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PhD studentship output

Bayesian Inference for Small Population Longevity Risk Modelling

Liang Chen (CERA), Prof Andrew Cairns, Dr Torsten Kleinow

Actuarial Research Centre, Institute and Faculty of Actuaries, Heriot Watt University, UK

Email: lc285@hw.ac.uk/A.J.G.Cairns@hw.ac.uk/t.kleinow@hw.ac.uk

Introduction

Model: For death counts $D(t,x)$ and Exposure $E(t,x)$:

$$\text{logit}\, q(\theta,t,x) = \kappa_t^{(1)} + \kappa_t^{(2)}(x-\bar{x}) + \kappa_t^{(3)}\left((x-\bar{x})^2 - \hat{\sigma}_x^2\right) + \gamma_c^{(4)}$$

$$D(t,x)|\theta \sim Poi(m(\theta_1,t,x)E(t,x))$$

$$m(\theta,t,x) = -\log[1-q(\theta,t,x)]$$

Background: For small population, modelling with Two Stage approach (fit time series for the maximum likelihood (MLE) estimates $\hat{\theta}$) leads to biased estimates of volatility.

- Large sampling variation -> significant noise for latent parameter estimation (Cairns, Blake, Dowd et al. 2011).
- Non-negligible bias to the parameter estimation of the projecting model (Chen, Cairns and Kleinow 2015).
- Over fits the short cohorts with only one observation (Cairns et al. 2009)

Motivation: Bayesian approach offers a way to reduce such bias by

- Combining Poisson and time series likelihood
- Using knowledge of larger population to choose more informative prior
- Balancing the short cohorts with time series likelihood

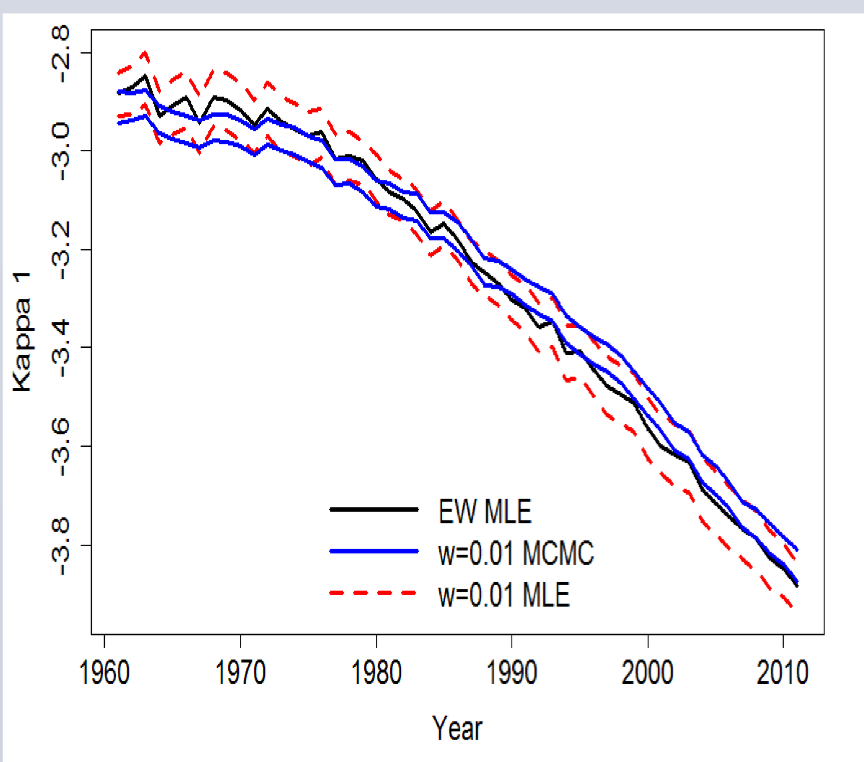
Data and Bootstrap Simulation

- Benchmark exposure $E_0(t,x)$ and corresponding death counts $D_0(t,x)$: Males in England and Wales (EW) from HMD, during 1961-2011, age 50-89 last birthday.
- $\hat{\theta}_0$: MLE for England and Wales
- $m(\hat{\theta},t,x)$: fitted death rates given $\hat{\theta}_0$
- Simulate $D_w(t,x)|\hat{\theta}_0 \sim \text{Poi}(m(\hat{\theta},t,x)wE_0(t,x))$, for $w = 0.01$
- $\hat{\theta}_w$: MLE for $D_w(t,x)$

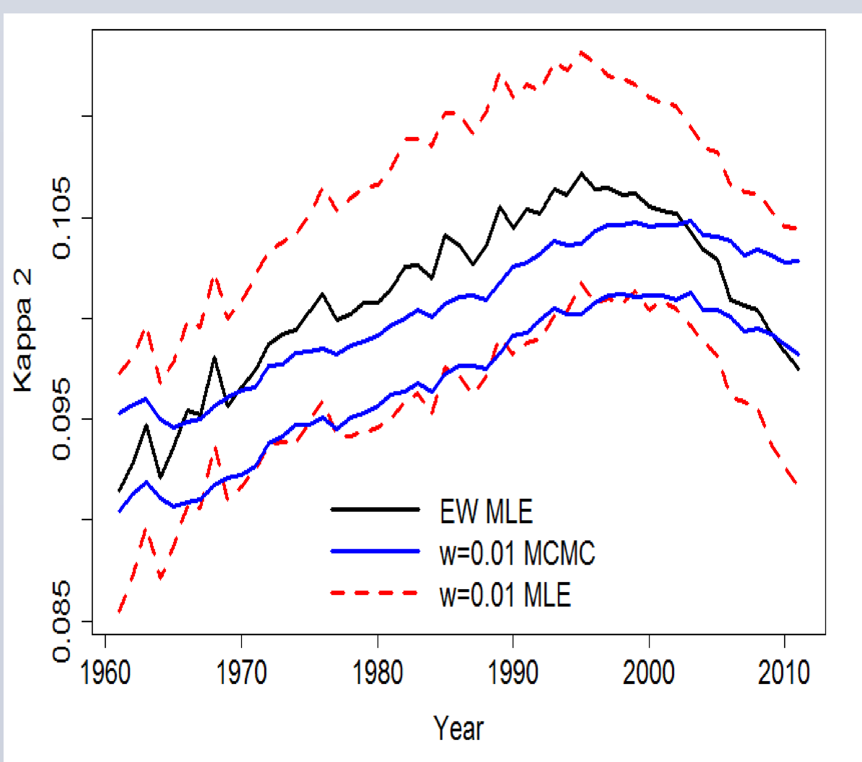
Prior Distributions for Latent and Hyper Parameters

- $\boldsymbol{\kappa_t} = \left(\kappa_t^{(1)}, \kappa_t^{(2)}, \kappa_t^{(3)}\right)^T$
- $\boldsymbol{\kappa_1} \propto$ Uniform Distribution
- $\boldsymbol{\kappa_t}|\boldsymbol{\kappa_{t-1}} \sim \boldsymbol{MVN}(\boldsymbol{\mu} + \boldsymbol{\kappa_{t-1}}, \boldsymbol{V_\epsilon})$, for $t > 1$
- $\boldsymbol{\mu} = (\mu_1, \mu_2, \mu_3)^T \propto$ Uniform Distribution
- $\boldsymbol{V_\epsilon} \propto$ Inverse Wishart $(\nu, \boldsymbol{\Sigma})$, independent of t
- MCMC-Mean: Fix the ***mean*** of the prior for $\boldsymbol{V_\epsilon}$ to the respected England and Wales' estimation
- MCMC-Mode: Fix the ***mode*** of the prior for $\boldsymbol{V_\epsilon}$ to the respected England and Wales' estimation
- $\gamma_c^{(4)}|\gamma_{c-1}^{(4)} \sim N(\alpha_\gamma \gamma_{c-1}^{(4)}, \sigma_\gamma^2)$, for $c > 2$
- $\gamma_1^{(4)} \sim N(0, \frac{\sigma_\gamma^2}{1-\alpha_\gamma^2})$
- $\alpha_\gamma \propto \left(1-\alpha_\gamma^2\right)^g$ for $|\alpha| < 1$
- $\sigma_\gamma^2 \sim$ Inverse Gamma(a_γ, b_γ)

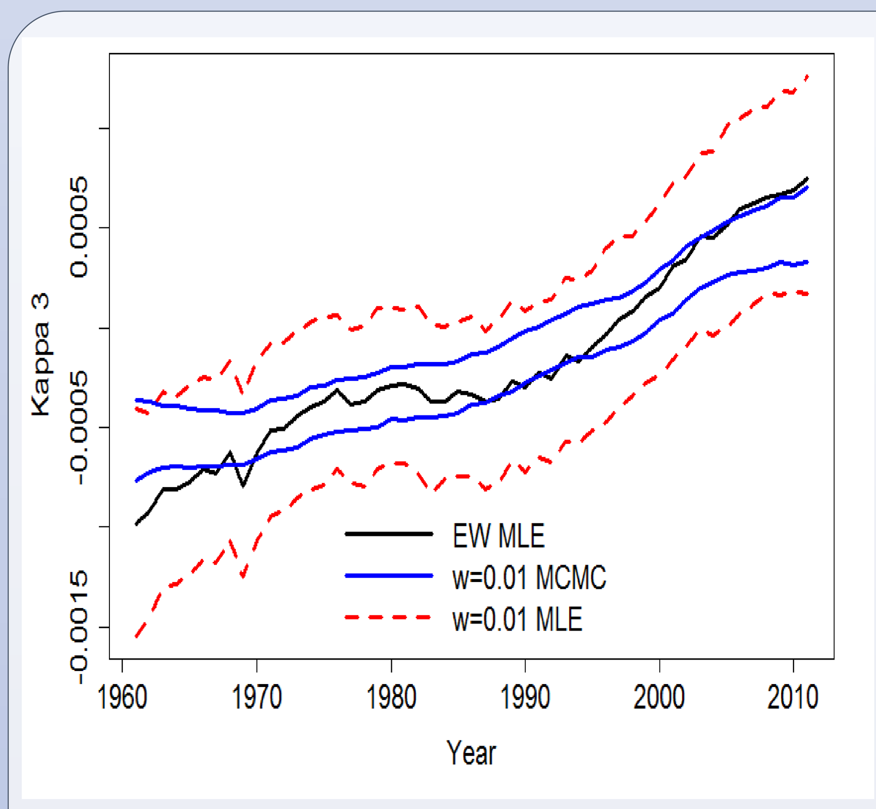
Credibility Interval for κ and γ



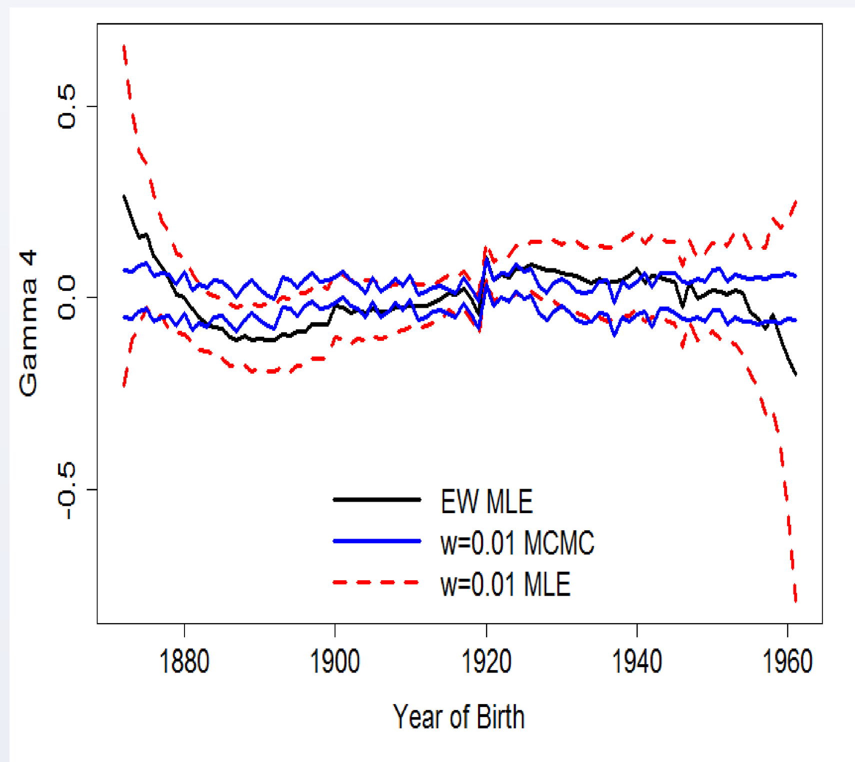
$\boldsymbol{\kappa^{(1)}}$



$\boldsymbol{\kappa^{(2)}}$



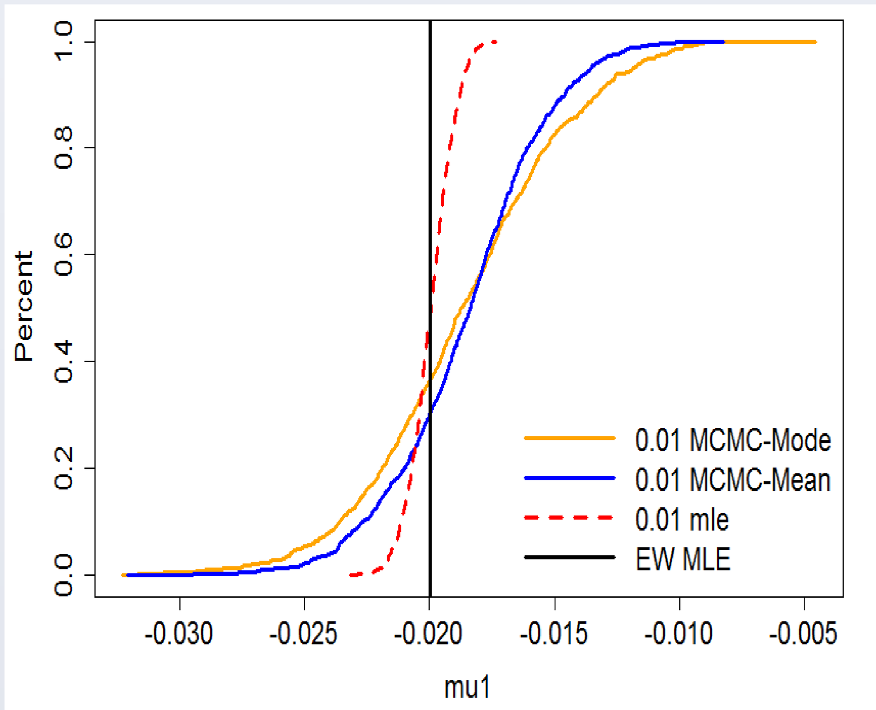
$\boldsymbol{\kappa^{(3)}}$



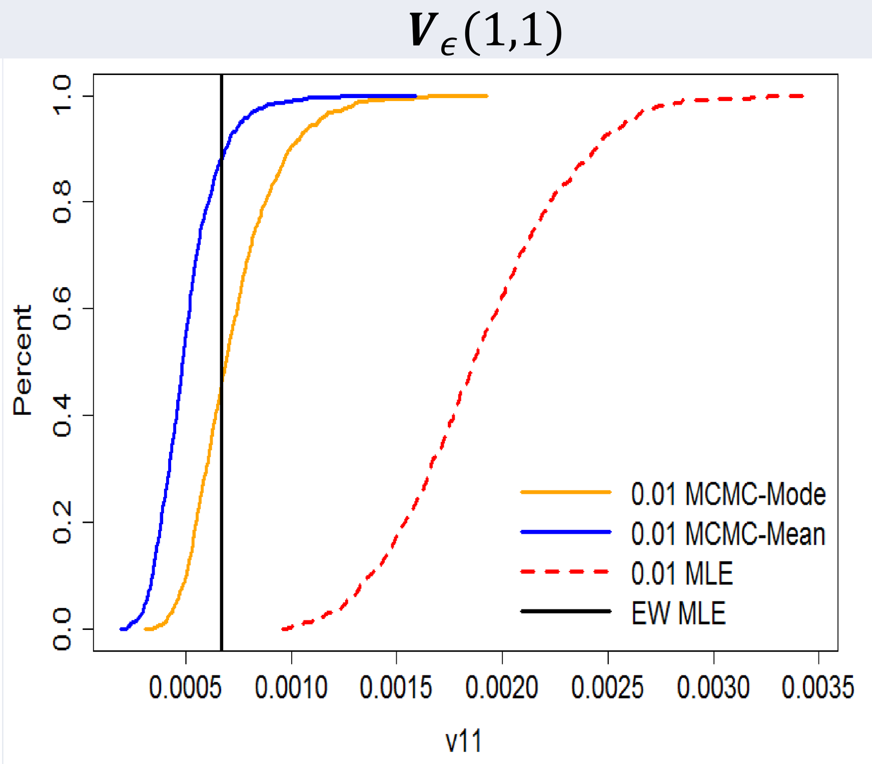
$\boldsymbol{\gamma^{(4)}}$

Posterior Distribution for μ and V_ϵ

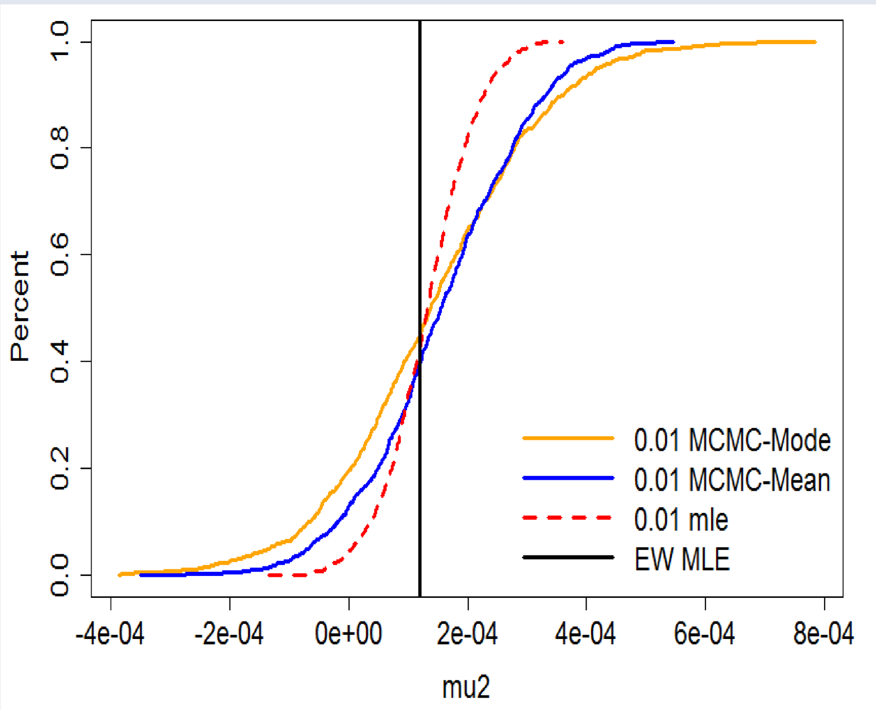
μ_1



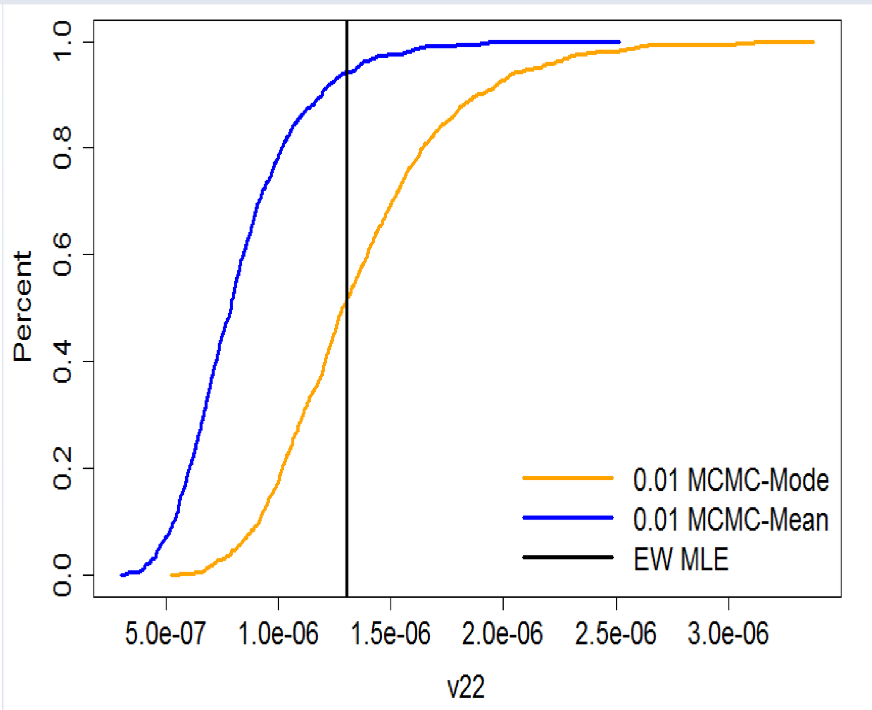
$\boldsymbol{V_\epsilon}(1,1)$



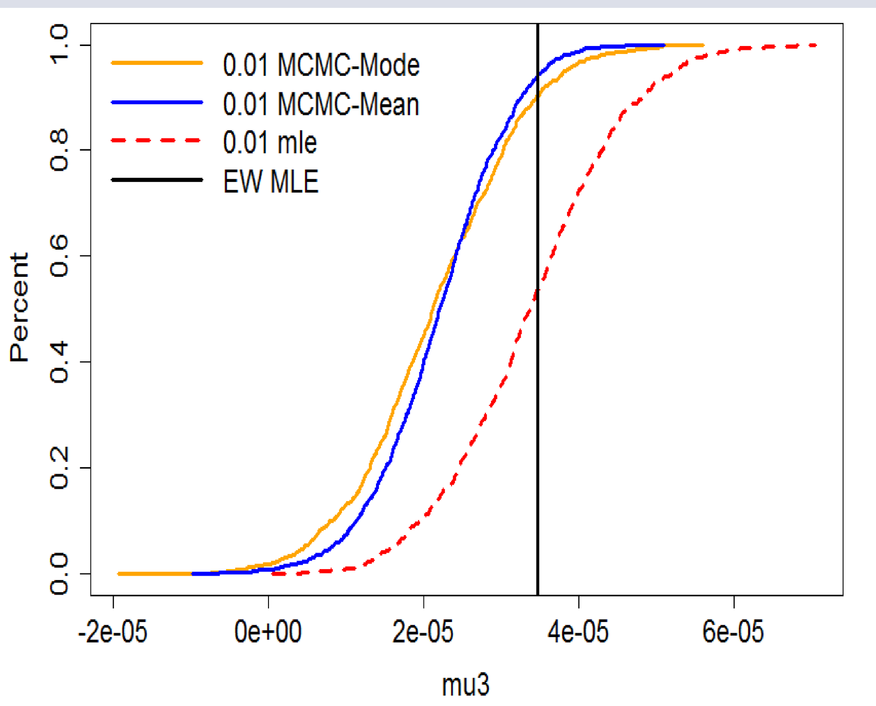
μ_2



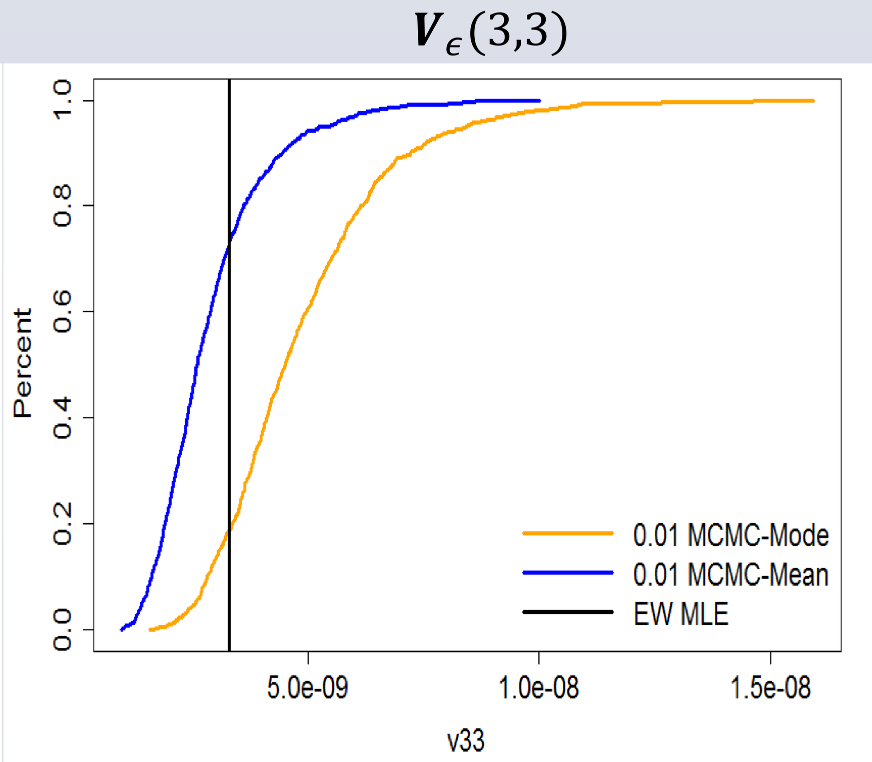
$\boldsymbol{V_\epsilon}(2,2)$



μ_3



$\boldsymbol{V_\epsilon}(3,3)$



Note: The MLEs of $V_\epsilon(2,2)$ and $V_\epsilon(3,3)$ for w=0.01 are also biased to the right of the true value (vertical line). We exclude them for a clear view of the posterior distribution.

Conclusions for Fitting Small Population Modelling

- The co-variance matrix estimated by MLE is significantly biased to the right of the assumed true value due to the Poisson model's over fitting.
- We combine the two stages into one by adding time series likelihood for the latent parameters and gained the posterior distribution with the MCMC procedure.
- The Bayesian method provides an improved fit to the hyper parameter $\boldsymbol{V_\epsilon}$.
- The low level information involved in short cohorts is balanced by the time series prior.
- The posterior distribution for small population is sensitive and fixing the mode of the prior for the co-variance matrix to the assumed true rates provides approximately unbiased fit to $\boldsymbol{V_\epsilon}$