpha_i^*(C_i)$ also approaches zero.

If the firm is constrained to accept ($\alpha_i = 1$) or reject ($\alpha_i = 0$) each project, then the solution to accept if and only if $A_i(1) - C_i > 0$. Thus many attractive, but highly risky projects are rejected.

Budget Constraints

The problem is reduced to

$$\begin{aligned} \max \sum_{i=1}^n [A_i(\alpha_i) - \alpha_i C_i] \quad \text{subject to} \\ \sum_{i=1}^n \alpha_i C_i \leq C \quad \text{and} \\ 0 \leq \alpha_i \leq 1 \quad \text{for all } i. \end{aligned}$$

The solution is obtained by Dynamic Programming. The Lagrangian is

$$A(\alpha_1, \dots, \alpha_n) - \lambda (\sum_{i=1}^n \alpha_i C_i - C)$$

The problem is

$$\max \lambda C + \sum_{i=1}^n [A_i(\alpha_i) - \alpha_i C_i (1 + \lambda)]$$

subject to $0 \leq \alpha_i \leq 1$ for all i .

λ fixed implies independent decisions.

The effect of the budget constraint is equivalent to each initial cost of each project being multiplied by $(1 + \lambda)$. Thus λ can be viewed as an increase in the cost of capital (c.f. solvency margins).

Initially we solve the unconstrained problem, since if

$$\sum_{i=1}^n \alpha_i^*(C_i) C_i \leq C$$

then it is also the solution of the constrained project. If the initial capital required by the

unconstrained solution exceeds the budget C , the problem can be solved in a one-dimensional search when λ is increased in value until

$$\sum_{i=1}^n \alpha_i^*(C_i(1 + \lambda)) C_i = C$$

As λ increases, the share of each payment may change from positive to zero participation. The λ value at which the i^{th} project drops out is found from

$$E(\tilde{z}_i) - C_i(1 + \lambda) = 0$$

The critical value is thus $\lambda^{(i)} = (E(\tilde{z}_i) - C_i)/C_i$

Example

Project 1

We use $A(\alpha) = -\frac{1}{r} \ln E\{e^{-r\alpha}\}$

If the profit is normally distributed with mean μ and variance σ^2

$$E\{e^{+r\alpha}\} = e^{+r\alpha\mu + \frac{1}{2}r^2\alpha^2\sigma^2}$$

$$A_1(\alpha) = \alpha\mu - \frac{1}{2}r\alpha^2\sigma^2$$

$$\begin{aligned}\alpha_1^*(C_1) &= 1 \text{ for } 0 < C_1 \leq \mu - r\sigma^2 \\ &= \frac{\mu - C_1}{r\sigma^2} \text{ for } \mu - r\sigma^2 < C_1 < \mu \\ &= 0 \text{ for } C_1 \geq \mu\end{aligned}$$

If $\mu = 4\text{million}$, $\sigma^2 = 200$ and $C_1 = 1.5\text{million}$

Then $A_1(\alpha_1) = 4\alpha_1 - 5\alpha_1^2$

Project 2

This Project has Gamma probability distribution of return.

$$f_2(Z_2) = a^b(Z_2)^{b-1}e^{-aZ_2}/\Gamma(b) \text{ for } Z_2 \geq 0$$

with mean b/a and variance b/a^2

Thus $A_2(\alpha) = \frac{b}{r} \ln(1 + \frac{r\alpha}{b})$

and $A_2'(\alpha) = \frac{b}{a+r\alpha}$

and $\begin{aligned}\alpha_2^*(C_2) &= 1 \text{ for } C_2 \leq (\frac{b}{a+r}) \\ &= \frac{1}{r}(\frac{b}{C_2} - a) \text{ for } (\frac{b}{a+r}) < C_2 < \frac{b}{a} \\ &= 0 \text{ for } C_2 \geq \frac{b}{a}\end{aligned}$

Let $b=0.8$, $a=0.02$. Then $b/a = 4$ million and $b/a^2 = 200$.

The initial cost is $C_2 = 1.6$ million

Then $A_2(\alpha_2) = 1.6 \ln(1 + \frac{5\alpha_2}{2})$

Using these two portfolios we now consider the Unconstrained and Constrained Position

Unconstrained

	Project 1	Project 2
$A_i(1)$	-1.0	+1.975
accept/reject	reject	accept
best share (α_i^*)	.25	.60
$A_i(\alpha_i^*)$.6875	1.4661
$\alpha_i^* C_i$.3750	0.9600
$A_i(\alpha_i^*) - \alpha_i^* C_i$.3125	0.5061

Thus we have an optimal portfolio of risk adjusted return of 2.1536 for an initial cost of 1.3350 and risk adjusted profit of .8186

If sharing was not feasible then we accept the second project only with a risk adjusted profit of $1.975 - 1.600 = .375$

Constrained

If our budget exceeds 1.3350 we have the optimal solution above.

If not we have the following

Constraint	λ	Project 1 share	Project 2 share
$C = 1.335$	0	.250	.600
$C = 1.0$.217	.218	.422
$C = .689$.5	.175	.267
$C = .310$	1	.100	.100
$C = .015$	1.6	.010	0

Project 2 becomes unviable at $\lambda = 1.5$, Project 1 becomes unviable at $\lambda = 1.667$