JIA 70 (1939) 0104-0105



# NOTATION FOR DIVIDED DIFFERENCES

BY HARRY FREEMAN, M.A., F.I.A., F.S.S.
*of the Government Actuary's Department*

THERE seems to be at present no universally recognized notation for divided differences. There are objections to the many forms in practice. For example, $\Delta' u_a$, representing $(u_b - u_a)/(b-a)$, (*Actuarial Mathematics*, p. 57, and Henry, *Calculus and Probability*, chap. VIII) has the disadvantage that the dash is apt to be confused with the index (cf. $\Delta'^2 u_a$ and $\Delta^{12} u_a$). A preferable form $\Delta(a, b)$, $\Delta^2(a, b, c)$ ... is clearer, but departs from the recognized symbolical notation in that the function $u_x$ on which the divided-difference symbol operates is not present; this renders the notation unsuitable for elementary work. The alternative notation adopted in *Actuarial Mathematics*, viz. as in (3) below, is simple, and where there is no ambiguity with $f(x, y)$, representing a function of two variables, there is no objection to its use. This notation does not, however, conform with the $\Delta$ notation for differencing when the intervals are equal and to that extent it is hardly satisfactory.

The following are some alternative notations that have been used:

| | Arguments | Function | Divided differences |
|---|---|---|---|
| (1) | $a, b, c$ ... | $u_a, u_b, u_c$ ... | $\Delta(a, b), \Delta(b, c), \Delta^2(a, b, c)$ ... |
| (2) | $a, b, c$ ... | $u_a, u_b, u_c$ ... | $(a, b), (b, c), (a, b, c)$ ... |
| (3) | $x_0, x_1, x_2$ ... | $f(x_0), f(x_1), f(x_2)$ ... | $f(x_0, x_1), f(x_1, x_2), f(x_0, x_1, x_2)$ ... |
| (4) | $x_0, x_1, x_2$ ... | $f(x_0), f(x_1), f(x_2)$ ... | $[x_0, x_1], [x_1, x_2], [x_0, x_1, x_2]$ ... |
| (5) | $a_0, a_1, a_2$ ... | $A_0, A_1, A_2$ ... | $\theta A_0, \theta A_1, \theta^2 A_0$ ... |

(1) D. C. Fraser, (2) W. F. Sheppard, (3) J. F. Steffensen, (4) Milne-Thomson, (5) De Morgan.

The notation to be adopted in the forthcoming text-book *Mathematics for Actuarial Students* is due to Dr A. C. Aitken, F.R.S. (*Proc. Roy. Soc. Edin.* Vol. LVIII, p. 175, 1938), and has all the advantages possessed by the ordinary difference symbol. In this notation Newton's Interpolation Formula, based on the given values $u_a$, $u_b$, $u_c$ ..., appears thus:

$$u_x = u_a + (x-a) \underset{b}{\Delta} u_a + (x-a)(x-b) \underset{bc}{\Delta}^2 u_a + \ldots,$$

from which the notation is obvious. When the arguments and their order are fixed, the shortened forms $\triangle u_a$, $\triangle^2 u_a$, $\triangle^3 u_a$ ... may be used for the *leading* differences of $u_a$. Aitken remarks (*loc. cit.*):

> The symbol of operation is here detached from, and prefixed to, the operand $u_a$, the arguments [of $u$] involved in the operation being indicated by the suffixes [of $\triangle$]. Admittedly this notation does not bring out the symmetrical nature of divided-differences; on the other hand it does bring out their operational analogy with ordinary differencing and differentiation.

It may be added that the symbol $\triangle$ is suggestive and easily recognized, and that in practice the notation, being simple to handle, is found to be very satisfactory.

[It seems probable that Aitken's convenient and distinctive notation will become standard. It is in any case desirable that writers in the *Journal* should adopt it as it is to be used in Mr Freeman's new text-book.—Eds. *J.I.A.*]