

LARGEST CLAIMS REINSURANCE PREMIUMS  
UNDER POSSIBLE CLAIMS DEPENDENCE

PROFESSOR DR ERHARD KEREMER

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# **LARGEST CLAIMS REINSURANCE PREMIUMS UNDER POSSIBLE CLAIMS DEPENDENCE.**

**ERHARD KREMER (HAMBURG)**

## **SUMMARY:**

Largest claims reinsurance covers are reconsidered. A general new net premium bound is derived and specialized to the situation of Pareto-distributed claims sizes.

Important is that the classical assumption of claims-size-independence is not basically assumed, the claims-sizes are allowed to be dependent. No assumptions on the dependence-structure are needed. In an example it is shown for the classical largest claims reinsurance cover that the new bound is quite useful.

Closing the note something is said also on the ECOMOR-treaty.

## 1. Introduction.

Reinsurance mathematics is one of the classical fields of mathematical risk theory. (see e.g. Bühlmann (1970)). First important contributions go back to the thirties (see e.g. Thesen (1937)), newest ones to the last few years (see e.g. Kremer (1998)).

Certain classical reinsurance treaties became of new interest to several researchers. E.g. the famous stoploss treaty was reinvestigated for many times (see e.g. De Vylder & Goovaerts (1983), Kremer (1990 c)) and a comprehensive new theory was developed for reinsurance treaties of the largest claims type (see e.g. Kremer (1984), (1985, (1986), (1988), (1990 a), (1990 b), (1992), (1994 a)).

Most results are based fundamentally on the assumption of independence of the individual claims sizes. Only little is known under allowing the claims sizes to be dependent.

A paper by Heilmann (1986) treats with the stoploss cover under relaxation of the independence assumption. Some results in Kremer (1988) hold also in case of dependence of the claims sizes for (generalized) largest claims reinsurance covers under conditions allowing the individual claims sizes to be dependent.

In the following note, that point is taken up. Certain handy, more strict upper bounds are given for the premiums of (generalized) largest claims covers under conditions of possible dependence of the individual claims sizes.

## 2. The treaty.

Let the random variables  $X_1, X_2, \dots$  on  $(\Omega, \mathcal{A}, P)$  denote the claims of a collective of risks and let  $N$  on  $(\Omega, \mathcal{A}, P)$  describe the number of claims. We order the claims in decreasing size resulting in the ordered claims:

$$X_{N:1} \geq X_{N:2} \geq \dots \geq X_{N:N}.$$

For given constants  $c_1, c_2, \dots$  such that:

$$\sum_{i=1}^n c_i \cdot y_i \in \left[ 0, \sum_{i=1}^n y_i \right]$$

holds true for all:

$$y_1 \geq y_2 \geq \dots \geq y_n \geq 0,$$

define the claims amount taken by the reinsurer according:

$$S_N = \sum_{i=1}^N c_i \cdot X_{N:i}. \quad (2.1)$$

Consequently the family  $(c_i, i \geq 1)$  defines a reinsurance treaty, which shall be called generalized largest claims reinsurance cover (GLCR).

The name is obvious from the fact that (2.1) reduces for the special choice:

$$\begin{aligned} c_1 &= c_2 = \dots c_p = 1 \\ c_j &= 0, \text{ for all } j > p \end{aligned} \quad (2.2)$$

to the claims amount of the (classical) largest claims reinsurance cover, like defined e.g. by Ammeter (1964). The (net) premium:

$$\mu(c_i, i \geq 1) := E(S_N)$$

is subject of many mathematical investigations (see e.g. Kremer (1984)). In all papers one assumes that:

- (A.1) the claims sizes are identically distributed with the same distribution function  $F$  on  $[0, \infty)$ ,
- (A.2) the claims number  $N$  is independent of the claims sizes  $(X_i, i = 1, 2, \dots)$ .

and in nearly all papers it is basic that:

- (A.3) the claims sizes  $X_1, X_2, X_3, \dots$  are independent.

For continuous  $F$  one knows under (A.1)–(A.3) that (see Kremer (1985)):

$$\mu(c_i, i \geq 1) = \sum_{i=1}^{\infty} \left( \frac{c_i}{(i-1)!} \right) \cdot \int_0^1 F^{-1}(t) \cdot (1-t)^{i-1} M^{(i)}(t) dt, \quad (2.3)$$

where:

$$F^{-1}(u) = \inf\{x : F(x) \geq u\}$$

is the pseudo-inverse of  $F$  and  $M^{(i)}$  is the  $i$ -th derivate of the probability generating function:

$$M(s) = \sum_{n=0}^{\infty} P(N = n) \cdot s^n$$

of the distribution of  $N$ . Clearly in case that (A.3) is not necessarily given, the formula (2.3) is not expected to hold true. What can one say then?

### 3. The premium bound.

For the sequel assume (A.1) and (A.2) but not necessarily (A.3). Denote the mean claims size with:

$$\mu = \int_0^{\infty} xF(dx),$$

assumed to exist and with:

$$\rho(P) = \left(\frac{1}{\mu}\right) \cdot \int_P^{\infty} (x - P)F(dx)$$

the (net) premium rate of an excess-of-loss cover with priority  $P \geq 0$ . Remember that the net premium of this reinsurance cover is just:

$$E(N) \cdot \int_P^{\infty} (x - P)F(dx)$$

under the given conditions. One has the

#### Theorem:

Let  $F$  be continuous and suppose that:

$$c_i \geq 0, \text{ for all } i \geq 1$$

Then the (net) premium  $\mu(c_i, i \geq 1)$  is bounded by:

$$\left[ \sum_{m=1}^{\infty} \left( \frac{c_m}{m} \right) \cdot \left( \sum_{n=m}^{\infty} n \cdot P(N=n) \cdot \rho(P_{mn}) \right) \right] \cdot \mu + \left[ \sum_{m=1}^{\infty} c_m \cdot \left( \sum_{n=m}^{\infty} P(N=n) \cdot P_{mn} \right) \right] \quad (3.1)$$

where the priorities are defined as:

$$P_{mn} = F^{-1}(1 - (m/n)).$$

■

### Proof:

According to Corollary 3(b) in Rychlik (1994) (with there  $f(x) \equiv x$ ) one has given  $N = n$  :

$$E(X_{n:m}) = E(X_{N:m} | N = n) \leq \left( \frac{n}{m} \right) \cdot \int_{P_{mn}}^{\infty} xF(dx). \quad (3.2)$$

Obviously:

$$\int_{P_{mn}}^{\infty} xF(dx) = \rho(P_{mn}) \cdot \mu + \left( \frac{m}{n} \right) \cdot P_{mn}. \quad (3.3)$$

Furthermore one has:

$$\begin{aligned} \mu(c_i, i \geq 1) &= E(E(R_N | N)) = \\ &= \sum_{n=1}^{\infty} P(N=n) \cdot \sum_{m=1}^n c_m \cdot E(X_{n:m}) \\ &= \sum_{m=1}^{\infty} c_m \cdot \sum_{n=m}^{\infty} P(N=n) \cdot E(X_{n:m}). \end{aligned} \quad (3.4)$$

Inserting (3.2), (3.3) into (3.4) gives at once the upper bound of the Theorem.

■

**Remark 1**

The upper premium bound is defined with help of the premiums of countable many excess-of-loss covers, what shows again the intimate connections between largest claims and excess-of-loss reinsurance that were already discovered by Benktander (1978). ■

One can expect to get simpler expressions for the upper bound of the theorem in case of additional, handy model assumptions. In many practical situations it will be adequate to assume that  $F$  is of the Pareto type, i.e. that

$$F(x) = 1 - (x/a)^{-\alpha}, \text{ for } x \geq a, \quad (3.5)$$

where  $a$  is a non-negative threshold and  $\alpha > 1$  the (unknown) parameter. One has the following:

**Corollary.**

The upper bound of the theorem reduces under (3.5) to

$$a \cdot \left[ \sum_{m=1}^{\infty} \left( \frac{c_m}{m^{1/\alpha}} \right) \right] \cdot E(N^{1/\alpha}) \cdot \left( \frac{\alpha}{\alpha-1} \right) - R \quad (3.6)$$

where the remainder term  $R$  is given according:

$$R = a \cdot \left[ \sum_{m=1}^{\infty} \left( \frac{c_m}{m^{1/\alpha}} \right) \cdot \sum_{n=1}^{m-1} P(N=n) \cdot n^{1/\alpha} \right] \cdot \left( \frac{\alpha}{\alpha-1} \right)$$

**Proof:**

One has simply:

$$F^{-1}(t) = a \cdot (1-t)^{-1/\alpha},$$

implying:

$$P_{mn} = a \cdot (m/n)^{-1/\alpha}. \quad (3.7)$$

Furthermore:

$$\mu = a \cdot \left( \frac{\alpha}{\alpha-1} \right)$$

and

$$\rho(P) = (P/a)^{1-\alpha}/\alpha. \quad (3.8)$$

Inserting (3.7), (3.8) into (3.1) implies after few routine calculations (3.6). ■

**Remark 2.**

Usually one will have a natural  $p$  such that:

$$c_m = 0, \quad \forall j > p. \quad (3.9)$$

In case that  $P(N < p)$  is sufficiently small (i.e.  $p$  is sufficiently small and  $E(N)$  sufficiently large) one has that the remainder term  $R$  is nearly zero and one can put:

$$R \approx 0. \quad (3.10)$$

■

**Remark 3.**

With the help of a Taylor-expansion of the function  $g(x) = x^{1/\alpha}$  one comes to the approximation:

$$E(N^{1/\alpha}) \approx (E(N))^{1/\alpha} \cdot \left( 1 - \left( \frac{1}{2} \right) \left( \frac{\alpha - 1}{\alpha^2} \right) \cdot \left( \frac{\text{Var}(N)}{E(N)^2} \right) \right). \quad (3.11)$$

Anyway, one gets with Jensen's inequality that:

$$E(N^{1/\alpha}) \leq (E(N))^{1/\alpha}. \quad (3.12)$$

Obviously (3.6) becomes quite handy in case of (3.10) and by using (3.11) or (3.12). ■

**4. Numerical example.**

Consider the special case (2.2), which is of type (3.9), and suppose that  $N$  is Poisson-distributed, i.e.:

$$P(N = n) = \left( \frac{\lambda^n}{n!} \right) \cdot \exp(-\lambda), \quad n = 0, 1, 2, 3, \dots,$$



where  $\lambda = E(N)$  is the distribution parameter. One knows that (for not small  $\lambda$ ) the premium rate of the treaty, i.e.:

$$\rho_p = \left( \frac{E(S_N)}{E(N) \cdot \mu} \right),$$

can be calculated in case of Pareto-distributed claims sizes (see (3.5)) nearly exact with the formula (see Ammeter (1964)):

$$\hat{\rho}_p = \left[ \frac{\Gamma(p+1-1/\alpha)}{\Gamma(p)} \right] \cdot \lambda^{1/\alpha-1}, \quad (4.1)$$

where  $\Gamma(y)$  is defined as:

$$\Gamma(y) = \int_0^{\infty} s^{y-1} \cdot \exp(-s) ds,$$

when in addition to (A.1),(A.2) also (A.3) is assumed. When (A.3) is dropped, then one can take the bound (3.6) with (3.10) (for smaller  $p$  and larger  $\lambda$ ) and (3.12), giving the premium rate bound:

$$\hat{\rho}_p = \left[ \sum_{m=1}^p m^{-1/\alpha} \right] \cdot \lambda^{1/\alpha-1} \quad (4.2)$$

which is even more handy than (4.1). When one takes a closer look into Kremer (1988), one can find out that the theorem 1 there also holds true in case that (A.3) is not necessarily given. This means that the there given premium rate bound:

$$\tilde{\rho}_p = (p/\lambda) + \left[ \frac{(p/\lambda)(1 - (p/\lambda))}{\alpha \cdot (\alpha - 2)} \right]^{1/2} \quad (4.3)$$

can be taken also in case one does not necessarily have independence of the claim sizes.

The question arises which one is the sharper bound, (4.2) or (4.3). Furthermore one would like to know how much maximally the premium rate can raise when one neglects the assumption (A.3). In the typical situations where one applies the (classical) largest claims cover one has  $\alpha \in [2.0, 3.0]$  and  $p \leq 10$ .

Numerical values for  $\hat{\rho}_p$ ,  $\hat{\hat{\rho}}_p$  and  $\tilde{\rho}_p$  are given for those practical situations in the following tables, where  $\lambda = 1000$  is chosen.

$\alpha = 2.01$
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	$\hat{\rho}_p$	$\hat{\hat{\rho}}_p$	$\tilde{\rho}_p$
$p = 1$	2.76%	3.11%	22.39%
2	4.14	5.31	31.71
3	5.18	7.11	38.88
4	6.05	8.67	44.92
5	6.81	10.06	50.25
6	7.49	11.34	55.07
7	8.12	12.52	59.51
8	8.70	13.62	64.64
9	9.25	14.67	67.50
10	9.76	15.66	71.18

$$\alpha = 2.50$$

	$\hat{\rho}_p$	$\hat{\hat{\rho}}_p$	$\tilde{\rho}_p$
$p = 1$	1.42%	1.58%	2.90%
2	2.27	2.79	4.20
3	2.95	3.81	5.19
4	3.54	4.72	6.05
5	4.07	5.55	6.81
6	4.55	6.32	7.51
7	5.01	7.05	8.16
8	5.44	7.74	8.77
9	5.85	8.40	9.35
10	6.24	9.03	9.90

$$\alpha = 3.00$$

	$\hat{\rho}_p$	$\hat{\hat{\rho}}_p$	$\tilde{\rho}_p$
$p = 1$	0.90%	1.00%	1.92%
2	1.50	1.79	2.78
3	2.01	2.49	3.46
4	2.45	3.12	4.04
5	2.86	3.70	4.57
6	3.24	4.25	5.06
7	3.60	4.77	5.51
8	3.95	5.27	5.94
9	4.27	5.76	6.35
10	4.59	6.22	6.74

Obviously the bounds (4.3) are of no great help for smaller values of  $\alpha$ . They are hit out for the relevant values of  $\alpha$  by the bounds (4.2), which seem to be quite strict. One can compute the ratios  $(\hat{\rho}_p/\hat{\rho}_p) = r_p$  for seeing how much the premium rate can maximally increase when deleting the assumption (A.3). One gets as results:

$$\alpha = 2.01$$

	p=1	2	3	4	5	6	7	8	9	19
$r_p$	1.13	1.29	1.37	1.44	1.48	1.52	1.54	1.57	1.59	1.61

$$\alpha = 2.50$$

	p=1	2	3	4	5	6	7	8	9	10
$r_p$	1.11	1.23	1.29	1.33	1.36	1.39	1.41	1.42	1.43	1.45

$$\alpha = 3.00$$

	p=1	2	3	4	5	6	7	8	9	10
$r_p$	1.11	1.19	1.24	1.27	1.29	1.31	1.32	1.33	1.34	1.35

The range of  $r_p$  is approximately the interval [1.1, 1.6], the mean value is about 1.35. These results might be very helpful for practical rating situation, where one does not know anything about the dependence of the original claims sizes, and one is not willing to assume (A.3).

## 5. ECOMOR

In the theorem of section 3 it is assumed basically that the coefficients  $c_i$  are all non-negative. Unfortunately there exists a special GLCR, that has a negative coefficient  $c_i$ . This is the so-called ECOMOR-cover that is defined by the choice

$$\begin{aligned} c_1 &= c_2 = \dots = c_{p-1} = 1, \quad c_p = 1 - p \\ c_j &= 0, \text{ for all } j > p. \end{aligned}$$

Obviously the net premium of the ECOMOR cover is just the net premium of the (classical) largest claims cover (GLCR with (2.2)) minus:

$$p \cdot E(X_{N:p})$$

Consequently one needs in addition to (3.1) (or (3.6)) only a lower bound on:

$$\begin{aligned} \mu_p &= E(X_{N:p}) \\ &= \sum_{n=p}^{\infty} P(N=n) \cdot E(X_{n:p}) \end{aligned}$$

A lower bound on  $E(X_{n:p})$  can be given again with Rychlik (1994), Corollary 3(a) (with there  $f(x) \equiv x$ ). With the priorities:

$$Q_{mn} = F^{-1}(1 - (m-1)/n)$$

One finally gets (like in the proof of the theorem):

$$\begin{aligned} \mu_p &\geq \left[ \sum_{n=p}^{\infty} P(N=n) \cdot \left( \frac{n}{n-p+1} \right) \cdot (1 - \rho(Q_{mn})) \right] \cdot \mu + \\ &+ \left[ \sum_{n=p}^{\infty} P(N=n) \cdot \left( 1 - \left( \frac{n}{n-p+1} \right) \right) \cdot Q_{mn} \right] \end{aligned}$$

Even under (3.5) this lower bound does not simplify considerably and remains comparably unhandy. Certain cruder simplifications turned out to be too crude for getting suitably strict handy bounds on the net premium of the ECOMOR. Concerning the ECOMOR the author restricts on the given remarks. As far as the author knows, the ECOMOR is not in current practical use, though it is an attractive alternative to the classical excess-of-loss treaty (see Kremer (1994 b)).

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**Author:**

E. Kremer

Institut für Mathematische Stochastik

Universität Hamburg

Bundesstraße 55

20146 Hamburg

F.R.G.