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# SOME RESULTS ON THE GOMPERTZ AND HELIGMAN AND POLLARD LAWS OF MORTALITY

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ABSTRACT

The 'law of mortality' proposed by Heligman and Pollard is compared with the law of Gompertz and with English Life Table No. 14. Some new mathematical results are derived, including specific equations for the curve of deaths. Some numerical illustrations are given.

KEYWORDS

Life Tables; Mathematics; Mortality

## 1. INTRODUCTION

Heligman & Pollard (1980) have proposed a 'law of mortality' which, at ages over 50, provides a most interesting alternative to the original law of Gompertz (1825). A main purpose of the present paper is to compare the properties of these two laws and to see how well they fit the data in E.L.T. No. 14, based on the mortality rates in 1980–82.

In this context, the word 'law' has a long history. It is here used in a statistical sense, to describe an observed regularity or pattern, particularly one which satisfies some simple mathematical formula. Whether such regularities exist, how closely they hold and whether they persist or just recur, are matters which can only be determined by observation. However, when either the Gompertz or the Heligman and Pollard law holds, it is possible (and indeed easy) to calculate, from just two parameters, estimates of all the entries in a life table from age 50 onwards. There are therefore plenty of ways in which the 'laws' can be checked.

## 2. MATHEMATICAL RELATIONSHIPS

*Notation*

We shall use the standard notation. If $l_x$ denotes the number of members of a population who survive to reach the exact age $x$, then $d_x = l_x - l_{x+1}$, $q_x = d_x/l_x$ and $p_x = 1 - q_x = l_{x+1}/l_x$. All the logarithms in the paper are natural logarithms.

*The Gompertz and Heligman and Pollard Laws*

In the course of studying mortality tables, Gompertz (1825) discovered the remarkable fact that the first differences of log $l_x$ were very close to a geometric progression. Nowadays his 'law of human mortality' is usually expressed in the continuous (though rather more complicated) form:

$$\mu_x = -\frac{1}{l_x}\left(\frac{dl_x}{dx}\right) = -\frac{d(\log l_x)}{dx} = A \cdot c^x \tag{1}$$

where $A$ and $c$ are constants. The left hand equation in (1) is simply the *definition* of the force of mortality ($\mu_x$) as the instantaneous rate of mortality at age $x$. The right hand equation in (1) is the actual 'law'. The constant $A$ reflects the general level of mortality in the population concerned. The constant $c$ (the Gompertz constant) reflects the rate at which the force of mortality increases with age; Gompertz attributed this increase to the physiological process of ageing.

Several more complicated 'laws' have since been proposed, attempting to fit the data in mortality tables more accurately over a wider range of ages. In particular, Heligman & Pollard (1980) have proposed a law covering all ages from birth onwards. Their full expression has three terms and eight parameters. However, above age 50 the first two terms can be neglected and we are left with just the term:

$$q_x/p_x = G \cdot H^x \tag{2}$$

where $G$ and $H$ are constants. For brevity, we shall describe (2) as 'the H&P law'.

We note that for a person who has just reached age $x$, the quantity $q_x/p_x$ is the odds on dying before reaching age $x+1$. The odds on dying within a year is a very direct measure indeed of the risk of mortality. Clearly it is closely related to the force of mortality, and the H&P law is very similar to the Gompertz law. The constants $c$ and $H$ both reflect the rate at which people become more liable to die as they become older.

We observe that (2) can be rewritten as:

$$\log(q_x/(1-q_x)) = \log G + x \cdot \log H. \tag{3}$$

This shows that if $q_x$ is plotted against $x$ we obtain a logistic curve, with $q_x$ tending asymptotically to 1 as $x$ tends to infinity.

Under the Gompertz law, $q_x$ also tends asymptotically to 1 as $x$ tends to infinity, but it does so more rapidly than on the logistic curve. Thus the chances of reaching very high ages indeed are greater under (2) than under (1).

*Life Tables from the Gompertz Law*

In this section we give for reference the standard formulae for the Gompertz law. We suppose that the law holds above some starting age $x = a$, where the rate of mortality is $q_a$. We assume some convenient (but essentially arbitrary) value for $l_a$. We have $p_a = 1 - q_a$.

The first differences of $\log l_x$ are given by:

$$\Delta(\log l_x) = \log l_{x+1} - \log l_x = \log p_x. \tag{4}$$

Thus Gompertz' finding that these first differences are in geometric progression means that:

$$\log p_{x+1} = c \cdot \log p_x \tag{5}$$

where $c$ is a constant. Also we have, by definition:

$$l_{x+1} = l_x \cdot p_x. \tag{6}$$

Using (5) and (6) we can calculate successive values of $p_x$ and $l_x$ by iteration, starting from $p_a$ and $l_a$; or we can sum the geometric progressions to obtain:

$$\log p_x = c^{x-a} \cdot \log p_a \tag{7}$$

$$\log(l_x/l_a) = \log p_a \cdot (c^{x-a} - 1)/(c-1). \tag{8}$$

Alternatively, we can integrate (1) to obtain:

$$l_x = K \cdot \exp(-Ac^x/\log c) \tag{9}$$

and hence

$$\log p_x = \Delta(\log l_x) = -Ac^x(c-1)/\log c \tag{10}$$

where $K$ is a constant of integration which is determined by $l_a$. In all cases we can derive $q_x$ and $d_x$ from $p_x$ and $l_x$. Thus we can construct a life table from the Gompertz law by using (5)–(8) if we are given $c$ and $p_a$, or by using (9)–(10) if we are given $c$ and $A$.

*Life Tables from the Heligman and Pollard Law*

For the H&P law, we deduce from (2) that:

$$1 - p_x = p_x \cdot GH^x$$

and hence

$$p_x = (1 + GH^x)^{-1}. \tag{11}$$

From this we can calculate all the required values of $p_x$, and hence $q_x$, for the life table. By using (6) we can then calculate all the values of $l_x$ for each exact year of age.

However, for calculating the expectation of life and the continuous curve of deaths, we shall also need to find $l_x$ for ages in between the exact years. In other words, we need an equivalent of (8) or (9). This is less straightforward than for the Gompertz law, but it can be done and there are two methods.

The first method gives an exact mathematical solution. It follows from (6) and (11) that $l_x$ must satisfy the functional equation:

$$l_{x+1}/l_x = (1 + GH^x)^{-1}. \tag{12}$$

It has been shown by Hayman & Thatcher (1990) that there is one and only one mathematical function which satisfies (12) and which is also monotonic for all values of $H$ and $x$. This unique function is the infinite product:

$$l_x = K \cdot \prod_{u=1}^{\infty} (1 + GH^{x-u})^{-1} \tag{13}$$

where $K$ is a constant which is determined by the starting value $l_a$, where $x$ is a continuous variable and where $u$ takes the successive integer values 1, 2, 3, . . . . Although the appearance of (13) may be unfamiliar at first sight, nevertheless the infinite product converges rapidly and (13) can be evaluated quite quickly on a computer.

The second method is an approximation, but a very good one. It is based on a finding about the second differences of log $l_x$, which are the same as the first differences of log $p_x$. Now by differentiating (3) and using $q_x = 1 - p_x$ we find that:

$$\frac{1}{p_x}\left(\frac{dp_x}{dx}\right) = -q_x \cdot \log H$$

for all values of $x$. From this it can be shown that:

$$-q_{x+1} \cdot \log H < \Delta(\log p_x) < -q_x \cdot \log H$$

and hence that

$$-q_{x+1} \cdot \log H < \Delta^2(\log l_x) < -q_x \cdot \log H. \tag{14}$$

This means that for the cases of actuarial interest, where $H$ is about 1.1, the second differences $\Delta^2$ (log $l_x$) will vary only very slowly and will be confined to a range between 0 and about $-0{\cdot}1$. (This is a complete contrast to the Gompertz law, for which the second differences $\Delta^2$ (log $l_x$) diverge and eventually tend to minus infinity.)

Given this convenient property of the second differences, it is possible to take advantage of the forward interpolation formula:

$$\log l_{x+t} = \log l_x + t \cdot \Delta(\log l_x) + \tfrac{1}{2}t(t-1) \cdot \Delta^2(\log l_x) + \ldots . \tag{15}$$

In view of (14), and for the limited purpose of interpolating log $l_x$ (and hence $l_x$) in the range $0 < t < 1$, we can ignore the third and later differences and can evaluate (15) by putting:

$$\Delta(\log l_x) = \log p_x \tag{16}$$

and

$$\Delta^2(\log l_x) \doteqdot -q_x \cdot \log H. \tag{17}$$

The method gives an excellent approximation for log $l_{x+t}$ and its accuracy has been confirmed by comparison with exact values of $l_{x+t}$ calculated directly from (13).

*Expectation of Life*

The expectation of life at age $x$ is given by:

$$\mathring{e}_x = \frac{1}{l_x}\int_x^{\infty} l_t dt \tag{18a}$$

$$= \frac{1}{l_x}\sum_{i=0}^{\infty} L_i \tag{18b}$$

where $L_i = \int_{x+i}^{x+i+1} l_t \cdot dt.$

We can evaluate the $L_i$ by the standard methods of numerical integration, sub-dividing the years of age into as many intervals as are needed to achieve the accuracy required.

For example, if the year is sub-divided into two equal intervals, Simpson's rule gives:

$$L_i \doteqdot (l_{x+i} + 4 \cdot l_{x+i+\frac{1}{2}} + l_{x+i+1})/6$$

and the values of $l_{x+i+\frac{1}{2}}$ can be found from (8) for the Gompertz law and from (15) for the H&P law.

At very high ages indeed, there are asymptotic formulae. For the Gompertz law, on substituting (9) in (18a) and then putting $z = Ac^t/\log c$, we find that:

$$\mathring{e}_x = \int_U^{\infty} e^{-z} z^{-1} dz / (e^{-U} \log c)$$

where $U = Ac^x/\log c = \mu_x/\log c$.

The integral in this expression is the exponential integral, a limiting form of the gamma function: it has a known asymptotic expansion which gives:

$$\int_U^{\infty} e^{-z} z^{-1}\, dz \sim e^{-U} U^{-1} \quad \text{as } U \to \infty.$$

It follows that:

$$\mathring{e}_x \sim \frac{1}{\mu_x} \quad \text{as} \quad x \to \infty. \tag{19a}$$

Using (1) and (10) this can also be written as:

$$\mathring{e}_x \sim \left[ -\log p_x \cdot \frac{\log c}{(c-1)} \right]^{-1} \quad \text{as} \quad x \to \infty. \tag{19b}$$

Here the symbol '$\sim$' means that the ratio of the two sides tends to 1 as $x$ tends to infinity.

For the H&P law, it has recently been shown by Hayman & Thatcher (1990) that:

$$\mathring{e}_x \sim (-\log p_x)^{-1} \quad \text{as} \quad x \to \infty. \tag{20}$$

It is notable that this does not depend on the value of $H$.

The results (19) and (20) apply only at very high ages indeed and are given largely for their theoretical interest. What they show is that at these very high ages the main determinant of the expectation of life, under both laws, is the value of $p_x$. This may perhaps have some application in debates about the span of human life.

*The Curve of Deaths for the Gompertz Law*

The figures of $d_x$ in a life table give a histogram, which shows the relative frequency of deaths at each age, if mortality follows the rates used to construct the table. This histogram is an approximation to a smooth frequency curve, the so-called 'curve of deaths', for which the $y$-coordinate is defined by:

$$y_x = \mu_x l_x = -\frac{dl_x}{dx}. \tag{21}$$

For the Gompertz law, $\mu_x$ is given by (1) and $l_x$ is given by (9), so by multiplying these together we find that the equation of the curve of deaths is given specifically by:

$$y_x = K \cdot Ac^x \cdot \exp(-Ac^x/\log c). \tag{22}$$

By using (10) we can also write this as:

$$y_x = -l_x \cdot \log p_x \cdot \log c/(c-1). \tag{23}$$

We can now find the age at which the curve reaches its peak, i.e. the modal age of death. From (22) we find that:

$$\frac{d\,(\log y_x)}{dx} = \log c - Ac^x.$$

This shows that the peak occurs at exactly that age $x$ where:

$$A \cdot c^x = \log c. \tag{24}$$

This provides a direct derivation of a result which has previously been given by Redington (1969). We can also go a little further. From (24), (10) and (7) we find that, at the peak:

$$c-1 = -\log p_x = (-\log p_a) \cdot c^{x-a}. \tag{25}$$

On taking logarithms of (25) it then follows that the peak of the Gompertz curve of deaths occurs at exactly the age:

$$x = a + \log\left(\frac{c-1}{-\log p_a}\right)\Big/\log c. \tag{26}$$

Moreover, we find from (23) and (25) that at the peak we have:

$$y_{\max} = l_x \cdot \log c. \tag{27}$$

*The Curve of Deaths for the H&P Law*

In order to find the curve of deaths for the H&P law, we note that (21) can be rewritten as:

$$y_x = -l_x \cdot \frac{d(\log l_x)}{dx}. \tag{28}$$

Now from (15) we can find the limit of $(\log l_{x+t} - \log l_x)/t$ as $t$ tends to zero. This gives:

$$\frac{d(\log l_x)}{dx} \doteqdot \Delta(\log l_x) - \tfrac{1}{2}\cdot\Delta^2(\log l_x)$$
$$\doteqdot \log p_x + \tfrac{1}{2}\cdot q_x \cdot \log H. \tag{29}$$

From (28) and (29) we see that the curve of deaths for the H&P law is given, to a very close approximation indeed, by:

$$y_x \doteqdot -l_x \cdot (\log p_x + \tfrac{1}{2}\cdot q_x \cdot \log H). \tag{30}$$

Next, by using the identity $d_x = l_x \cdot q_x$ in conjunction with (2), it is not difficult to show that:

$$d_{x-1} < d_x \quad \text{if} \quad 1 < H \cdot p_x$$
$$d_{x+1} > d_x \quad \text{if} \quad 1 > H \cdot p_{x+1}.$$

It follows, as an exact result, that the peak value of $d_x$ will occur at the exact age $x$ where:

$$p_x > 1/H \qquad \text{and} \qquad p_{x+1} < 1/H. \tag{31}$$

From this we may take it that the peak of the curve (30) must lie close to, if not exactly at, the age $x$ where:

$$p_x = 1/H. \tag{32}$$

This age we can now find. When (32) is satisfied we shall have $q_x/p_x = H-1$. Hence from (2) we shall have:

$$q_x/p_x = GH^x = (q_a/p_a)\cdot H^{x-a} = H-1.$$

Hence, on taking logarithms of the last two terms:

$$x = a + \log\left(\frac{H-1}{q_a/p_a}\right)\bigg/\log H. \tag{33}$$

At this age, i.e. where $p_x = 1/H$, it follows from (30) that at the peak we have:

$$\begin{aligned} y_{\max} &\doteqdot l_x \cdot \log H \cdot (1 - \tfrac{1}{2} \cdot q_x) \\ &\doteqdot l_x \cdot \log H \cdot ((H+1)/2H). \end{aligned} \tag{34}$$

Thus the co-ordinates of the peak of the curve of deaths for the H&P law are given, approximately, by (33) and (34).

## 3. FITTING THE 'LAWS' TO THE DATA

In order to see whether a given life table shows the regularities which are predicted by one of the 'laws', we first need to estimate the two parameters for the law concerned. This can be done in several ways.

*Method A*

The first (and easiest) method is to reduce the problem to that of fitting a straight line by simple least squares. For the Gompertz law, this can be done as follows. If we put $y = \log(-\log p_x)$, then (7) shows that:

$$\begin{aligned} y &= \log(-\log p_x) = (x-a) \cdot \log c + \log(-\log p_a) \\ &= ex + f \end{aligned} \tag{35}$$

where $e = \log c, f = \log(-\log p_a) - a \cdot \log c$.

For each value of $x$ we can calculate the corresponding value of $y$ from the values of $q_x$, and hence $p_x$, in our given life table. If we now fit the straight line (35) by simple least squares, we obtain estimates of $e$ and $f$, from which we can easily derive estimates of $c$ and $p_a$.

The same method can be applied to the H&P law. If we put $y = \log(q_x/p_x)$, then (2) shows that (35) holds, this time with $e = \log H, f = \log G$. We can now fit (35) by simple least squares and so derive estimates of $G$ and $H$.

*Method B*

A quite different method, with theoretical attractions, is to find the values of the parameters which give the closest fit to the distribution of ages at death, as given by the figures of $d_x$ in the life table. This can be done, for example, by minimising chi-squared.

*Method C*

Alternatively, we can find the values of the parameters which give the closest fit to the distribution of ages at death, using the method of maximum likelihood.

*Method D*

In the case of the H&P law, there is also the method which was used by Heligman and Pollard themselves, and following them by Forfar & Smith (1987). These authors fitted the full Heligman and Pollard expression with all eight parameters, using a computer programme which minimises the sum of squares:

$$\Sigma(\hat{q}_x/q_x - 1)^2$$

where $q_x$ is the observed and $\hat{q}_x$ the fitted value. The parameters were fitted to mortality rates from birth to age 85.

Table 1 shows the results obtained by fitting the H&P law to E.L.T.s Nos. 10–14 by all four methods. For the first three columns of the table, the simple H&P law (2) was fitted to the data in the life tables at ages 50–90 inclusive. The reason for this choice of ages was that the simple law (2) was not originally expected to apply below about age 50 (though in fact it continues to apply at lower ages than this for females); and that above age 90 the E.L.T.s make use of extrapolations. However, even after these exclusions, there were still 41 observations available to fit two parameters. The author is much indebted to

Table 1. *Estimates of* G *and* H *fitted by four methods*

| | Method A | Method B | Method C | Method D |
|---|---|---|---|---|
| $G \times 10^5$ for males | | | | |
| E.L.T. No. 10 | 10·1326 | 9·4309 | 9·3199 | 12·875 |
| E.L.T. No. 11 | 7·5661 | 7·8299 | 7·8118 | 5·4925 |
| E.L.T. No. 12 | 7·6970 | 8·2788 | 8·0589 | 4·0468 |
| E.L.T. No. 13 | 7·7250 | 8·1619 | 8·0097 | 3·7853 |
| E.L.T. No. 14 | 5·6467 | 5·8543 | 5·7837 | 2·955 |
| $G \times 10^5$ for females | | | | |
| E.L.T. No. 10 | 5·7617 | 5·0956 | 4·9795 | 4·8421 |
| E.L.T. No. 11 | 2·2894 | 1·9262 | 1·8791 | 2·1927 |
| E.L.T. No. 12 | 1·8520 | 1·6018 | 1·5739 | 1·2486 |
| E.L.T. No. 13 | 2·2835 | 1·8984 | 1·8324 | 2·8313 |
| E.L.T. No. 14 | 2·1563 | 1·7429 | 1·6971 | 2·243 |
| *H* for males | | | | |
| E.L.T. No. 10 | 1·0968 | 1·0979 | 1·0981 | 1·0930 |
| E.L.T. No. 11 | 1·1003 | 1·0998 | 1·0998 | 1·1052 |
| E.L.T. No. 12 | 1·0989 | 1·0981 | 1·0985 | 1·1090 |
| E.L.T. No. 13 | 1·0980 | 1·0975 | 1·0978 | 1·1093 |
| E.L.T. No. 14 | 1·1011 | 1·1008 | 1·1009 | 1·1113 |
| *H* for females | | | | |
| E.L.T. No. 10 | 1·1013 | 1·1031 | 1·1034 | 1·1036 |
| E.L.T. No. 11 | 1·1121 | 1·1147 | 1·1150 | 1·1125 |
| E.L.T. No. 12 | 1.1131 | 1.1152 | 1.1155 | 1.1184 |
| E.L.T. No. 13 | 1.1083 | 1.1110 | 1·1115 | 1·1047 |
| E.L.T. No. 14 | 1·1075 | 1·1105 | 1·1109 | 1·1065 |

Miss Amal Soliman for making the calculations for Methods B and C. The final column of the table shows for comparison the values found by Forfar & Smith (1987) by Method D, fitting the full Heligman and Pollard expression with eight parameters to ages 0–85 inclusive, i.e. 86 observations.

In the case of males, Methods B and C gave almost identical results and these were all very close to those given by Method A. However, Method D seemed to be out of line; over the period 1931–1981 it gave a considerably larger fall in $G$ and a larger rise in $H$ than the other methods; also, the results did not always fit the E.L.T.s quite so well at ages over 50. For females the contrasts were less clear cut, but again Method D showed somewhat larger fluctuations than the others. It must be remembered that it is a difficult thing to fit eight parameters simultaneously to only 86 observations. For a limited purpose like the present paper, where the interest is confined to ages 50 and over, there may be an advantage in using only two parameters, unencumbered by the data at ages 0–49.

In the case where the largest difference occurred between Methods A, B and C, it was found that Methods B and C gave a very slightly better fit to the E.L.T. at ages 70–90, where the bulk of the deaths are concentrated, and a very slightly worse fit at ages below 70 and above 90. Here 'very slightly' means that at ages up to 80, the differences between the estimates of $q_x$ given by Method A on the one hand, and Methods B and C on the other, were only of the order of ·001.

In view of these findings, it was decided to use Method A to find the parameters for the numerical illustrations which follow. This is by far the simplest method and it can readily be used to anyone who wishes to fit either the Gompertz or the H&P laws to a given life table: whereas the other methods are far more complicated and do not appear to have decisive advantages at ages over 50. Also, by using Method A in the illustrations, it is possible for the reader to see how this method performs. Applied to E.L.T. No. 14 at ages 50–90 inclusive, the method gives the following parameters for 1980–82:

For the Gompertz law: $c=1{\cdot}0979$, $p_{50}={\cdot}992930$ for males;
$c=1{\cdot}1052$, $p_{50}={\cdot}996394$ for females.

For the H&P law: $G=5{\cdot}6467\times10^{-5}$, $H=1{\cdot}1011$ for males:
$G=2{\cdot}1563\times10^{-5}$, $H=1{\cdot}1075$ for females.

## 4. NUMERICAL ILLUSTRATIONS

*Life Tables*

Using these parameters, and the formulae derived above, we can construct quite easily the abridged life tables in Tables 2 and 3. Since we are only dealing with persons aged 50 and over, Table 2 shows the ratio $l_x/l_{50}$, which is the proportion of persons reaching age 50 who survive to reach age $x$. From this, one can readily obtain values of $l_x$, as required, for any desired value of $l_{50}$.

Tables 2 and 3 deserve careful study, for they show in a very direct and compact way how well the two 'laws' fit the current E.L.T. Perhaps the first reaction must be one of surprise, that so many of the figures are so close to each

Table 2. *Estimates of* $l_x$ *and* $q_x$

| | Estimates of $l_x/l_{50}$ | | | Estimates of $q_x$ | | | |
|---|---|---|---|---|---|---|---|
| Age | Gompertz | H&P | E.L.T. No. 14 | Gompertz | H&P | E.L.T. No. 14 | |
| Males 1980–82 | | | | | | | |
| 50 | 1·000 | 1·000 | 1·000 | ·007 | ·007 | ·006 | |
| 55 | ·958 | ·958 | ·961 | ·011 | ·011 | ·011 | |
| 60 | ·894 | ·895 | ·897 | ·018 | ·018 | ·018 | |
| 65 | ·801 | ·801 | ·801 | ·028 | ·029 | ·029 | |
| 70 | ·672 | ·671 | ·667 | ·045 | ·046 | ·047 | |
| 75 | ·509 | ·505 | ·497 | ·071 | ·072 | ·074 | |
| 80 | ·326 | ·321 | ·312 | ·110 | ·111 | ·113 | See |
| 85 | ·160 | ·158 | ·153 | ·170 | ·169 | ·166 | note (a) |
| 90 | ·051 | ·052 | ·053 | ·257 | ·247 | ·227 | ·227 |
| 95 | ·008 | ·010 | ·012 | ·378 | ·347 | ·290 | ·308 |
| 100 | ·000 | ·001 | ·002 | ·531 | ·462 | ·381 | ·409 |
| 105 | ·0000 | ·0000 | ·0001 | ·701 | ·582 | ·524 | ·528 |
| Females 1980–82 | | | | | | | |
| 50 | 1·000 | 1·000 | 1·000 | ·004 | ·004 | ·004 | |
| 55 | ·978 | ·978 | ·977 | ·006 | ·006 | ·006 | |
| 60 | ·943 | ·943 | ·940 | ·010 | ·010 | ·010 | |
| 65 | ·887 | ·888 | ·886 | ·016 | ·016 | ·015 | |
| 70 | ·803 | ·802 | ·807 | ·026 | ·027 | ·024 | |
| 75 | ·681 | ·679 | ·692 | ·043 | ·044 | ·041 | |
| 80 | ·519 | ·516 | ·532 | ·070 | ·071 | ·070 | See |
| 85 | ·332 | ·329 | ·336 | ·113 | ·112 | ·119 | note (a) |
| 90 | ·158 | ·159 | ·153 | ·179 | ·174 | ·185 | ·185 |
| 95 | ·047 | ·050 | ·047 | ·278 | ·260 | ·249 | ·260 |
| 100 | ·006 | ·008 | ·009 | ·415 | ·370 | ·323 | ·358 |
| 105 | ·0002 | ·0005 | ·0009 | ·587 | ·494 | ·478 | ·479 |
| 110 | ·00000 | ·00001 | ·00001 | ·768 | ·619 | ·674 | |

The table is calculated from the following parameters:
Gompertz: $c = 1{\cdot}0979$, $p_{50} = {\cdot}992930$ for males; $c = 1{\cdot}1052$, $p_{50} = {\cdot}996394$ for females.
H&P law: $G = 5{\cdot}6467 \times 10^{-5}$, $H = 1{\cdot}1011$ for males; $G = 2{\cdot}1563 \times 10^{-5}$, $H = 1{\cdot}1075$ for females.

*Note:* (a) The figures in the final column are estimates based on death registrations only.

other. Clearly the regularity which was first observed by Gompertz is still flourishing. It is also remarkable that so much information can be calculated from just two parameters.

On points of detail, for males the H&P law is slightly closer to E.L.T. No. 14 than is the Gompertz law, at all the ages in Table 2. In the case of females, the Gompertz law fits, if anything, very slightly better than the H&P law at ages 50–90, but above age 95 it under-estimates the number of survivors.

Table 2 was calculated by using equations (7) and (8) for the Gompertz law, and (6) and (11) for the H&P law. Table 3 was calculated from equation (18), with Simpson's rule, and using (8) to interpolate for the Gompertz law and (15) for the H&P law.

Table 3. *Expectations of life*

| Age | Gompertz | H&P | E.L.T. No. 14 | |
|---|---|---|---|---|
| | Estimates of $\mathring{e}_x$ | | | |
| Males 1980–82 | | | | |
| 50 | 24·4 | 24·4 | 24·3 | |
| 55 | 20·3 | 20·3 | 20·1 | |
| 60 | 16·6 | 16·5 | 16·4 | |
| 65 | 13·2 | 13·2 | 13·0 | |
| 70 | 10·3 | 10·2 | 10·1 | |
| 75 | 7·7 | 7·7 | 7·7 | |
| 80 | 5·7 | 5·7 | 5·8 | See note (a) |
| 85 | 4·0 | 4·1 | 4·3 | 4·3 |
| 90 | 2·8 | 2·9 | 3·3 | 3·2 |
| 95 | 1·9 | 2·1 | 2·6 | 2·4 |
| 100 | 1·2 | 1·5 | 1·9 | 1·8 |
| 105 | 0·8 | 1·1 | 1·2 | 1·3 |
| Females 1980–82 | | | | |
| 50 | 29·3 | 29·3 | 29·4 | |
| 55 | 24·9 | 24·8 | 25·0 | |
| 60 | 20·7 | 20·7 | 20·9 | |
| 65 | 16·8 | 16·8 | 17·0 | |
| 70 | 13·3 | 13·3 | 13·4 | |
| 75 | 10·2 | 10·2 | 10·2 | |
| 80 | 7·6 | 7·6 | 7·5 | See note (a) |
| 85 | 5·5 | 5·6 | 5·4 | 5·2 |
| 90 | 3·8 | 4·0 | 3·9 | 3·9 |
| 95 | 2·6 | 2·8 | 3·0 | 2·8 |
| 100 | 1·7 | 2·0 | 2·2 | 2·0 |
| 105 | 1·1 | 1·4 | 1·4 | 1·4 |
| 110 | 0·7 | 1·0 | 0·8 | 1·0 |

The table is calculated from the same parameters as Table 2.

*Note:* (a) The figures in the final column are estimates based on death registrations only.

*Ages over 90*

Although Tables 2 and 3 have been extended to very high ages, it must be remembered that the figures above age 90 are subject to certain qualifications. We are here extrapolating the Gompertz and H&P laws beyond the range to which the parameters were fitted. Moreover, the figures in E.L.T. No. 14 are also dependent on extrapolations. There are considerable difficulties in analysing the data at the highest ages, largely owing to uncertainties about the exact numbers at risk.

For centenarians it is possible to make some independent estimates, based on death registrations only, by the method of extinct generations. This has been used by Humphrey (1970) and Thatcher (1987), who found that in the range of ages 90–105 the data could be represented reasonably well by segments of Gompertz curves with the constants $c = 1{\cdot}0740$ for males and 1·0804 for females. The

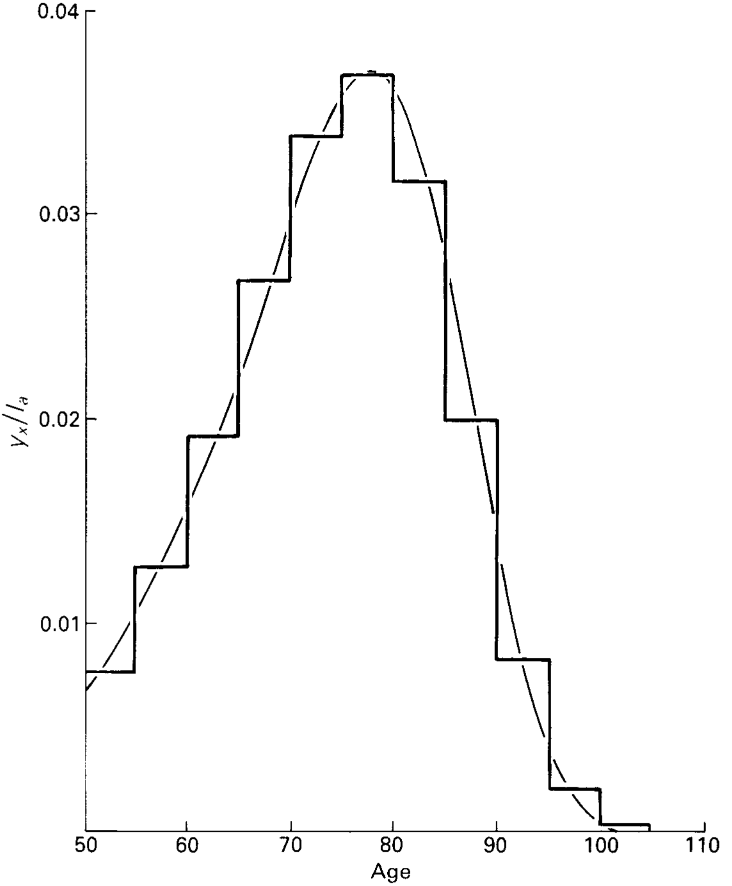


Figure 1. Curve of deaths from the H&P law and histogram from E.L.T. No. 14. Males, 1980–82.

resulting estimates of $q_x$ and $e_x$ in 1981 are reproduced in the last columns of Tables 2 and 3.

We may note that over limited segments, it is possible to fit mortality rates almost equally well by either a Gompertz or an H&P law, provided that the parameters are suitably chosen; but in general, such curves will cease to fit so well outside the range to which the parameters were fitted. What is remarkable about Tables 2 and 3 is that they show that the H&P law, with parameters chosen to fit the data at ages 50–90, continues to fit the data extremely well—certainly in the case of females—at ages far over 90 and still using the *same* parameters. Thus the law continues to hold at much higher ages than were claimed by its authors. This seems an important finding.

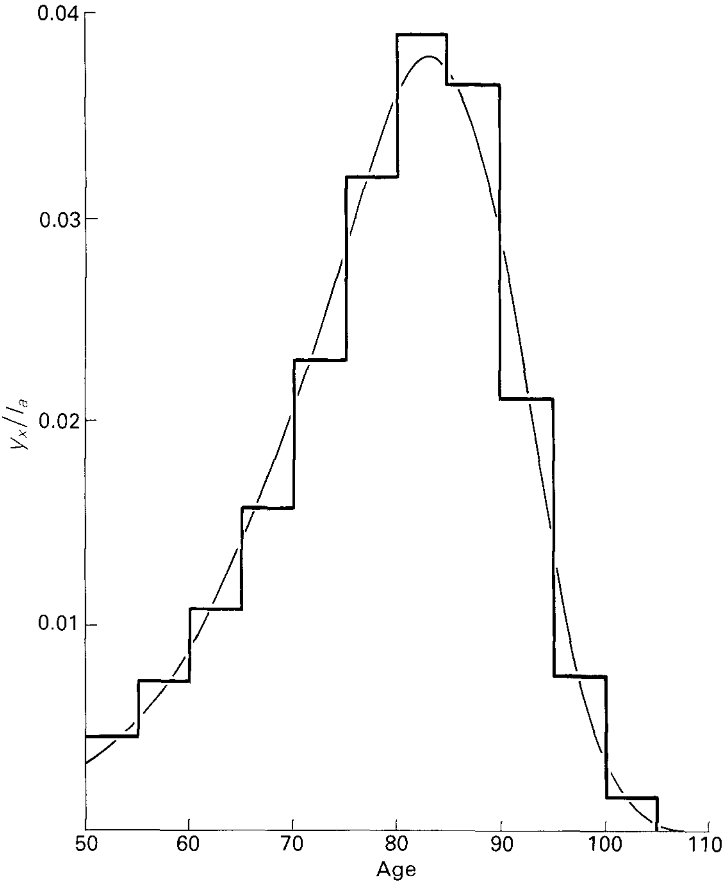


Figure 2. Curve of deaths from the H&P law and histogram from E.L.T. No. 14. Females, 1980–82.

On the other hand, the observed mortality rates at ages 90 and over, as shown in the last two columns of Table 2, all fall below those given by the Gompertz law with parameters fitted to ages 50–90.

*The Curve of Deaths*

The continuous curve of deaths, as given by the H&P law, is plotted on Figures 1 and 2, for males and females, respectively. This is calculated from equation (30). The quantity actually plotted is $y_x/l_a$, so that the scale is independent of the choice of $l_a$ or the absolute number of deaths. The Figures also show, for comparison, histograms of the ages at death as given by E.L.T. No. 14. This is shown for 5-year intervals of age; the height of the line plotted between the exact ages $x$ and $x+5$ is $(l_x - l_{x+5})/(5 \cdot l_{50})$. It will be seen that the agreement between the theoretical curves and the histograms is good, at least over 5-year intervals.

According to equations (26) and (33), the peak of the Gompertz curve of deaths occurs at ages 78·1 for males and 83·7 for females; and on the H&P curve at ages near 77·8 for males and 83·4 for females.

The position of the peak can be sensitive to relatively small changes in the parameters $p_a$ or $G$ (which particularly affect the age at the peak) and $c$ or $H$ (which particularly affect its height). These effects can be quantified by using equations (26), (27), (33) and (34).

## 5. CONCLUDING REMARKS

The fact that the Heligman and Pollard law gives a good fit to current life tables was, of course, established by its authors, using Australian data. Forfar & Smith (1987) have since fitted the full Heligman and Pollard model to all the E.L.T.s Nos 1–14, back to the year 1841. Thus the fact that the law holds is not 'news'. Nevertheless, readers may find it helpful to see for themselves, from the Tables and Figures of this paper, quite how well it fits the current E.L.T. and also how close it is to the original law of Gompertz, above the age of 50.

It is also hoped that the new mathematical results given in the paper will be helpful to others who may wish to make use of the 'laws'.

There is a final point which seems relevant. This paper, like many others before it, has used a mathematical approach. This is, after all, the most efficient way to fit parameters and to calculate life tables and curves of deaths. However, it is important not to lose sight of the advantages of a graphical examination of the data. For this purpose, though, it is rather important to draw the right graph. At present it is the common practice to plot graphs of log $q_x$ against $x$; but at higher ages these give a curve, which is not very informative to the eye. However, when the simple Heligman and Pollard law (2) holds, a graph of log $(q_x/p_x)$ will be a straight line. The human eye can recognise this easily. The moral of this is that if we plot log $(q_x/p_x)$, *instead* of log $q_x$, then we can see immediately which are the ranges of ages where the H&P law holds.

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