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Joint Life Income in a Pooled Annuity Fund

by
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The Optimising Future Pensions research was commissioned by the Institute and Faculty of Actuaries' (IFoA) Actuarial Research Centre that has sponsored a number of large research collaborations with academia in recent years on topics such as longevity and morbidity risk, population ageing and big health. The ARC programme currently consists of two remaining academic programmes which conclude in 2023. The IFoA's research and thought leadership programme has broadened its approach to reflect current strategy and themes, and invests in industry-led, cutting-edge research through its thought leadership activity, policy reports, and member-led research initiatives; for instance, its work relating to the Great Risk Transfer and Poverty Premium.

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Abstract

A pooled annuity fund provides a regular income to its participants. So far as we are aware, the literature on pooled annuity funds has focused on payments to a single life. In practice, joint life annuities, which provide a regular payment to a couple until both have died, are also popular. This is attractive as a surviving spouse can continue to receive an income, which would not happen under a single life payment.

Here how to provide a joint life income from a pooled annuity fund is detailed. This involves the calculation of the longevity credit appropriate for a joint life. It is also shown that this joint life income can not be replicated using two single life incomes.

The method proposed means that pooled annuity funds can offer a range of income benefits to their participants, which should increase their attractiveness.

Keywords: tontine; longevity risk; mortality; retirement; decumulation.

1 Introduction

The provision of a lifetime income to an annuitant's partner is an important benefit offered as part of some life annuities. It is particularly important for women, who typically have smaller pensions than their male partners and are likely to out-live them. Around one third of UK annuity purchases include such a benefit (ABI, 2013), showing that there is a strong demand for them. Such annuities are called joint life annuities, since their payout depends on the life status of both the annuitant and the annuitant's partner. The vast majority of UK defined benefit schemes include a similar

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income benefit. In summary, these joint life income benefits are attractive and seen in practice. Therefore, they should be a standard benefit offered in vehicles like pooled annuity funds.

A pooled annuity fund offers a means of paying a lifetime income to their participants, while avoiding the costly guarantees in life annuities. They do this by pooling each participant's longevity risk directly with the other participants, rather than indirectly through an insurance company. They lie between life annuities and income drawdown in terms of the risk borne by the participants in the fund. How to operate them, in terms of how to allocate the money in the fund among the participants, is the focus of many papers. However, to date, only the payment of a single life income to each participant has been proposed. None of the academic literature has considered how to pay out a joint life income.

It is shown here how to pay out a joint life income from a pooled annuity fund. The important contribution is showing how to allocate the money in the fund among the participants so that the desired joint life income can be provided. Furthermore, the amount of income paid to the couple, depending on which of them is alive, can be tailored to their own personal circumstances. It is also seen that the joint life income benefit is a richer structure than the single life income benefit, in the sense that it cannot be replicated by a combination of single life benefits. The approach set out in this paper could also be extended to include more than two lives; for example, children as well as partners.

Section 2 gives the overview of the longevity risk-sharing field, which pays a single life income to the participants in the risk-sharing scheme. An exposition of how to allocate the money in the fund among these participants to give them a single life income is in Section 3. The allocation is a sharing of the account values of those who have died among the surviving participants, with the share of funds paid to each participant called a longevity credit. Using this exposition as a road-map, the allocation of funds to give participants a joint life income is set out in Section 4. For a joint life income, the longevity credits arise upon the death of either of the couple, and not just when both of them die. However, not all of the account value is surrendered to the pool when only one of them dies. It is shown here how much should be given up by each couple when one of them dies. All the calculations are shown over one time period. They can then be repeated for subsequent time periods.

2 Literature review

Structures in which longevity risk is pooled directly among participants have increasingly gained industry and academic interest in recent years. Various ways of doing longevity risk-sharing have been proposed. Some of these work for single-cohort pools in which every member is an independent and identical copy of each other (for example, Milevsky and Salisbury 2015; Stamos 2008) and some are intended for multi-cohort pools (for example, Piggott et al. 2005; Stamos 2008; Sabin 2010; Qiao and Sherris 2013; Donnelly et al. 2014; Milevsky and Salisbury 2016). Some calculate what proportion of the funds of those who have died should be received by each participant – whether explicitly (Stamos, 2008; Sabin, 2010; Donnelly et al., 2014) or implicitly (Piggott et al., 2005; Qiao and Sherris, 2013). Both of these explicit and implicit schemes then use an annuity value to calculate the income paid out.

In a general framework, Denuit and Robert (2021) present various fair linear risk-sharing rules, and a conditional mean risk-sharing rule and study their convergence. Interestingly, Weinert and Gründl (2021) derive a distribution to model the longevity credits paid from the pooled annuity fund, rather than modelling directly the mortality experience of the pool of participants.

In this paper, a longevity risk-sharing method is derived by consideration of the longevity credits implied by the expected present value of a life annuity. Studying this for a joint life annuity allows us to propose a longevity credit calculation. Broadly, it is seen that the money given up the pool by those who experience a death is shared out among the surviving participants. Importantly, how much can be given up upon one or more deaths is determined: this is the key to the

In the single life income case, attention is drawn to two methods of sharing the money given up, as longevity credits, so that it is an actuarially fair allocation. The allocation is actuarially fair if, over a fixed time period, the expected gain due to receiving longevity credits is equal to the expected loss if death occurs and some or all of the account value is given up to the other participants. This is an important concept, to avoid a participant gaining at the expense of other participants.

One actuarially fair allocation is by Sabin (2010), who shows that the characteristics of the participants – their chance of dying and account value - must satisfy certain mathematical conditions for the longevity credits to be actuarially fair. In his scheme, the payments are made to the surviving participants at the end of each period of longevity risk-sharing. On the downside, the calculation of the longevity credits in Sabin (2010) is extremely complex.

The other actuarially fair allocation is by Donnelly et al. (2014). They show that, if the longevity credits are paid to the participants who were in the scheme at the start of each period of longevity risk-sharing (with the longevity credit payments made at the end of each period), then no conditions are needed on the participants. This means that longevity credits are paid not only to the surviving participants but also to the estates of those who have just died. The calculation is very simple, being the product of the account value at risk of being given up to the other participants and the probability of dying over the period of longevity risk-sharing in question. However, the downside of this particular calculation is that the survivors in the fund have a lower income than, for example, that under the allocation of Sabin (2010). They have a lower income because money leaves the fund through payments to the estates of those who have just died.

Trading off ease-of-calculation against paying maximising the income of the surviving participants means that 'perfect' actuarial fairness is given up. But, while it may not hold exactly, it should hold approximately.

Milevsky and Salisbury (2015) take a different tack and calculate the payout from a pooled annuity fund which maximizes the expected discounted value of lifetime consumption. In their approach, the optimal payout to participants varies according to the utility-maximization problem considered, as can be seen in various papers employing this approach (for example, Chen et al. 2020, 2021). it may be argued whether aiming to pay a constant income is more attractive to potential customers than, for example, a power utility-maximized income. However, the latter approach is consistent with using expected utility theory to assess the attractiveness of a particular income stream.

Complementary to the literature on the attractiveness of pooled annuity funds for individuals who are not too risk averse, Chen et al. (2018) consider an individual receiving income from a tontine (which is synonymous for a pooled annuity fund in this context) up to some fixed age, followed by income from a deferred life annuity. They determine the age at which to switch from income paid from the tontine to income paid from the life annuity, which is optimal according to their CRRA utility function-based criterion.

The demand for pooled annuity fund compared to life annuities has been studied by various authors (for example, Piggott et al. 2005; Valdez et al. 2006; Donnelly et al. 2013; Hanewald et al. 2013; Milevsky and Salisbury 2016; Chen et al. 2021). The results show that the attractiveness of pooled annuity funds increases as the risk aversion of the retiree reduces. This is because the less risk averse retirees are happier to bear the volatility of pooled annuity funds in exchange for their higher expected return. Pooled annuity funds become increasingly preferred to life annuities as the loadings on the latter increase.

Hanewald et al. (2013) go back to financial economic basics to show the demand for pooled annuity funds relative to typical post-retirement investments, when systematic longevity risk is present. One of their findings is that people with a bequest motive will buy less of the life annuity and replace it with a risk-free bond, in order to provide the bequest. As the loadings on the life annuity increase, they will also replace it with investment in a pooled annuity fund.

The provision of a bequest in conjunction with an income benefit is studied by several authors. Chen and Rach (2022) allow for a bequest through bundling a life insurance contract with a pooled annuity fund. They take the perspective of the insurer rather than the individual. They find that their bundled product may lead to lower risk margins than the products sold separately. This is a different product to that in the present paper, since the bequest is provided through a separate contract in Chen and Rach (2022) and is not affected by the pooled annuity fund as it is in this paper.

To provide both an income and a bequest, Zhou et al. (2021) propose a portfolio consisting of a life annuity contract and a product that pays out the return earned on a principal amount while leaving the principal untouched (called the "natural income"). The bequest benefit is then the principal amount. This set-up allows the individual to choose their own attractive combination of the two products. Due to natural hedging, they find that the interest rate risk can be effectively hedged by coupon-bearing government bonds.

Bernhardt and Donnelly (2019) propose and study optimal strategies for the pooled annuity fund with integrated bequest which is analysed in this paper. They determine optimal investment and consumption strategies when there is no idiosyncratic longevity risk. Their model does not allow for systematic longevity risk. They find that, the stronger the bequest motive, the more money that is allocated to the bequest account. Dagpunar (2021) extends the analysis by allowing the proportion of longevity credits funnelled to the bequest account to vary. He suggests a way of choosing the parameter that indicates the strength of the bequest motive.

3 Longevity credits for a single life income

In a pooled annuity fund, the participants each have an account value. Each participant's account earns investment returns and longevity credits, the latter being discussed next, and withdrawals of money are made for the participant to live on.

In a pooled annuity fund which provides only single life income, the longevity credit paid to a participant is the share of the account values of those who have recently died. When the participant themselves dies, their account value is shared out among all the surviving participants. This can be viewed as a subsidy from the shorter-lived to the longer-lived or as everyone in the fund becoming a beneficiary of each other. There are several methods proposed to operate a pooled annuity fund which provides only single life income.

In this section, it is shown how to calculate the longevity credit for joint lives. To our knowledge, this has not been done before. As is seen below, the calculation of the longevity credits are more complex while both partners in the couple remain alive. When one of the couple dies, the survivor receives a single life income and, as will be shown, the longevity credit calculation is the same as in a single life income fund.

To motivate the approach, the derivation of the longevity credit in a fund which provides a

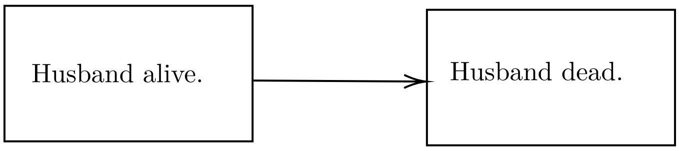

Figure 3.1.1: The single life model over one time period, at the start of which the participant is age x.

single life income is presented first. This gives not only a pathway to deriving the longevity credit for joint lives, but also gives a single life longevity credit calculation which is consistent with the latter. This is important since many different methods of calculating longevity credits have been proposed and it is not a priori evident which is suitable for this setting.

To derive the longevity credits for a joint life income, a discrete-time multiple state model is presented first. The account value released upon transition between the states in the model is derived from the expected present value of a joint life annuity.

3.1 Model of the single life status

Consider first a single life, who participates in a pooled annuity fund. Time and ages are measured in years. The participant is age x at time 0, and is alive at that time.

The model of the life of the participant is shown in Figure 3.1.1.

An important principle is that a participant in the longevity risk-sharing method considered here, can only gain longevity credits in a state if they risk some of their account value in that state. The gain should be proportional to the account value (i.e. the *sum-at-risk*) and the probability of the risk occurring, i.e. the death of the participant in this case. This is a linear risk-sharing principle, which are discussed in Denuit and Robert (2021).

While risk-sharing rules in the single life case are well-known (e.g. XXX), it is not obvious what is the sum-at-risk in the joint life case. The single life is explored first to understand the approach before it is extended to the joint life case.

3.2 Calculation of the single life expected longevity credit

To calculate the expected longevity credit, consider the change in the expected present value (EPV) of a single life annuity, paid annually in advance, from age x to age $x+1$. The difference in the EPVs is attributed to a gain due to investment returns - which are assumed to be constant - and the gain due to a longevity credit. This latter gain is, by assumption, conditional on the participant being alive at age $x+1$. The longevity credit gain is an expected longevity credit; it assumes that idiosyncratic longevity risk has been completely diversified away. There is no systematic longevity risk.

Let $r > -1$ be the annual, constant, effective rate of interest. For each $k \geq 0$, let ${}_kp_x$ be the probability of the husband, who is age x at time 0, survives to age $x+k$. Then define the EPV of a single life annuity which pays \$1 per annum annually in advance to the husband, who is age x at time 0, as

$$\ddot{a}_x = \sum_{k=0}^{\infty} (1+r)^{-k} \, {}_kp_x.$$

Suppose that the husband has an account value equal to $\ddot{a}_x$ at time 0. Using that $(\ddot{a}_x - 1)(1+r)$ is the accumulation at time 1 of the husband's account value at time 1 and assuming that he is alive at time 1, the expected value of the longevity credit paid at time 1 to his account is the difference

$$\ddot{a}_{x+1} - (\ddot{a}_x - 1)(1+r) = (\ddot{a}_x - 1)(1+r)\left(\frac{1}{p_x} - 1\right) = (\ddot{a}_x - 1)(1+r)\,\frac{1-p_x}{p_x}.$$

Substitution of the well-known and easily derived relationship

$$\ddot{a}_{x+1} = \frac{(\ddot{a}_x - 1)(1+r)}{p_x},$$

gives that the expected value of the longevity credit paid at time 1 is

$$(\ddot{a}_x - 1)(1+r)\,\frac{1-p_x}{p_x}.$$

Each of the survivors to time 1 is expected to gain the latter amount at time 1, in addition to the investment return rate r earned on their account value.

For any individual who dies before time 1, their account value falls to zero at time 1 as it is shared out among the survivors.

3.3 Interpretation of the single life expected longevity credit

Knowing the expected value of the longevity credit is helpful because it indicates the sum-at-risk for the individual in that state. To see this, suppose that there are ℓ_0 individuals in the fund at time 0, each with the same distribution of future mortality. If deaths occur in line with this distribution, then the number of deaths expected to occur from time 0 to time 1 is $\ell_0(1-p_x)$ and the expected number of survivors at time 1 is $\ell_0 p_x$. The expected longevity credit paid at time 1 to a surviving participant's account is

$$\begin{aligned}
&(\ddot{a}_x - 1)(1+r)\,\frac{1-p_x}{p_x} \\
=&\ell_0(1-p_x)\,(\ddot{a}_x - 1)(1+r)\,\frac{(\ddot{a}_x - 1)(1+r)(1-p_x)}{\ell_0 p_x\,(\ddot{a}_x - 1)(1+r)(1-p_x)} \\
=&\text{Total value of accounts released by expected deaths over the first year} \\
&\cdot\frac{\text{Account value at time 1 of the surviving participant} \times \text{Their risk of death}}{\text{Sum of the account value at time 1 of each survivor at time 1} \times \text{Their risk of death}},
\end{aligned} \tag{1}$$

Interpreting the above expressions, the sum-at-risk is the account value at time 1 of each participant, $(\ddot{a}_x - 1)(1+r)$. The risk materialises among those who die between time 0 and time 1. The total of the latter's account values is shared out among the surviving $\ell_0 p_x$ participants at time 1 in proportion to their own expected loss: the product of their individual account value (their sum-at-risk) and their probability of dying (the probability of the risk occurring).

It is not evident from what is written here, why the expression above should include the expected loss $(\ddot{a}_x - 1)(1+r)(1-p_x)$ in the fraction. Indeed, it is not needed in this case, in which each participant has the same account value and the same probability of death while they continue to live. The motivation for this expression are explored in the next section.

3.4 A calculation of the single life longevity credit

So far, the expected longevity credit has been calculated. What if deaths do not turn out as expected? For practical purposes, the longevity credit paid to each survivor needs to be based on the actual deaths and not the expected number.

Suppose that there are $\mathcal{L}_0 > 1$ participants in the single life income fund at time 0 who each join at time 0 with $\$F(0_-) > 0$. Between time 0 and time 1, $N_1 < \mathcal{L}_0$ of them die, leaving $\mathcal{L}_1 = \mathcal{L}_0 - N_1$ alive at time 1.

Let the account value at time 1, after investment returns have been added and before longevity credits are added, of each survivor at time 1 be $F(1)$. Effectively, $F(1)$ is the sum-at-risk of each participant. Assume that each participant has the same probability $1 - p_x$ of dying between time 0 and time 1. Thus $1 - p_x$ is the probability that the sum-at-risk is lost.

The total account value released by deaths between time 0 and time 1 is $N_1 F(1)$. This sum of money is shared out among the survivors at time 1. In line with the expression (1), the longevity credit paid at time 1 to each of these survivors is

$$\frac{F(1)(1-p_x)}{\mathcal{L}_1\, F(1)(1-p_x)}\, N_1 F(1). \tag{2}$$

Generalising to the case in which participants are heterogeneous and further motivated by Donnelly et al. (2014), suppose that there are again $\mathcal{L}_0 > 1$ participants in the single life income fund at time 0. However, they join with different amounts of money and they have different chances of dying. Specifically, suppose that the account value at time 1, of Participant i is $F_i(1)$ and they have a chance $q_i \in (0, 1]$ of dying between time 0 and time 1, for $i = 1, 2, \ldots, \mathcal{L}_0$. Let the set of the indices of the surviving participants at time 1 be denoted by $\mathcal{A}_0$ and that of the dead participants by $\mathcal{A}_3$.

Then the longevity credit paid to each Participant $i \in \mathcal{A}_0$ is

$$\frac{q_i\, F_i(1)}{\sum_{j\in\mathcal{A}_0} q_j\, F_j(1)} \sum_{\ell\in\mathcal{A}_3} F_\ell(1). \tag{3}$$

This longevity credit calculation prioritises maximising the longevity credit to survivors over actuarial fairness. To be specific, using the same approach to calculating the expected loss of each participant, an actuarially fair version of this longevity credit would require division over all $j \in \{1, 2, \ldots, \mathcal{L}_0\}$ rather than, as in expression (3), over $j \in \mathcal{A}_0$. In the homogenous case, when $q_i = q_j$ for all i, j, this means the actuarially fair version divides by $\mathcal{L}_0$ rather than $\mathcal{L}_1$. Since $\mathcal{L}_0 \geq \mathcal{L}_1$, a smaller longevity credit is paid to the surviving participants when using the actuarially fair version. In the homogeneous case, the actuarially fair version is, on average, $(1 - q_1)$ of the longevity credit calculated via expression (2). At higher ages, as the chance of dying q_1 becomes larger, this translates into a non-negligible difference in the income paid to surviving participants. Since the focus of the pooled annuity fund considered here is to maximise the income to the survivors, actuarial fairness is not required of the longevity credit calculations proposed in this paper.

These last two expressions are used to motivate the longevity credit calculation for a joint life income. They are important because they motivates the more general form of the joint life income longevity credit, which can be used for couples with heterogeneous financial and mortality characteristics.

4 The joint life longevity credit

Now suppose that a large number of couples join the fund. Each couple consists of a husband, age x at time 0, and a wife, age y. They each want a joint life income. Specifically, they want to be paid \$1 per annum annually in advance while they are both alive. If the wife dies and the husband is alive, then he wants to be paid $\$\alpha \geq 0$ per annum annually in advance until her death. If the

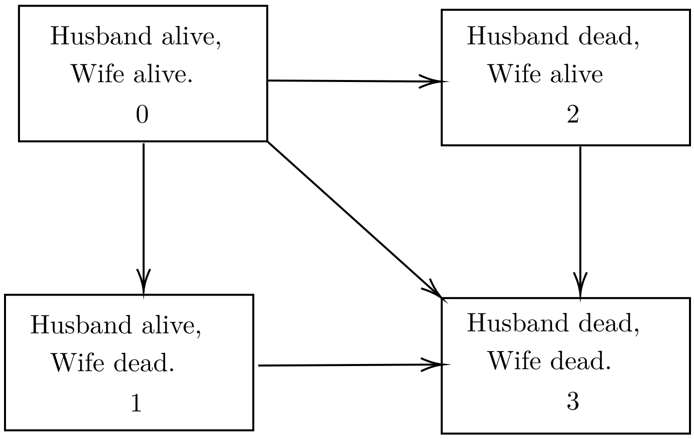

Figure 4.1.1: The discrete-time joint life model, at the start of which the husband is age x and the wife is age y, showing the possible transitions between states.

husband dies and the wife is still alive, then she wants to be paid $\$\beta \geq 0$ per annum annually in advance until her death. This is a typical ratio of benefits for both life annuity contracts and UK defined benefit pension schemes, where usually $\alpha = 1$ and $\beta \in \{1/2, 2/3, 1\}$, or vice versa.

The risk-sharing principle that follows from the derivation below is that participants share risk only with those in the same state as themselves. For example, while the couple is alive, then share their longevity risk only with other couples. Longevity credits are gained for a living pair of husband and wife when either one or both partners in other couples die. However, once one partner in the couple dies, the surviving partner shares longevity risk only with similar survivors. For example, if a wife dies, her widower shares longevity risk only with other widowers. He receives a longevity credit only when one of other widowers dies.

4.1 Model of the joint life status

The joint life status of the couple can be in one of four states at any future time $t > 0$. These four states are shown in Figure 4.1.1.

The benefits paid to the couple depend on which state they are in. With all payments made annually in advance, the desired annual payments are

- $1 while in State 0, i.e while both partners are alive,
- $\$\alpha$ while in State 1, i.e while the husband is alive and the wife is dead,
- $\$\beta$ while in State 2, i.e while the husband is dead and the wife is alive, and
- $0 while in State 3, i.e when both the husband and the wife are dead.

Except for the null payment in State 3, these payments are not guaranteed in a pooled annuity fund. They are adjusted as the mortality and investment experience emerges.

The couple starts in State 0 at time 0, when both the husband and the wife are alive. If the couple moves into State 1 or State 2, then there is only one of them left alive. In that case, the calculation of the longevity credit is calculated similarly to equation (2).

As described above, a couple in State 0 receives longevity credits when one or more of the other couples move out of State 0. The calculation of the expected amount of longevity credit received by each couple in State 0 is shown here. The amount can be decomposed to show how much comes from transitions into each of State 1, 2 and 3. These expected values can then be used to motivate the calculation for the longevity credit when the actual number of transitions are known.

4.2 The joint life income cannot be replicated using two single life incomes

Before proceeding to the calculation of the joint life longevity credits, consider first if the joint life income can be replicated using two single life incomes. It is seen that it cannot.

Suppose that a couple have an amount of money $\$F(0) = \gamma \ddot{a}_x + \epsilon \ddot{a}_y$ at time 0. The husband in the couple takes $\gamma \ddot{a}_x$ and joins a pooled annuity fund to get a single life income of γ per annum. The wife takes the residual money, $\epsilon \ddot{a}_y$ and also joins a pooled annuity fund to get a single life income of ϵ per annum. The joint life status of the couple can still be considered using the model in Figure 4.1.1. Since it is the ratios between the payments in different states which are of interest then, without loss of generality, set $\epsilon + \gamma = 1$ to normalise the payments in each state.

The total expected payments to the husband and wife, from their separate single life incomes, are:

- \$1 while in State 0, i.e while both partners are alive,
- $\$\gamma$ while in State 1, i.e while the husband is alive and the wife is dead,
- $\$1 - \gamma$ while in State 2, i.e while the husband is dead and the wife is alive, and
- \$0 while in State 3, i.e when both the husband and the wife are dead.

Thus, when only a single life income is available, the couple has only one choice available to them. They can decide how much of the annual income paid in State 0, when both are alive, can be paid to the husband if the wife predeceases him. The remaining income stream is paid to the wife, if the husband dies before her.

Suppose the same couple is able to choose a joint life income from a pooled annuity fund. They want that, in State 1, the husband gets α of the income paid in State 0. In State 2, the wife gets β of the income paid in State 0. As before, nothing is paid in State 3, when both the husband and the wife are dead. Let a represent the EPV at time 0 of the desired joint income.

Then the total expected payments to the husband and wife, from this joint life income, are:

- $\$F(0)/a$ while in State 0, i.e while both partners are alive,
- $\$\alpha\, F(0)/a$ while in State 1, i.e while the husband is alive and the wife is dead,
- $\$\beta\, F(0)/a$ while in State 2, i.e while the husband is dead and the wife is alive, and
- \$0 while in State 3, i.e when both the husband and the wife are dead.

Thus, with the joint life income offered, the couple can choose the desired income level in State 1 and in State 2. They do not have to make any trade-offs between what the widow and widower may get. For example, they could choose $\alpha = 2/3$ and $\beta = 2/3$ so that they will each have 2/3 of the joint income received while they were both alive. In comparison, the two single life incomes could, at best, offer $\gamma = 1/2$ of the total income received while they were both alive.

Furthermore, setting $\alpha = \beta = \gamma$, it can be seen that the joint life income can be used to give the sum of the individual single life incomes. Thus proposing a joint life income is a genuine improvement to the possible benefits offered by a pooled annuity fund.

4.3 Calculation of the joint life longevity credits

The expected value of the longevity credit is calculated first, using the same technique as for the single life case. The expected value motivates a more useful formula for the longevity credits paid to couples in State 0.

Assume that the future lifetime of the husband and wife are independent random variables. Define the EPV of a joint life annuity which pays \$1 per annum annually in advance to husband and wife while both are alive (i.e. while they are in State 0) and nothing otherwise as

$$\ddot{a}_{x:y} = \sum_{k=0}^{\infty}(1+r)^{-k}\, {}_kp_x\, {}_kp_y.$$

The EPV of a reversionary annuity which pays \$1 per annum annually in advance to the husband while he is alive, conditional on the wife being dead during the payment time, (i.e. while in State 1) and nothing otherwise is

$$\ddot{a}_{y|x} = \sum_{k=0}^{\infty}(1+r)^{-k}\, {}_kp_x\, \left(1 - {}_kp_y\right).$$

Correspondingly, the EPV of a reversionary annuity which pays \$1 per annum annually in advance to the wife while she is alive, conditional on the husband being dead during the payment time, (i.e. while in State 2) and nothing otherwise is

$$\ddot{a}_{x|y} = \sum_{k=0}^{\infty}(1+r)^{-k}\, {}_kp_y\, \left(1 - {}_kp_x\right).$$

The next proposition gives the formula for the expected value of the longevity credits, for a joint life income.

Proposition 4.1. *Let $q_x := 1 - p_x$ and $q_y := 1 - p_y$. Suppose that each couple in State 0 at time 0 has an account value of \$$(\ddot{a}_{x:y} + \alpha\ddot{a}_{y|x} + \beta\ddot{a}_{x|y})$ and wishes to be paid*

- *\$1 per annum annually in advance in State 0, i.e. while both are alive,*
- *\$$\alpha \geq 0$ per annum annually in advance in State 1, i.e. while the husband is alive and the wife is dead,*
- *\$$\beta \geq 0$ per annum annually in advance in State 2, i.e. while the husband is dead and the wife is alive, and*
- *Nothing in State 3, i.e. when both the husband and the wife are dead.*

Based on the EPV of the above payments calculated at the annual effective rate of interest $r > -1$, the expected value of the longevity credit paid to each couple which remains in State 0 at time 1 is

$$\begin{aligned} &\frac{p_x q_y}{p_x\, p_y}\left[\left(\ddot{a}_{x:y} + \alpha\ddot{a}_{y|x} + \beta\ddot{a}_{x|y} - 1\right)(1+r) - \alpha\, \ddot{a}_{x+1}\right] \\ +&\frac{q_x p_y}{p_x\, p_y}\left[\left(\ddot{a}_{x:y} + \alpha\ddot{a}_{y|x} + \beta\ddot{a}_{x|y} - 1\right)(1+r) - \beta\, \ddot{a}_{y+1}\right] \\ +&\frac{q_x q_y}{p_x\, p_y}\left(\ddot{a}_{x:y} + \alpha\ddot{a}_{y|x} + \beta\ddot{a}_{x|y} - 1\right)(1+r). \end{aligned}$$

There is no longevity credit paid at the end of year of the transition, to the couples who transition out of State 0.

Proof. As the payments are in advance, \$1 is paid to the couple at time 0 and the EPV of the future payments immediately falls to $\ddot{a}_{x:y} + \alpha\ddot{a}_{y|x} + \beta\ddot{a}_{x|y} - 1$. This value accumulates with interest to time 1 to $(1+r)(\ddot{a}_{x:y} + \alpha\ddot{a}_{y|x} + \beta\ddot{a}_{x|y} - 1)$.

The expected value of the longevity credit at time 1, for couples who are still in State 0 at time 1, is the difference between the EPV of the payment starting at time 1 and this accumulated value at time 1, i.e.

$$\begin{aligned}&\ddot{a}_{x+1:y+1} + \alpha\ddot{a}_{y+1|x+1} + \beta\ddot{a}_{x+1|y+1} - (1+r)\left(\ddot{a}_{x:y} + \alpha\ddot{a}_{y|x} + \beta\ddot{a}_{x|y} - 1\right) \\ =&\ddot{a}_{x+1:y+1} - (1+r)\ddot{a}_{x:y} + \alpha\left(\ddot{a}_{y+1|x+1} - (1+r)\ddot{a}_{y|x}\right) + \beta\left(\ddot{a}_{x+1|y+1} - (1+r)\ddot{a}_{x|y}\right) + 1 + r.\end{aligned}$$

Recall the well-known recursive relationships (for example, Dickson et al. 2013, Chapter 5.11)

$$\ddot{a}_{x+1} = \frac{1+r}{p_x}\left(\ddot{a}_x - 1\right), \qquad \ddot{a}_{y+1} = \frac{1+r}{p_y}\left(\ddot{a}_y - 1\right), \qquad \ddot{a}_{x+1:y+1} = \frac{1+r}{p_x\, p_y}\left(\ddot{a}_{x:y} - 1\right)$$

and

$$\ddot{a}_{y|x} = \ddot{a}_x - \ddot{a}_{x:y} \qquad \text{and} \qquad \ddot{a}_{x|y} = \ddot{a}_y - \ddot{a}_{x:y}.$$

Substitution of these relationships and the application of $1 - p_x\, p_y = p_x\, q_y + q_x\, p_y + q_x\, q_y$, and some algebra gives the result.

□

The sources of the longevity credit in Proposition 4.1 are the three possible transitions out of State 0. For example, the probability $p_x q_y$ is the probability of a couple transitioning into State 1, the state in which the husband is alive and the wife is dead. The other two probabilities, $p_y q_x$ and $q_x q_y$, correspond to transitions into State 2 and 3, respectively.

Proposition 4.1 is important because, even though it is based around expected values, it shows a way of sharing longevity credits for couples in State 0. It also shows that the two-state alive-dead model, such as the one shown in Figure 3.1.1, is not a rich enough model for the joint life longevity credit calculation.

The coefficient of each probability shows how much of the account value of each transitioning couple is given up by them. For example, for transitions to State 1, which means that the husband is alive but the wife has died, the amount

$$\left(\ddot{a}_{x:y} + \alpha\ddot{a}_{y|x} + \beta\ddot{a}_{x|y} - 1\right)(1+r) - \alpha\ddot{a}_{x+1}$$

is given up by the transitioning couple. It is shared out among the couples who stay in State 0 at time 1. The amount is the difference between what the transitioning couple would have needed if they had stayed in State 0 at time 1 and what they need (or, rather, what the widower needs) in State 1 at time 1. The first term, $\left(\ddot{a}_x + \alpha\ddot{a}_{y|x} + \beta\ddot{a}_{x|y} - 1\right)(1+r)$, is the accumulated amount at time 1 of the couple's account value, given that they were in State 0 at time 0. The second term, $\alpha\,\ddot{a}_{x+1}$, is the account value that the new widower needs to have, in order to have a payment of $\$\alpha$ per annum for the rest of his life.

4.4 A calculation of the joint life longevity credit

The expression in Proposition 4.1 suggests how to calculate longevity credits for the couples in State 0. It shows that it matters which person in a couple dies, and indicates how to share the 'surplus' account value of a couple in which one or both die, among the surviving couples in State 0. The surplus account value is the account value which the couple no longer needs to support the desired income payment in their new state.

Broadly, Proposition 4.1 motivates the longevity credit for couples in State 0 at time n being calculated as

$$\sum_{k=1}^{3} \frac{\text{Number of transitions to State } k \text{ over the year to time } n}{\text{Number of couples in State 0 at time } n}$$
$$\cdot \text{ Account value not needed at time } n \text{ in State } k \text{ by a couple transitioning from State 0.}$$

How to express this mathematically is detailed in the next section. However, the approach has limitations which render it impratical for real world applications. The obvious drawback is that it assumes all couples are the same at time 0. For example, they bring the same amount of money to the fund, the husband is age x, the wife is age y. All wives have the same future lifetime distribution, and similarly for the husbands. Therefore, Section 4.4.2 adjusts the longevity credit calculation in Section 4.4.1 to allow for real life complications.

4.4.1 A joint life longevity credit calculation for homogeneous couples

Here the longevity credit calculation when the couples are identical at time 0 is set out. It is then generalised in Section 4.4.2 to allow for couples to have different characteristics at time 0. The desired annual payments are, for each couple, the same as before: \$1 in State 0, \$$\alpha$ in State 1 and \$$\beta$ in State 2. In State 3, the couple is dead and no payments are made.

Suppose that there are $\mathcal{L}^{(0)}(0) > 1$ couples in the fund, who are all in State 0, at time 0. In the year after joining, some of the husbands may die and some of the wives may die. Mathematically, suppose that $N^{(0,k)}(1)$ of the initial couples move to State $k \in \{1, 2, 3\}$ at time 1, in which $\sum_{k=1}^{3} N^{(0,k)}(1) < \mathcal{L}^{(0)}(0)$. This leaves $\mathcal{L}^{(0)}(1) = \mathcal{L}^{(0)}(0) - \sum_{k=1}^{3} N^{(0,k)}(1)$ couples in State 0 at time 1. Transitions occur only at time $1, 2, 3, \ldots$.

As an aside, if there are no couples left in State 0 at time 1, i.e. $\mathcal{L}^{(0)}(1) = 0$, it is assumed that the transitioning couples do not give up any of their account value. In particular, for a couple in which both of them dies over the year to time 1, their estate is paid their account value.

At time 0_-, each couple has money \$$F(0_-)$, which they transfer into an account in the pooled annuity fund at time 0. They want to have

$$C(0) = \frac{F(0_-)}{\ddot{a}_{x:y} + \alpha \ddot{a}_{y|x} + \beta \ddot{a}_{x|y}}$$

per annum paid annually in advance to them while they are both alive. If the wife dies and the husband is still alive, they want her to receive \$$\alpha C(0)$ per annum paid annually in advance until her death, in which $\alpha > 0$. If the husband dies and the wife is still alive, they want her to receive \$$\beta C(0)$ per annum paid annually in advance until her death, in which $\beta > 0$. These income amounts are subject to change over time, as the mortality and investment experience, which emerges over time, differs from expectations.

Immediately upon joining the fund, each couple is paid \$$C(0)$ and their account value falls to

$$F(0) := F(0_-) - C(0) = F(0_-) \frac{\ddot{a}_{x:y} + \alpha \ddot{a}_{y|x} + \beta \ddot{a}_{x|y} - 1}{\ddot{a}_{x:y} + \alpha \ddot{a}_{y|x} + \beta \ddot{a}_{x|y}}.$$

Let the account value at time 1 of the each couple in State 0, after investment returns have been added and before the longevity credits at time 1 are calculated, be $F(1_-)$.

For a couple in State 1 at time 1, i.e. in which the wife dies and the husband survives, the new widower expects to have an annual income of \$$\alpha\, C(0)$. Using expected values, this means that he requires a fund value of \$$\alpha\, C(0)\, \ddot{a}_{x+1}$ at time 1. Thus an amount

$$F(1_-) - \alpha\, C(0)\, \ddot{a}_{x+1}$$

is given up by him, to be shared out among those remaining in State 0 at time 1. In effect, this is the sum-at-risk for transitions from State 0 to State 1. However, it is a value at time 1, rather than a value at time 0. Although the time 0 value could be used, it makes more sense to use the time 1 value since it represents the amount being given up by those couples who transition out of State 0 at time 1.

The expression in Proposition 4.1 shows that a fraction $1/(p_x p_y)$ is paid to each couple in State 0 at time 1 of the expected per couple sum-at-risk released at time 1 by transitions into State 1. Outside of the latter's expected value setting, the actual sum-at-risk released at time 1 by transitions into State 1 replaces the expected sum-at-risk. Since $N^{(0,1)}(1)$ couples transition to State 1 at time 1, there will be $N^{(0,1)}(1)$ times the amount $F(1_-) - \alpha\, C(0)\, \ddot{a}_{x+1}$ shared out among the couples in State 0 at time 1.

This suggests that the longevity credit paid to each couple in State 0 at time 1 due to other couples transitioning to State 1 is

$$\frac{N^{(0,1)}(1)}{\mathcal{L}^{(0)}(1)} \left(F(1_-) - \alpha\, C(0)\, \ddot{a}_{x+1}\right).$$

Similar reasoning gives that a longevity credit of amount

$$\frac{N^{(0,2)}(1)}{\mathcal{L}^{(0)}(1)} \left(F(1_-) - \beta\, C(0)\, \ddot{a}_{y+1}\right)$$

should be paid to each couple in State 0 at time 1, due to other couples transitioning to State 2. A longevity credit of amount

$$\frac{N^{(0,3)}(1)}{\mathcal{L}^{(0)}(1)} F(1_-)$$

should be paid to each couple in State 0 at time 1, due to other couples transitioning to State 3.

In total, each couple in State 0 at time 1 is paid a longevity credit of amount

$$\begin{aligned} \mathrm{LC}^{(0)}(1) := \frac{1}{\mathcal{L}^{(0)}(1)} \Big\{ & N^{(0,1)}(1) \left(F(1_-) - \alpha\, C(0)\, \ddot{a}_{x+1}\right) \\ & + N^{(0,2)}(1) \left(F(1_-) - \beta\, C(0)\, \ddot{a}_{y+1}\right) \\ & + N^{(0,3)}(1)\, F(1_-) \Big\} \end{aligned} \tag{4}$$

at time 1 and hence their fund value increases to $F(1) := F(1_-) + \mathrm{LC}^{(0)}(1)$.

At time 1, a new income calculation is done for the couples in State 0. The income paid to them at time 1 is

$$C(1) = \frac{F(1_-) + \mathrm{LC}^{(0)}(1)}{\ddot{a}_{x+1} + \alpha \ddot{a}_{y+1|x+1} + \beta \ddot{a}_{x+1|y+1}}$$

and their fund value at time 1 is

$$F(1) = F(1_-) + \mathrm{LC}^{(0)}(1) - C(1) = \Big(F(1_-) + \mathrm{LC}^{(0)}(1)\Big) \frac{\ddot{a}_{x+1} + \alpha \ddot{a}_{y+1|x+1} + \beta \ddot{a}_{x+1|y+1} - 1}{\ddot{a}_{x+1} + \alpha \ddot{a}_{y+1|x+1} + \beta \ddot{a}_{x+1|y+1}}.$$

The longevity credit is repeated at time 2, using the updated income value $C(1)$ and the number of transitions to each state between time 1 and time 2. For example, let the account value of couples in State 0 just before time 2 be $F(2_-)$. Suppose there are $N^{(0,k)}(2) - N^{(0,k)}(1)$ transitions from State 0 to State $k \in \{1, 2, 3\}$ between time 1 and time 2. Then the longevity credit paid to each couple in State 0 at time 2 is

$$\mathrm{LC}^{(0)}(1) := \frac{1}{\mathcal{L}^{(0)}(2)} \Big\{ \Big(N^{(0,1)}(2) - N^{(0,1)}(1)\Big) \left(F(2_-) - \alpha\, C(1)\, \ddot{a}_{x+2}\right) + \Big(N^{(0,2)}(2) - N^{(0,2)}(1)\Big) \left(F(2_-) - \beta\, C(1)\, \ddot{a}_{y+2}\right) + \Big(N^{(0,3)}(2) - N^{(0,3)}(1)\Big)\, F(2_-) \Big\}.$$

4.4.2 A joint life longevity credit for heterogeneous couples

In practice, the couples who join a pooled annuity fund to get a joint life income will bring different fund values when they join and will be of different ages compared to other couples in the fund. A longevity credit calculation must take account of these realities. In this section, an expression for calculating the longevity credit when couples are heterogeneous is developed.

Turn to the single life longevity credit calculation for motivation for a suitable joint life longevity credit calculation. There have been various ways proposed to share longevity risk in a single life setting. Some of them are actuarially fair over fixed time periods. In other words, the expected gain from longevity risk-sharing – here, the payment of a longevity credit upon survival over each year – equals the expected gain from longevity risk-sharing – the loss of the account value when a single life dies. Some of the longevity risk methods are only approximately actuarially fair, but in practice, this is good enough. The circumstances under which they fail to be close to actuarially fair are the circumstances when pooling is inadequate, for example when there are few participants in the fund and the fund is very heterogeneous in terms of the participants' mortality and wealth.

A common thread can be seen in the many proposed ways of calculate a single life longevity credit. It is that the share of the funds of those who have died should be in proportion to the expected loss of each participant. The expected loss is the product of the sum-at-risk and the probability of the risk materialising; in the case, the chance of the single life dying. There are some variations, for example Qiao and Sherris (2013) divide the sum-at-risk by the probability of the risk *not* materialising to define the proportionate share of a participant. Nonetheless, this is approximately the same as the expected loss.

It is also evident that, as there are many ways of calculating the longevity credit for a single life income, there will be many ways for calculating it for a joint life income. Here, following the expected loss approach, first adjust expression (4) to allow for the expected loss. The expected loss of a couple in State 0 at time 0 for a possible transition to State $k \in \{1, 2, 3\}$ at time 1 is

$$\{\text{Couple's probability of moving to State } k\} \cdot \{\text{Couple's sum-at-risk if moving to State } k\}.$$

For example, the expected loss of a couple in State 0 at time 0 for a possible transition to State 1 at time 1 is $p_x\, q_y\, (F(1_-) - \alpha\, C(0)\, \ddot{a}_{x+1})$.

Consider the expression (4) for the longevity credit when the homogeneous couples. To incorporate the expected loss means that the term $1/\mathcal{L}^{(0)}(1)$, the inverse of the number of surviving couples in State 0 at the end of the year, is adjusted to

$$\frac{\{\text{Couple's expected loss if move to State } k\}}{\sum_{\text{Surviving couples}} \{\text{Each surviving couple's expected loss if move to State } k\}}.$$

Since the number of surviving couples in State 0 at time 1 is $\mathcal{L}^{(0)}(1)$ and each couple in State 0 has, at time 0, the same expected loss if they transition to a fixed state $k \in \{1, 2, 3\}$ at time 1, then the above fraction can be simplified to $1/\mathcal{L}^{(0)}(1)$ for each of the three possible values of k.

In terms of the notation developed in Section 4.4.1, the longevity credit, paid to each surviving couple when all couples are homogeneous copies of each other at time 0, is

$$
\begin{aligned}
\mathrm{LC}^{(0)}(1) =& \frac{p_x\, q_y\, (F(1_-) - \alpha\, C(0)\, \ddot{a}_{x+1})}{\mathcal{L}^{(0)}(1)\, p_x\, q_y\, (F(1_-) - \alpha\, C(0)\, \ddot{a}_{x+1})} \left\{ N^{(0,1)}(1)\, (F(1_-) - \alpha\, C(0)\, \ddot{a}_{x+1}) \right\} \\
&+ \frac{q_x\, p_y\, (F(1_-) - \beta\, C(0)\, \ddot{a}_{y+1})}{\mathcal{L}^{(0)}(1)\, q_x\, p_y\, (F(1_-) - \beta\, C(0)\, \ddot{a}_{y+1})} \left\{ N^{(0,2)}(1)\, (F(1_-) - \beta\, C(0)\, \ddot{a}_{y+1}) \right\} \\
&+ \frac{q_x\, q_y\, F(1_-)}{\mathcal{L}^{(0)}(1)\, q_x\, q_y\, F(1_-)} \left\{ N^{(0,3)}(1)\, F(1_-) \right\}
\end{aligned}
$$

The first line in the last expression can be understood as follows. The first, fractional, term shows the proportion that a particular couple, which is in State 0 at time 1, should get of the total sum-at-risk released by actual transitions (by other couples) into State 1 at time 1. The second term, in curly brackets, gives the total sum-at-risk released by actual transitions (by other couples) into State 1 at time 1. The explanation of the second and third lines follows similarly.

Following the expected loss approach, the longevity credit for couples in State 0 at time n could be calculated as follows. Suppose at time 0 that there are N couples in State 0. Couple i consists of a husband of age x_i and a wife of age y_i at time 0, for $i = 1, 2, \ldots, N$. Let $p^{(f)}(i; x)$ and $p^{(m)}(i; x)$ denote the probability of survival from age x to age $x + 1$ for the wife and husband, respectively. Define the corresponding probabilities of mortality, $q^{(f)}(i, y) := 1 - p^{(f)}(i, y)$ and $q^{(m)}(i, x) := 1 - p^{(m)}(i, x)$. It is assumed, as before, that the future lifetimes of the couple are independent random variables.

Let $\ddot{a}^{(f)}_{i;x}$ and $\ddot{a}^{(m)}_{i;x}$ represent the EPV of a single life annuity payment, of \$1 per annum paid annually in advance, starting from age x, to the wife and husband, respectively. Let $\ddot{a}_{i;x,y}$ represent the EPV of a joint life annuity payment, of \$1 per annum paid annually in advance, starting when the husband is age x and the wife is age y and stopping when one or both of them die. The EPVs $\ddot{a}_{i;y|x}$ and $\ddot{a}_{i;x|y}$ of the reversionary life annuities are written similarly.

Couple i bring an amount $\$F_i(0_-)$ to the fund and their account value at time n is denoted $F_i(n)$ for $n \geq 0$. The amount of income withdrawn at time n is $C_i(n)$, for $n = 0, 1, 2, \ldots$. They wish to have an annual payment paid annually in advance of: (i) $\$C_i(0)$ while they are both alive; (ii) $\$\alpha_i\, C_i(0)$ while the husband is alive and the wife is dead; (iii) $\$\beta_i\, C_i(0)$ while the husband is dead and the wife is alive; and (iv) No payments if both are dead, in which $\alpha_i \geq 0$ and $\beta_i \geq 0$.

The initial income paid out to Couple i at time 0 is

$$
C_i(0) := \frac{F_i(0_-)}{\ddot{a}_{i;x_i,y(i)} + \alpha \ddot{a}_{i;y_i|x_i} + \beta \ddot{a}_{i;x_i|y_i}}.
$$

For Couple i, their expected loss for a transition at time 1 to State 1 – when the husband is alive and the wife is dead – is

$$
p^{(m)}(i; x_i)\, q^{(f)}(i; y_i)\, \left(F_i(1_-) - \alpha_i\, C_i(0)\, \ddot{a}_{i;x_i+1}\right).
$$

As before, the sum-at-risk is its value at time 1, at the time the longevity credit is paid out, rather than the value at time 0.

Suppose that at time 1, the set $\mathcal{A}^{(k)}$ holds the indices of the couples who are in State k at time 1, for $k \in \{0, 1, 2, 3\}$. The index of each couple is in exactly one of these four sets. Then for each

$i \in \mathcal{A}^{(0)}$, Couple i gets a longevity credit at time 1 of

$$\frac{p^{(m)}_{i,x_i}\, q^{(f)}_{i,y_i} \Big(F_i(1_-) - \alpha_i\, C_i(0)\, \ddot{a}^{(m)}_{i;x_i+1}\Big)}{\sum_{j\in\mathcal{A}^{(0)}} p^{(m)}_{j,x_j}\, q^{(f)}_{j,y_j} \Big(F_j(1_-) - \alpha_j\, C_j(0)\, \ddot{a}^{(m)}_{j;x_j+1}\Big)} \Bigg\{ \sum_{\ell\in\mathcal{A}^{(1)}} \Big(F_\ell(1_-) - \alpha_\ell\, C_\ell(0)\, \ddot{a}^{(m)}_{\ell;x_\ell+1}\Big) \Bigg\}$$
$$+ \frac{q^{(m)}_{i,x_i}\, p^{(f)}_{i,y_i} \Big(F_i(1_-) - \beta_i\, C_i(0)\, \ddot{a}^{(f)}_{i;y_i+1}\Big)}{\sum_{j\in\mathcal{A}^{(0)}} q^{(m)}_{j,x_j}\, p^{(f)}_{j,y_j} \Big(F_j(1_-) - \beta_j\, C_j(0)\, \ddot{a}^{(f)}_{j;y_j+1}\Big)} \Bigg\{ \sum_{\ell\in\mathcal{A}^{(2)}} \Big(F_\ell(1_-) - \beta_\ell\, C_\ell(0)\, \ddot{a}^{(f)}_{\ell;y_\ell+1}\Big) \Bigg\}$$
$$+ \frac{q^{(m)}_{i,x_i}\, q^{(f)}_{i,y_i}\, F_i(1_-)}{\sum_{j\in\mathcal{A}^{(0)}} q^{(m)}_{j,x_j}\, p^{(f)}_{j,y_j}\, F_j(1_-)} \Bigg\{ \sum_{\ell\in\mathcal{A}^{(3)}} F_\ell(1_-) \Bigg\}.$$

For a couple $\ell \in \mathcal{A}^{(1)}$ who moves from State 0 to State 1, they do not receive any longevity credit. Rather, an amount $\$\Big(F_\ell(1_-) - \alpha_\ell\, C_\ell(0)\, \ddot{a}^{(m)}_{\ell;x_\ell+1}\Big)$ is taken out of their account, leaving the widower with an account value of $F_\ell(1) = \alpha_\ell\, C_\ell(0)\, \ddot{a}^{(m)}_{\ell;x_\ell+1}$. The amount taken out of their account is shared out, in line with the longevity credit calculation above. A similar transfer of money occurs for couples who move from State 0 to State 2. For couples in which both partners die over the year, i.e. who move from State 0 to State 3, all of their account value is shared out among the couples in State 0.

With the expression for the longevity credit above, a pooled annuity fund can offer its participants a tailored benefit. Couples can choose their own values of α_i and β_i to meet their personal circumstances. Participants with different fund values and different future distributions of death can pool their longevity risk together to get their desired benefits.

It is clear that, under the proposed calculation, couples in State 0 can pool their longevity risk only with other couples in State 0. They cannot pool their longevity risk with the widowed individuals in State 1 and State 2. However, the widowed individuals can pool their longevity risk with other widowed individuals. More generally, they can pool their longevity risk with any individual in State 1 or State 2, using a longevity credit calculation like that in expression 3.

5 Conclusion

For pooled annuity funds to compete with life annuities, they need to provide similar benefits. This paper contributes to the literature on pooled annuity funds by showing how a joint life income can be provided. It requires a suitable longevity credit calculation, which is proposed here in a general setting. There are no restrictions on the mortality and financial characteristic of the couples who are to be paid a joint life income. The absence of restrictions if the method set out here is to be used in practice.

As has been shown, the joint life income benefit provides a richer menu of benefits to the couple than available to them if they chose two single life incomes. This means that they can choose the benefits that suit their own personal financial circumstances the best. Again, this flexibility increases the attractiveness of pooled annuity funds offering such benefits.

Additionally, this menu of benefits is richer than the joint life income benefits seen in life annuities and defined benefit pension plans. These latter products and schemes pay the annuitant or member a specified income while the member is alive. If they die, their spouse receives a specified fraction of the joint income, typically half or two-thirds of it. Why does the member get the full amount of the benefit while they are alive, and their spouse get less than that? In general, their financial needs as widowed individuals will be similar. Under the joint life income proposed in this paper, the couple can choose how much to receive when only one of them is left alive.

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