JIA 13 (1866-1867) 325-358

JOURNAL

OF THE

INSTITUTE OF ACTUARIES.

On the Law of Mortality. *By* WILLIAM MATTHEW MAKEHAM, *Fellow of the Institute of Actuaries.*

[Read before the Institute, 29th April, 1867.]

IN the following pages I shall have frequent occasion to avail myself of a term which the progress of the analysis of life contingencies has rendered indispensable, but which is not found in any of the standard elementary works in that science. I think, therefore, that I cannot better commence this paper than by an attempt to give an explanation of the expression "force of mortality," sufficiently ample to obviate any difficulties which might otherwise be experienced on this score.

In the subjoined table L_x denotes the number living at age x in a mortality table, and ΔL_x the difference corresponding to an increment of Δx in the age—in this case 10 years. The other characters will be explained further on.

x. (1.)	L_x. (2.)	$-\Delta L_x$. (3.)	$-\frac{1}{L_x}\cdot\frac{\Delta L_x}{\Delta x}$. (4.)	$\frac{L_x - L_{x+1}}{L_x}$. (5.)	$-\frac{1}{L_x}\cdot\frac{dL_x}{dx}$. (6.)	x. (7.)
20	9626·100	754·934	·00784	·00775	·00775	20
30	8871·166	818·836	·00923	·00875	·00872	30
40	8052·330	998·874	·01241	·01108	·01097	40
50	7053·456	1370·050	·01942	·01653	·01626	50
60	5683·406	1918·892	·03376	·02919	·02868	60
70	3764·514	2190·060	·05818	·05826	·05780	70
80	1574·454	1350·552	·08578	·12310	·12614	80
90	223·902	221·368	·09887	·25824	·28647	90
100	2·534	2·534	·10000	·49913	·66265	100

The second column then, headed L_x, contains the numbers living at successive decennial intervals of age; the third $(-\Delta L_x)$ shows the decennial decrements; the fourth $\left(-\frac{1}{L_x}\cdot\frac{\Delta L_x}{\Delta x}\right)$ gives the ratio which the average annual decrement of each decennial period bears to the number living at the commencement of the period; while the fifth column $\left(\frac{L_x - L_{x+1}}{L_x}\right)$ exhibits the series known as the rate of mortality—*i.e.*, the ratio of the number actually dying in one year to the number living at the commencement of that year.

Thus, out of 9626·100 persons living at age 20, 754·934 die before attaining the age of 30—which is at the *average* rate of 75·4934 per annum. Dividing this last by 9626·100, we have ·00784 for the ratio which the average number of annual deaths bears to the number living at the commencement of the decennial interval.

Comparing this with the corresponding number in column 5, we see that it is but little in excess of the rate of mortality for the least of the 10 ages included in the given interval—the rate of mortality at age 20 being ·00775. The same comparative deficiency in the average annual proportion of deaths of the decennial interval will be observed upon comparing the two columns for the respective ages of 30, 40, 50, and 60. At the age of 70 and upwards, however, the deficiency in question becomes much more apparent—the average annual rate of decrease, during the 10 years, being then actually *less* than the rate of mortality for the youngest age of the decennial period; and so rapidly does this go on, that at the age of 100 the former is little more than one-fifth of the latter.

It is scarcely necessary to point out that this result is owing to the fact that as the numbers living are gradually diminishing by the effect of mortality, the number of deaths must also experience a corresponding diminution; and the longer the interval between the ages observed, the greater must be the disturbance caused by this incessant reduction in the number exposed to the risk of death. My object, however, in calling attention to a phenomenon which must be sufficiently familiar to all who have given the least consideration to the subject, is to show how extremely inadequate the average annual number of deaths—taken for several years together—becomes as a measure of the actual *intensity* or *force* of the operating causes by which the decrements of life are produced.

Now, it will be observed that the function $\frac{L_x - L_{x+1}}{L_x}$ (representing the rate of mortality), of which the values are shown in col. 5, is of precisely the same form as the function $-\frac{1}{L_x}\cdot\frac{\Delta L_x}{\Delta_x}$ $\left(\text{or } -\frac{1}{L_x}\cdot\frac{\Delta L_x}{10}\right)$, from which the values in col. 4 are computed. For the latter is equivalent to $\frac{L_x - L_{x+10}}{10 L_x}$, which becomes identical with the former by substituting 1 for 10 as the increment of x. But it is evident that the same causes which render the series in col. 4 so imperfect a measure of the force of mortality must also tend (in a less degree) to make the series in col. 5 unfit for the same purpose. If, therefore, instead of making Δx (or the increment of age) $=1$ in the formula $-\frac{1}{L_x}\cdot\frac{\Delta L_x}{\Delta x}$, we make it a fraction (say ·5, for instance), we shall obtain a still better expression for the measure in question. And if we go still further, and diminish Δx without limit, we shall evidently get rid of the disturbing element altogether, and thus obtain a perfect measure of the actual intensity or force of mortality at each age.

This is, in effect, the process adopted in the construction of col. 6; $\frac{dL_x}{dx}$ (which in the language of the differential calculus is called the differential coefficient of L_x) being the limit of the ratio of the infinitely small decrement of L_x to the infinitely small increment of x, by which it is produced. The means by which the limit in question may be computed will sufficiently appear in the course of the following pages.

The considerations to which I have thus ventured to call attention afford, I think, a sufficient explanation of the nature of the function known as the "force of mortality," as well as of the reason for its designation; and at the same time convey some idea of the importance which such a function is calculated to possess in the investigation of the nature of the law of mortality, and the analysis of life contingencies generally.

I proceed now to establish some theorems in connection with the expression $-\frac{1}{L_x}\cdot\frac{dL_x}{dx}$, which will be of service to us in the sequel; premising that when finite differences are used they are (unless the contrary be expressed) to be taken from the series L_x, L_{x+n}, L_{x+2n}, &c.—in other words, n is substituted for Δx.

I. The usual expansion of $\frac{dL_x}{dx}$ in terms of the finite differences is (De Morgan's *Diff. and Int. Calculus.*, p. 264)

$$n\cdot\frac{dL_x}{dx}=\Delta L_x-\tfrac{1}{2}\Delta^2L_x+\tfrac{1}{3}\Delta^3L_x-\ldots.$$

Transforming the second member of this equation into the system of central differences (see Mr. Woolhouse's paper on Interpolation, *Assurance Magazine*, vol. xi.), we have

$$n\frac{dL_x}{dx}=\frac{\Delta L_x+\Delta L_{x-n}}{2}-\frac{1}{2.3}\,\frac{\Delta^3L_{x-n}+\Delta^3L_{x-2n}}{2}$$
$$+\frac{1.2}{3.4.5}\cdot\frac{\Delta^5L_{x-2n}+\Delta^5L_{x-3n}}{2}-\ldots.$$

Hence, neglecting third and higher differences, we get for an approximate expression for the force of mortality $-\frac{\Delta L_x+\Delta L_{x-n}}{2nL_x}$. If n be taken $=1$, we have the formulæ given by Mr. Woolhouse in his excellent paper above referred to.

The following still more rapidly converging series may be used with advantage when the values of $L_{x+n:2}$, $L_{x+3n:2}$, &c. ($n:2$ denoting the quantity $\frac{n}{2}$, &c.) are contained in the table:—

$$n\frac{dL_x}{dx}=\Delta L_{x-n:2}-\frac{1}{24}\Delta^3L_{x-3n:2}+\frac{3}{640}\Delta^5L_{x-5n:2}-\ldots.*$$

II. If the force of mortality be denoted by F_x, we shall have $-dL_x=L_xF_xdx$. Now it will be shown further on, that whether a population be stationary or fluctuating—and by whatever law, in the latter case, the fluctuations may be governed—the ratio of the the annual average number of deaths, during a period of n years, between the ages v and $v+n$, to the mean or average population during the same period and between the same ages, is accurately represented by the formula

$$\frac{-\int_0^n\frac{dL_{v+x}}{dx}C_xdx}{\int_0^n L_{v+x}.C_x.dx},$$

where C_x is a function of x depending upon the nature of the fluctuations of the population during the period and between the ages observed, and which becomes a constant and disappears when the

* The higher differential coefficients may be similarly developed. For instance, we have $\frac{n^2}{2}\cdot\frac{d^2L_x}{dx^2}=\frac{\Delta^2L_{x-n}}{2}-\frac{1}{2.3}\,\frac{\Delta^4L_{x-2n}}{4}+\frac{1.2}{3.4.5}\cdot\frac{\Delta^6L_{x-3n}}{6}-\ldots.$ Of course L_x may here stand, generally, for a function of x.

population is stationary. But from the equation $-d\mathrm{L}_x = \mathrm{L}_x\mathrm{F}_x dx$ we have $-d\mathrm{L}_{v+x}\mathrm{C}_x = \mathrm{L}_{v+x}\mathrm{F}_{v+x}\mathrm{C}_x dx$, and therefore $-\int_0^n \frac{d\mathrm{L}_{v+x}}{dx}\mathrm{C}_x dx = \int_0^n \mathrm{L}_{v+x}\mathrm{F}_{v+x}\mathrm{C}_x dx$. Now if F_{v+x} be supposed constant between the limits of integration, and equal to its value at the mean age $v+\frac{1}{2}n$, the second member of the last equation becomes $\mathrm{F}_{v+\frac{1}{2}n}\int_0^n \mathrm{L}_{v+x}\mathrm{C}_x dx$, and we have

$$\mathrm{F}_{v+\frac{1}{2}n} = \frac{-\int_0^n \frac{d\mathrm{L}_{v+x}}{dx}\mathrm{C}_x dx}{\int_0^n \mathrm{L}_{v+x}\mathrm{C}_x dx},$$

so that the ratio of the average annual number of deaths to the mean population between given ages approximately represents the force of mortality at the mean age.

III. Our opening remarks will have made it sufficiently apparent that the force of mortality expresses the rapidity with which, at any given instant of time, a body of individuals of a given age are diminishing by death. Being a function of the age it is incessantly varying in passing from one age to the next, but it may be accurately defined as the ratio between the number living at the given age and the number of deaths which would take place in one year, supposing the force of mortality to remain constant during that period, and each vacancy arising by death to be filled up as it occurs by the substitution of a life of the same age. When speaking of the general mortality, irrespective of the particular causes, it may be designated as the "total" force of mortality, to distinguish it from the "partial" forces, by which latter term I propose to designate the ratio above defined, when we are considering the deaths arising from one or more particular diseases only. The total force of mortality at age x I have denoted by F_x, and similarly the several partial forces will be represented by F_x', F_x'', &c.

IV. Let the total force of mortality in a given body of individuals be composed of the two distinct and independent partial forces F_x' and F_x''. And as L_x denotes the number living at age x in a body subject to the total force of mortality, let L_x' and L_x'' respectively represent the number living in a body subject to each one of the partial forces only. And let it be supposed that the decrement $-\Delta\mathrm{L}_x$ (corresponding to any small increment of time Δ_x) takes place at the commencement of the interval, and the decrement $-\Delta\mathrm{L}_x$ at the termination thereof. Then the total decrement of L_x (supposing both causes of mortality to be in operation together) will be

$$-\Delta L_x = -L_x \times \frac{\Delta L'_x}{L'_x} - L_x\left(1 - \frac{\Delta L'_x}{L'_x}\right)\frac{\Delta L''_x}{L''_x}$$
$$= -L_x\left(\frac{\Delta L'_x}{L'_x} + \frac{\Delta L''_x}{L''_x}\right) + L_x \cdot \frac{\Delta L'_x}{L'_x} \cdot \frac{\Delta L''_x}{L''_x}.$$

Now, by taking Δx sufficiently small, the supposition as to the time at which the decrements take place may be made as near the truth as we please. Let Δx be diminished without limit, in which case $\Delta L'_x . \Delta L''_x$ becomes infinitely small in comparison with $\Delta L'_x$ and $\Delta L''_x$, and the term in which it is a factor may therefore be neglected without affecting the truth of the equation. Dividing the remaining terms by $L_x dx$, we have

$$-\frac{dL_x}{L_x dx} = -\frac{dL'_x}{L'_x dx} - \frac{dL''_x}{L''_x dx}$$
$$\text{or } F_x = F'_x + F''_x.$$

Hence, generally the total force of mortality is equal to the sum of the several partial forces.

V. Integrating both sides of the equation $\frac{dL_x}{L_x dx} = \frac{dL'_x}{L'_x dx} + \frac{dL''_x}{L''_x dx}$, we have $\log L_x = \log L'_x + \log L''_x + c$, whence $gL_x = L'_x L''_x$. But a table of the values of gL_x, and one of L_x, are, for all practical purposes, the same. From this it appears that if we have tables of L'_x, L''_x, L'''_x, &c., showing the numbers living at successive ages in several bodies of individuals, each of which is supposed to be subject only to one given cause of death, we may construct a table of the numbers living in a body subject to all or to any given number of such causes by merely multiplying the number living at each age, in the separate tables, into each other. Henceforth, then, L_x will denote $L'_x . L''_x \ldots$

VI. Representing by $-\Delta' L_x$ the number dying from one particular cause (viz., that corresponding to L'_x) out of L_x persons subject simultaneously to several causes of mortality, let it be required to find an expression for its value. The probability of a life aged v dying in the instant immediately following the expiration of the time x is found by multiplying the probability of the life surviving the period x into the probability of a life aged $v+x$ dying instantaneously. We have, therefore, for the probability required, $-\frac{L_{v+x}}{L_v} \cdot \frac{dL'_{v+x}}{L'_{v+x}}$, or $-\frac{1}{L_v} \cdot L''_{v+x} . dL'_{v+x}$. Integrating from $x=0$

to $x=n$, we have for the probability of the life dying within n years, from the given cause, $-\frac{1}{L_v}\int_0^n L''_{v+x}.\frac{dL'_{v+x}}{dx}dx$, or $-\frac{1}{L_v}\int_v^{v+n}L''_x\frac{dL'_x}{dx}dx$, and multiplying by L_v, we find that

$$-\Delta' L_v = -\int_v^{v+n} L''_x.\frac{dL'_x}{dx}dx.$$

Hence also
$$-\Delta'' L_v = -\int_v^{v+n} L'_x.\frac{dL''_x}{dx}dx.$$

Adding together the corresponding sides of these equations, the sums should be equal. Now $\int L''_x.dL'_x + \int L'_x\, dL''_x = L'_x L''_x = L_x$, and taking the limits $x=v$ to $x=v+n$, we get $L_{v+n}-L_v$ or ΔL_v. Therefore

$$-\Delta' L_v - \Delta'' L_v = -\Delta L_v,$$

the two members of which equation are evidently identical.

The value of $-\Delta' L_x$ may be deduced in a somewhat different form, for $-\frac{L_{v+x}}{L_v}\frac{dL'_{v+x}}{L'_{v+x}} = \frac{L_{v+x}F'_{v+x}}{L_v}dx$. Integrating and multiplying by L_v gives us

$$-\Delta' L_v = \int_v^{v+n} L_x F'_x dx.$$

VII. The probability of dying (from the given cause) in the infinitely small interval of time dx being $-\frac{dL'_x}{L'_x}$, the infinitely small decrement $(-d'L_x)$ from the deaths resulting from this particular force of mortality, in the number L_x, is $-L_x.\frac{dL'_x}{L'_x}$ or $L_x F'_x dx$. And from the equation $-d'L_x = F'_x L_x dx$ we get

$$F'_x = -\frac{d'L_x}{L_x dx},$$

and also $-\Delta' L_v = \int_v^{v+n} L_x F'_x dx$, the result already arrived at by a different process in (VI.).

VIII. From the equation $-d'L_x = L_x F'_x dx$ we get the approximate equation $F'_{v+\frac{1}{2}n} = \frac{\int_0^n \frac{d'L_{v+x}}{dx}.C_x dx}{\int_0^n L_{v+x}C_x dx}$ by the same process as that adopted in II. We see then that the theorem established in that article, viz., " That the average annual deaths in a given population (whether stationary or fluctuating) between the ages x and $x+n$, divided by the average number living between those ages, approximately represents the force of mortality at the mean age $x+\frac{1}{2}n$," is applicable as well to "partial" forces as to the "total" force of mortality.

In establishing the preceding theory of partial forces, it has been tacitly assumed that the numbers of the population are affected only by the births on the one hand and by the deaths on the other. But in all practical applications of the subject another important element, viz. migration, presents itself, so that it becomes necessary to show that the theorems hold good when that element is taken into consideration.

Let P_0 denote the number of annual births—which we will first suppose to be constant—and of these let there be P_n who will survive and remain in the country until the completion of n years of age. To simplify the case, let us assume, in the first instance, that there are no immigrants. Then if the law of mortality and migration remain constant (varying only with the age) during the period required for a whole generation to become extinct, the population will then have reached the *stationary state*. The number thenceforth annually completing the age x will be P_x, and $-\Delta P_x$ or $P_x - P_{x+1}$ will represent the number annually disappearing by death and emigration between the ages x and $x+1$.

If, again, we take the case of a population increasing in geometrical progression—by reason of the annual births exceeding the annual disappearances in a constant ratio to the existing population, the numbers annually completing each successive year of age will be in the following proportions, viz., P_x, $P_{x+1}r^{-1}$, $P_{x+2}r^{-2}$, . . . where P_x, P_{x+1}, P_{x+2} . . . are the corresponding numbers in the case of a stationary population.

And as in the preceding theorems relative to the decrements of life no supposition was made as to the nature of the law of mortality, it is evident that they must all hold good in the case of the decrements resulting from death and emigration together—which for the sake of distinction we may term the decrements of population. And further, as we have seen that by means of the theory of partial forces of mortality we may investigate the law which regulates the deaths from one or more particular diseases, notwithstanding that the population may also be affected by deaths arising from other diseases; so we may with equal facility perform such investigations notwithstanding that the population may be affected by the disappearance of individuals by emigration, and this even although we may have no record of the number annually emigrating at each age.

To complete the subject we have now to include the case in which *im*migration as well as emigration takes place. If we assume that at every age the emigrants exceed the *im*migrants, the case has just been disposed of, for the whole of the immigrants may be supposed to take the place of an equal number of emigrants; and the *effective* force of emigration at each age is represented by the excess of the latter over the former. But it is evident, nevertheless, that emigration and immigration may also be considered as distinct forces, the one being a *positive* and the other a *negative* force; and this consideration leads us naturally to the case in which the immigrants may exceed the emigrants, which we may see is met by extending our conceptions of the force of migration as one which may be either positive or negative.

This mode of treating the subject gets rid of the necessity of investigating separately the effect of migration, which Mr. Milne has done, in considerable detail, in his able and elaborate work on life contingencies; and also renders unnecessary the limitation which the eminent writer referred to has introduced in some of his theorems, viz., that the population subject to investigation shall not be affected by migration. (See Milne, articles 184 and 188.)

We are now in a position to proceed with the investigation of the subject which forms the title of the present paper.

The formula which I have already had occasion to bring under the notice of this Institute, as representing very closely the normal law of human mortality, is derived from a modification of Mr. Gompertz's simple and highly ingenious theory that the power to oppose destruction loses equal proportions in equal times. The modification which I propose to introduce consists in the limitation of the theory to a *portion only* of the partial forces of mortality; and the assumption that the remaining partial forces operate (in the aggregate) with equal intensity at all ages. This modification will, I think, lose none of its claims to an impartial consideration, from the fact that it in no way interferes with the philosophical principle upon which Mr. Gompertz has shown his theory to be based: a feature which distinguishes his formula from all others which have hitherto been proposed, and which doubtless accounts for the favourable reception it has met with from the highest scientific authorities; from those in fact best qualified, from their habits of thought, to distinguish the "general" from the "particular"—the permanent and essential law from the temporary and accidental circumstances by which that law is obscured and modified.

From one or two quarters, indeed, Mr. Gompertz's hypothesis has met with an opposition which I can account for only by supposing the existence of a vague suspicion that the hypothesis in question is nothing more than a mere abstract theory, not derived from experiment, but to support which, on the contrary, it is necessary to do violence to the facts supplied from actual observation. How much opposed such a suspicion is to the truth, the following extracts from Mr. Gompertz's writings will testify:—"This equation between the number living and the age," he says in his first paper, "is deserving of attention, because it appears corroborated during a long portion of life by experience; as I derive the same equation from various published tables of mortality, during a long period of man's life, which experience therefore proves that the hypothesis approximates to the law of mortality during the same portion of life; *and in fact the hypothesis itself was derived from an analysis of the experience here alluded to.*" And again, speaking of the modification introduced by himself, for the purpose of extending his formula to the whole duration of life, he says, after giving the result of his experiments, "And I consider the equation quoted sufficiently interesting for my endeavour to discover the cause of its existence;" and further on he adds, "And contemplating on this law of mortality, I endeavoured to inquire if there could be any physical cause for its existence."

These extracts prove conclusively that this eminent man followed no such erroneous method as that supposed; but that, on the contrary, he confined himself strictly to the *interpretation* of the facts supplied by experiment. His method is in perfect accordance with the precepts of our great English philosopher, Bacon, of whose highly figurative and graphic style we are reminded by Mr. Gompertz's writings—a feature which renders their perusal quite refreshing in this oppressively logical age.

In the supplement to the 25th Annual Report of the Registrar-General are given—(1) the mean population of England (in decennial intervals of age) during the years 1851 to 1860, (2) the average annual number of deaths from different causes during the same period, at the corresponding ages, and (3) the ratios of the latter to the former, which we have shown to be the approximate value of the several partial forces which together make up the aggregate or total force of mortality. We have here, then, the means of testing the value of the proposed modification of Mr. Gompertz's theory (so far as the necessarily uncertain nature of the data will admit) by comparing the results with those deduced from the theory as originally propounded.

The formula for F_x, according to Mr. Gompertz's theory, is Bq^x. For this I propose to substitute $A+Bq^x$, where A is the sum of certain partial forces which we assume to be, in the aggregate, of equal amount at all ages. The quantity Bq^x may also consist of the aggregate of several forces of a similar nature. So that we may put

$$F_x=(a+a'+a''+\ldots)+(b+b'+b''+\ldots)q^x,$$

where $a+a'+a''+\ldots=A$, and $b+b'+b''+\ldots=B$.

I do not profess to be able to separate the whole category of diseases into the two classes specified—viz., diseases depending for their intensity solely upon the gradual diminution of the vital power, and those which depend upon other causes, the nature of which we do not at present understand. I apprehend that medical science is not sufficiently advanced to render such a desideratum possible of attainment at present. I propose only at present to show that there are certain diseases—and those too of a well-defined and strictly homogeneous character—which follow Mr. Gompertz's law far more closely than the aggregate mortality from all diseases taken together. I shall then have given sufficient reason for the substitution of the form $Bq^x+\phi(x)$ for the force of mortality in lieu of Bq^x: the proof that the terms included in $\phi(x)$ form, *in the aggregate*, a constant quantity, I shall leave until we come to the examination of data more satisfactory than the returns of population and the public registers of deaths.

The two following tables are taken from the supplement before referred to. They give, first, the number of annual deaths (from all causes) to 1,000,000 living; and secondly, the number of annual deaths from certain specified causes to the same number living. The causes of death, as well as the ages for which they are given, have of course been selected as the most favourable exponents of the law of geometrical progression; but it will be observed that the former embrace all the principal vital organs of the body, and the latter include the whole of the period from early manhood to the confines of extreme old age.

The column headed "total force of mortality" should form a geometrical progression if Gompertz's law were applicable thereto. That it does not, however, form such a progression, is evident by inspection; the rate of increase in the earlier terms being less than

50 per cent., and gradually increasing until it exceeds 100 per cent. A similar result is found in all the known tables when the law is applied to the *total* force of mortality, the remedy for which (in constructing mortality tables by Mr. Gompertz's formula) is usually sought in a change of the constants of the formula after certain intervals. It is this gradual but constant variation of the rate of increase *in one direction*, and the fact of its being uniformly found in *all* tables, that show unmistakeably that if the law itself be true, its application stands imperatively in need of some modification.

Male Life, 1851–60.

Ages.	Total Force of Mortality.	Partial Forces of Mortality.					
		Lungs.	Heart.	Kidneys.	Stomach and Liver.	Brain.	Sum of five preceding Columns.
25–34	9,574	772	514	174	464	638	2,562
35–44	12,481	1,524	1,002	292	890	1,180	4,888
45–54	17,956	3,092	1,898	471	1,664	1,990	9,115
55–64	30,855	6,616	4,130	937	3,032	4,097	18,812
65–74	65,332	13,416	8,714	2,453	4,837	9,831	39,251

Female Life, 1851–60.

Ages.	Total Force of Mortality.	Partial Forces of Mortality.					
		Lungs.	Heart.	Kidneys.	Stomach and Liver.	Brain.	Sum of five preceding Columns.
25–34	9,925	582	603	109	570	532	2,395
35–44	12,147	1,049	1,118	151	937	872	4,127
45–54	15,198	2,062	2,064	212	1,608	1,681	7,627
55–64	27,007	5,027	4,558	317	2,967	3,818	16,687
65–74	58,656	11,016	8,916	485	4,692	8,905	34,014

The modification which I have suggested, viz., there are certain partial forces of mortality (how many I do not pretend to say) which increase in intensity with the age in a constant geometrical ratio, while there are also certain other partial forces which do not so increase, may be tested by an examination of the six columns which follow that of the *total* force above referred to. The tendency to a geometrical progression is more or less apparent in all of them; the average rate of increase being such that the force of mortality somewhat more than doubles itself in 10 years.

It should be observed that, in addition to the diseases of the particular organs specified, other diseases of a kindred nature are also included under each of the above five partial forces. Possibly if more detailed information were accessible, we might be able to trace the geometrical character during a still more extended period of life. This, at least, I find to be the case in reference to one particular disease, viz., bronchitis, which in the preceding tables is included in the class of "lung diseases." Now it so happens that the deaths from bronchitis alone, for a long series of years, are given in the 26th Annual Report of the Registrar-General, from which the materials for the following table are taken. The number living is supposed to be 100,000, instead of 1,000,000 as in the two preceding tables.

Ages.	1848 to 1854. (7 Years.)		1855 to 1857. (3 Years.)		1858 to 1863. (6 Years.)	
	Males.	Females.	Males.	Females.	Males	Females.
15–25	8	9	9	9	9	9
25–35	17	16	21	22	22	21
35–45	42	34	55	45	59	50
45–55	107	85	133	112	151	126
55–65	259	218	333	316	379	351
65–75	589	525	801	697	876	834
75–85	1,027	906	1,463	1,325	1,614	1,479

In the preceding examination of the results of the Registrar-General's returns of deaths, I have confined myself to the object of proving that Gompertz's law is traced much more distinctly in the deaths arising from certain specified diseases, than in the deaths arising from all causes together. If I have succeeded in this object (and I think it can scarcely be denied that I *have* succeeded), I have justified the introduction of an additional term in the formula representing the total force of mortality; but I have as yet brought forward nothing to show that such additional term is a constant in respect of the age, and varying only with the peculiar characteristics which distinguish different sets of observations from each other.

The several observations, however, which I now proceed to examine, if they do not enable us (like the former) to test particular terms of the function referred to, yet they will nevertheless afford a very satisfactory criterion of the complete expression. Not only, therefore, do they form by themselves (on account of their unquestionable accuracy and trustworthiness) ample evidence of the truth of the supposed law of mortality, but they also supply the deficiency, above adverted to, in the preceding investigation, as regards the requisite proof of the constancy of the term representing the aggregate of the remaining partial forces of mortality.

Commencing with the very valuable observations on the "Peerage Families" (both sexes), I find, by dividing the entire period of life into intervals of 14 years—neglecting, however, the first—the following results :—

$$\begin{array}{lll}
\log L_{14} = \cdot 99034 & & \\
 & -\cdot 05068 & \\
\log L_{28} = \cdot 93966 & & -\cdot 00716 \\
 & -\cdot 05784 & \\
\log L_{42} = \cdot 88182 & & -\cdot 02559 \\
 & -\cdot 08343 & \\
\log L_{56} = \cdot 79839 & & -\cdot 11395 \\
 & -\cdot 19738 & \\
\log L_{70} = \cdot 60101 & & -\cdot 41273 \\
 & -\cdot 61011 & \\
\log L_{84} = \bar{1}\cdot 99090 & &
\end{array}$$

The tendency to a geometrical progression in the four terms of the second order of differences is sufficiently apparent. In order, however, to show this more distinctly, I have devised the following method of correcting the series $\log L_x$ so that the four terms in question shall form a perfect geometrical progression.

If the series consist of *five* terms, and consequently the second order of differences of *three*, the latter may be converted into a pure geometrical progression by substituting for the original series another of the following form, viz.,

$$\log L_0 + p,\ \log L_n - p,\ \log L_{2n} + p,\ \log L_{3n} - p,\ \log L_{4n} + p,$$

where p is derived from the equation

$$4p = \frac{(\Delta_n^2)^2 - \Delta_0^2 \times \Delta_{2n}^2}{\Delta_0^2 + 2\Delta_n^2 + \Delta_{2n}^2}.*$$

This method, it is true, changes the value of the radix of the table, but I see no necessity for making a distinction between that and other terms of the series; for in comparing the terms of the *altered* with those of the *original* series, the object is to ascertain their bearing with respect to the original series *generally*, and not to any one term in particular. Secondly, by the method adopted, the first differences (which are the logarithms of the probabilities of living n years) are increased or diminished by an uniform quantity; whereas by omitting the correction in $\log L_0$, the first term of the first order of differences would be increased or diminished by one-half of the quantity introduced into the remaining terms. Lastly, the equation for p would be of the second order, instead of the simple one given above.

* The differences are those of the function $\log L_x$.

Again, if the series consist of *six* terms—in which case there will be *four* terms in the second order of differences—the required effect may be produced by substituting for $\log L_x$ the series

$$\log L_0+(v-w),\ \log L_n-(v-w),\ \log L_{2n}+v,\ \log L_{3n}-v,$$
$$\log L_{4n}+(v+w),\ \log L_{5n}-(v+w),$$

v and w being determined from the equations

$$2w=\frac{AC-B^2}{A+2B+C} \text{ and } 8v=\frac{A'C'-B'^2}{A'+2B'+C'},$$

where

$$A=\Delta_0^2+\Delta_n^2,\quad B=\Delta_n^2+\Delta_{2n}^2,\quad C=\Delta_{2n}^2+\Delta_{3n}^2,$$

and

$$A'=\Delta_0^3+4w,\quad B'=\Delta_n^3,\quad C'=\Delta_{2n}^3-4w.$$

Here again, by involving the corrections symmetrically, we obtain for the unknown quantities simple instead of complicated quadratic equations.

Applying these formulæ to the series at page 338, we have $2w=\cdot002651$, and $8v=\cdot007725$. The transformed series therefore becomes—

$\log L'_{14}=\cdot989980$				
	$-\cdot049960$			
$\log L'_{28}=\cdot940020$		$-\cdot007274$	$\log=\bar{3}\cdot86177$	
	$-\cdot057234$			$\cdot58737$
$\log L'_{42}=\cdot882786$		$-\cdot028128$	$\log=\bar{2}\cdot44914$	
	$-\cdot085362$			$\cdot58733$
$\log L'_{56}=\cdot797424$		$-\cdot108760$	$\log=\bar{1}\cdot03647$	
	$-\cdot194122$			$\cdot58737$
$\log L'_{70}=\cdot603302$		$-\cdot420572$	$\log=\bar{1}\cdot62384$	
	$-\cdot614694$			
$\log L'_{84}=\bar{1}\cdot988608$				

The logarithms of the third series, and their differences, show that the transformed series fulfils the required conditions.

I have now to show that this result has been attained without a greater alteration of the original series than is warranted by the probable errors of the latter. In the following table the first column contains the age, the second the natural numbers corresponding to the original series $\log L_x$, the third gives the decrement (deduced from the original data) of the year immediately following, while the fourth and fifth contain respectively the transformed series (denoted by L'_x)* and the amount by which it differs from the original series in the second column.

* Hitherto the accent has been used to distinguish the "partial" from the "total" forces of mortality, but as we have now done with this branch of the subject, no confusion will be caused by using it to denote (as it will be used henceforth) the *corrected* values of the function to which it is applied.

x.	L_x.	D_x.	L'_x.	$L_x - L'_x$.
14	9780·0	41·4	9771·9	+ 8·1
28	8702·8	69·8	8710·0	− 7·2
42	7617·6	78·3	7634·6	−17·0
56	6286·2	150·3	6272·3	+13·9
70	3990·3	220·3	4011·5	−21·2
84	979·3	152·2	974·1	+ 5·2
		712·3		72·6

Comparing columns 3 and 5 together, term by term, we find that in one instance only (viz. at age 42) does the alteration made in the numbers living exceed one-fifth [·2171] of the corresponding yearly decrement; while from the sums of the same columns it appears that the *average* alteration is little more than one-tenth of the *average* decrement. We may, therefore, say the limit of the variation of the two series (cols. 2 and 4) is about one-fifth of a year.

If we turn now to the original table of the "Peerage Families," we shall find that the series showing the annual rate of mortality at successive ages exhibits considerable irregularity—in one instance (age 57) being one-half only of the mortality for the ages immediately preceding and following it. Now, I think few persons who have had much experience in the construction of mortality tables will venture to maintain that a table which contains such irregularities as these can be relied upon within anything like one-fifth of a year. I say *few* persons, because I am not sure but that some have an idea that the standard of accuracy is always to be found in the numbers determined from actual experience, however limited in extent, and however great may be the difference in the conditions under which the observations are made and those affecting the lives to which they are to be applied as the basis of computation.

Let us now examine the important observations conducted by Mr. A. G. Finlaison upon the lives generally known as the "Government Annuitants," male and female. Dividing the whole period of life comprised in these observations—viz., from one year upwards—as before, into intervals of 14 years, and neglecting the first, we have

$$\begin{array}{lll}
\log L_{15} = 3{\cdot}98821 & & \\
 & -{\cdot}06068 & \\
\log L_{29} = 3{\cdot}92753 & & -{\cdot}00633 \\
 & -{\cdot}06701 & \\
\log L_{43} = 3{\cdot}86052 & & -{\cdot}02670 \\
 & -{\cdot}09371 & \\
\log L_{57} = 3{\cdot}76681 & & -{\cdot}10771 \\
 & -{\cdot}20142 & \\
\log L_{71} = 3{\cdot}56539 & & -{\cdot}44978 \\
 & -{\cdot}65120 & \\
\log L_{85} = 2{\cdot}91419 & &
\end{array}$$

Proceeding as before, we get the following transformed series ($\log L'_x$), which fulfils the given condition of a perfect geometrical progression in the four terms comprising the second order of differences:—

$$\begin{array}{lllll}
\log L'_{15}=3{\cdot}988254 & & & & \\
 & -{\cdot}060768 & & & \\
\log L'_{29}=3{\cdot}927486 & & -{\cdot}006357 & \log \bar{3}{\cdot}80325 & \\
 & -{\cdot}067125 & & & +{\cdot}61616 \\
\log L'_{43}=3{\cdot}860361 & & -{\cdot}026267 & \log \bar{2}{\cdot}41941 & \\
 & -{\cdot}093392 & & & +{\cdot}61621 \\
\log L'_{57}=3{\cdot}766969 & & -{\cdot}108548 & \log \bar{1}{\cdot}03562 & \\
 & -{\cdot}201940 & & & +{\cdot}61618 \\
\log L'_{71}=3{\cdot}565029 & & -{\cdot}448538 & \log \bar{1}{\cdot}65180 & \\
 & -{\cdot}650478 & & & \\
\log L'_{85}=2{\cdot}914551 & & & &
\end{array}$$

In this case the alterations introduced in the process of transformation are, in each of the six terms, perfectly insignificant, as the following table of the values of L_x, L'_x, and $L_x-L'_x$, will show:—

x.	L_x.	L'_x.	$L_x-L'_x$.
15	9732·2	9733·2	−1·0
29	8463·1	8462·3	+ ·8
43	7253·0	7250·4	+2·6
57	5845·3	5847·5	−2·2
71	3676·1	3673·1	+3·0
85	820·7	821·4	− ·7

It was originally my intention to extend the examination of the two preceding observations to the first 14 items of each series, which have been omitted in the foregoing comparison, with the view of deducing a formula which should be applicable to the whole period of life. But as such extension has no direct bearing upon the object which I have more immediately in view—viz., to establish a method for shortening the labour of forming tables of annuities on several lives, and for which object I find I shall have but little space to spare—I must postpone to a future opportunity the examination of so interesting and important a subject as the mortality of infancy and childhood.

In the records which we have next to examine, consisting of Mr. Hodgson's important observations on the clergy, we have necessarily no data for ages under 24. This circumstance has rendered it necessary to reduce the interval to 13 years.

$$\begin{array}{lll}
\log L_{24}=4{\cdot}00213 & & \\
 & -\ {\cdot}03057 & \\
\log L_{37}=3{\cdot}97156 & & -{\cdot}02059 \\
 & -\ {\cdot}05116 & \\
\log L_{50}=3{\cdot}92040 & & -{\cdot}07252 \\
 & -\ {\cdot}12368 & \\
\log L_{63}=3{\cdot}79672 & & -{\cdot}22938 \\
 & -\ {\cdot}35306 & \\
\log L_{76}=3{\cdot}44366 & & -{\cdot}74045 \\
 & -1{\cdot}09351 & \\
\log L_{89}=2{\cdot}35015 & &
\end{array}$$

The following are the adjusted series $\log L'_x$, and the comparative table of L_x and L'_x:—

$$\begin{array}{lllll}
\log L'_{24}=4\cdot001668 & & & & \\
 & -\cdot029646 & & & \\
\log L'_{37}=3\cdot972022 & & -\cdot022186 & \log=\bar{2}\cdot34608 & \\
 & -\cdot051832 & & & +\cdot50779 \\
\log L'_{50}=3\cdot920190 & & -\cdot071428 & \log=\bar{2}\cdot85387 & \\
 & -\cdot123260 & & & +\cdot50780 \\
\log L'_{63}=3\cdot796930 & & -\cdot229968 & \log=\bar{1}\cdot36167 & \\
 & -\cdot353228 & & & +\cdot50778 \\
\log L'_{76}=3\cdot443702 & & -\cdot740366 & \log=\bar{1}\cdot86945 & \\
 & -1\cdot093594 & & & \\
\log L'_{89}=2\cdot350108 & & & &
\end{array}$$

x.	L_x.	L'_x.	$L_x - L'_x$.
24	10049·2	10038·5	+ 10·7
37	9366·1	9376·1	− 10·0
50	8325·3	8321·3	+ 4·0
63	6262·1	6265·1	− 3·0
76	2777·5	2777·8	− ·3
89	223·9	223 9	·0

The numbers exposed to risk during the first five or six years of this table are so inconsiderable, that I should have preferred to exclude them, if it were not that the initial age is already somewhat advanced. In the next case, consisting of the observations known as the "Experience of the Seventeen Offices," Tables D(4) and E, I have neglected the ages under 20 and over 80, on account of the comparative insignificance of the numbers at risk at the excluded ages.

$$\begin{array}{lll}
\log L_{20}=3\cdot97023 & & \\
 & -\cdot04191 & \\
\log L_{32}=3\cdot92832 & & -\cdot00948 \\
 & -\cdot05139 & \\
\log L_{44}=3\cdot87693 & & -\cdot03332 \\
 & -\cdot08471 & \\
\log L_{56}=3\cdot79222 & & -\cdot09970 \\
 & -\cdot18441 & \\
\log L_{68}=3\cdot60781 & & -\cdot30823 \\
 & -\cdot49264 & \\
\log L_{80}=3\cdot11517 & &
\end{array}$$

By means of the formula previously used, we obtain the following values of $\log L'_x$:—

$$\begin{array}{lllll}
\log L'_{20}=3\cdot969914 & & & & \\
 & -\cdot041278 & & & \\
\log L'_{32}=3\cdot928636 & & -\cdot010580 & \log=\bar{2}\cdot02449 & \\
 & -\cdot051858 & & & +\cdot48803 \\
\log L'_{44}=3\cdot876778 & & -\cdot032548 & \log=\bar{2}\cdot51252 & \\
 & -\cdot084406 & & & +\cdot48810 \\
\log L'_{56}=3\cdot792372 & & -\cdot100144 & \log=\bar{1}\cdot00062 & \\
 & -\cdot184550 & & & +\cdot48809 \\
\log L'_{68}=3\cdot607822 & & -\cdot308114 & \log=\bar{1}\cdot48871 & \\
 & -\cdot492664 & & & \\
\log L'_{80}=3\cdot115158 & & & &
\end{array}$$

Taking out the natural numbers corresponding to the above, and comparing them with those of the original series, we have—

x.	L_x.	L'_x.	$L_x - L'_x$.
20	9337·5	9330·7	+6·8
32	8478·5	8484·7	−6·2
44	7532·3	7529·7	+2·6
56	6197·5	6199·7	−2·2
68	4053·3	4053·4	− ·1
80	1303·7	1303·6	+ ·1

The differences (L_x—L'_x) are in this case also evidently unimportant; they do not in fact, in any instance, exceed one-tenth of the yearly decrement.

I have only one more instance to give—viz., the observations on the males of the Friendly Societies, by Mr. A. G. Finlaison. For the same reason as in the preceding case, the examination is restricted to the ages commencing with 20 and ending with 80. Here we have

$$\begin{array}{lll}
\log L_{20}=3\cdot79612 & & \\
 & -\cdot03947 & \\
\log L_{32}=3\cdot75665 & & -\cdot00922 \\
 & -\cdot04869 & \\
\log L_{44}=3\cdot70796 & & -\cdot02993 \\
 & -\cdot07862 & \\
\log L_{56}=3\cdot62934 & & -\cdot08296 \\
 & -\cdot16158 & \\
\log L_{68}=3\cdot46776 & & -\cdot24299 \\
 & -\cdot40457 & \\
\log L_{80}=3\cdot06319 & &
\end{array}$$

which is transformed into

$$\begin{array}{lllll}
\log L'_{20}=3\cdot795915 & & & & \\
 & -\cdot039053 & & & \\
\log L'_{32}=3\cdot756862 & & -\cdot010068 & \log=\bar{2}\cdot00294 & \\
 & -\cdot049121 & & & +\cdot46028 \\
\log L'_{44}=3\cdot707741 & & -\cdot029055 & \log=\bar{2}\cdot46322 & \\
 & -\cdot078176 & & & +\cdot46034 \\
\log L'_{56}=3\cdot629565 & & -\cdot083861 & \log=\bar{2}\cdot92356 & \\
 & -\cdot162037 & & & +\cdot46036 \\
\log L'_{68}=3\cdot467528 & & -\cdot242062 & \log=\bar{1}\cdot38392 & \\
 & -\cdot404099 & & & \\
\log L'_{80}=3\cdot063429 & & & &
\end{array}$$

The observations which we are now examining are chiefly valuable on account of the large number of the lives observed, and I have availed myself of this circumstance to compare the results of the original observations year by year with those of the formula. The first table contains the numbers living and yearly decrements (L_x and D_x), the second the rate of mortality (m_x), and the third the "expectation" or mean duration of life (E_x).

x.	L_x.	L'_x.	D'_x.	$L_x - L'_x$.	x.	L_x.	L'_x.	D'_x.	$L_x - L'_x$.
18	6333·0	6338·6	44·1	− 5·6	62	3659·0	3674·2	112·3	− 15·2
19	6289·7	6294·5	44·0	− 4·8	63	3555·3	3561·9	116·8	− 6·6
20	6253·5	6250·5	44·0	+ 3·0	64	3439·2	3445·1	121·1	− 5·9
21	6201·8	6206·5	44·1	− 4·7	65	3326·7	3324·0	125·5	+ 2·7
22	6156·0	6162·4	44·2	− 6·4	66	3212·1	3198·5	129·9	+ 13·6
23	6108·4	6118·2	44·2	− 9·8	67	3069·2	3068·6	134·2	+ ·6
24	6063·0	6074·0	44·3	− 11·0	68	2936·0	2934·4	138·1	+ 1·6
25	6020·5	6029·7	44·5	− 9·2	69	2801·6	2796·3	141·9	+ 5·3
26	5978·1	5985·2	44·7	− 7·1	70	2642·5	2654·4	145·3	− 11·9
27	5937·9	5940·5	44·9	− 2·6	71	2497·0	2509·1	148·2	− 12·1
28	5893·9	5895·6	45·2	− 1·7	72	2341·1	2360·9	150·6	− 19·8
29	5849·7	5850·4	45·4	− ·7	73	2192·9	2210·3	152·3	− 17·4
30	5804·3	5805·0	45·9	− ·7	74	2055·9	2058·0	153·3	− 2·1
31	5754·9	5759·1	46·2	− 4·2	75	1905·2	1904·7	153·4	+ ·5
32	5710·2	5712·9	46·7	− 2·7	76	1756·3	1751·3	152·5	+ 5·0
33	5670·6	5666·2	47·2	+ 4·4	77	1593·1	1598·8	150·6	− 5·7
34	5623·2	5619·0	47·8	+ 4·2	78	1459·6	1448·2	147·6	+ 11·4
35	5576·9	5571·2	48·4	+ 5·7	79	1285·6	1300·6	143·3	− 15·0
36	5529·6	5522·8	49·2	+ 6·8	80	1156·6	1157·3	138·0	− ·7
37	5486·4	5473·6	50·0	+ 12·8	81	1000·0	1019·3	131·5	− 19·3
38	5438·0	5423·6	50·8	+ 14·4	82	881·8	887·8	123·9	− 6·0
39	5385·2	5372·8	51·8	+ 12·4	83	..	763·9	115·3	
40	5331·4	5321·0	52·9	+ 10·4	84	..	648·6	105·8	
41	5270·4	5268·1	54·1	+ 2·3	85	..	542·8	95·6	
42	5217·2	5214·0	55·3	+ 3·2	86	.	447·2	85·2	
43	5162·9	5158·7	56·8	+ 4·2	87	.	362·0	74·4	
44	5104·6	5101·9	58·2	+ 2·7	88	..	287·6	63·8	
45	5044·3	5043·7	59·9	+ ·6	89	..	223·8	53·5	
46	4983·6	4983·8	61·7	− ·2	90	..	170·3	43·9	
47	4922·9	4922·1	63·5	+ ·8	91	..	126·4	35·1	
48	4857·0	4858·6	65·7	− 1·6	92	..	91·3	27·2	
49	4792·1	4792·9	67·9	− ·8	93	..	64·1	20·6	
50	4721·7	4725·0	70·2	− 3·3	94	..	43·5	14·9	
51	4651·6	4654·8	72·9	− 3·2	95	.	28·6	10·6	
52	4581·3	4581·9	75·5	− ·6	96	..	18·0	7·1	
53	4501·1	4506·4	78·5	− 5·3	97	..	10·9	4·6	
54	4427·2	4427·9	81·6	− ·7	98	..	6·3	2·8	
55	4345·2	4346·3	84·8	− 1·1	99	..	3·5	1·7	
56	4259·3	4261·5	88·3	− 2·2	100	..	1·8	·9	
57	4158·1	4173·2	91·9	− 15·1	101	..	·9	·5	
58	4058·3	4081·3	95·7	− 23·0	102	..	·4	·2	
59	3947·4	3985·6	99·7	− 38·2	103	..	·2	·1	
60	3853·1	3885·9	103·7	− 32·8	104	..	·1	·1	
61	3746·8	3782·2	108·0	− 35·4	105	..	·0	·0	

x.	m_x.	m'_x.	x.	m_x.	m'_x.	x.	m_x.	m'_x.
18	·68	·692	41	1·01	1·026	64	3·27	3·516
19	·58	·698	42	1·04	1·061	65	3·45	3·777
20	·83	·704	43	1·13	1·100	66	4·45	4·061
21	·74	·711	44	1·18	1·142	67	4·34	4·371
22	·77	·716	45	1·20	1·187	68	4·58	4·708
23	·74	·723	46	1·22	1·237	69	5·68	5·074
24	·70	·730	47	1·34	1·292	70	5·51	5·473
25	·70	·739	48	1·34	1·351	71	6·24	5·908
26	·67	·746	49	1·47	1·416	72	6·33	6·379
27	·74	·755	50	1·48	1·487	73	6·24	6·892
28	·75	·766	51	1·51	1·565	74	7·33	7·448
29	·77	·778	52	1·75	1·649	75	7·82	8·053
30	·85	·789	53	1·64	1·741	76	9·29	8·708
31	·78	·803	54	1·85	1·842	77	8·38	9·419
32	·69	·817	55	1·98	1·952	78	11·92	10·189
33	·84	·833	56	2·38	2·072	79	10·03	11·023
34	·82	·851	57	2·40	2·203	80	13·54	11·925
35	·85	·870	58	2·73	2·345	81	11·82	12·900
36	·78	·890	59	2·39	2·501	82	20·36	13·953
37	·88	·913	60	2·76	2·670	83	15·93	15·088
38	·97	·938	61	2·34	2·855	84	12·50	16·311
39	1·00	·965	62	2·83	3·057	85	18·75	17·627
40	1·14	·994	63	3·27	3·277	86	14·63	19·040

x.	E_x.	E'_x.	$E_x - E'_x$.	x.	E_x.	E'_x.	$E_x - E'_x$.
18	44·79	44·78	+ ·01	50	22·00	22·04	− ·04
19	44·09	44·09	·00	51	21·32	21·35	− ·03
20	43·34	43·39	+ ·05	52	20·63	20·68	− ·05
21	42·69	42·69	·00	53	19·98	20·01	− ·03
22	42·00	41·99	+ ·01	54	19·30	19·34	− ·04
23	41·32	41·29	+ ·03	55	18·64	18·69	− ·05
24	40·62	40·58	+ ·04	56	18·00	18·04	− ·04
25	39·90	39·87	+ ·03	57	17·41	17·40	+ ·01
26	39·18	39·16	+ ·02	58	16·81	16·77	+ ·04
27	38·44	38·45	− ·01	59	16·26	16·15	+ ·11
28	37·72	37·73	− ·01	60	15·63	15·54	+ ·09
29	36·99	37·02	− ·03	61	15·05	14·94	+ ·11
30	36·28	36·30	− ·02	62	14·38	14·35	+ ·03
31	35·58	35·58	·00	63	13·77	13·77	·00
32	34·85	34·86	− ·01	64	13·21	13·20	+ ·01
33	34·09	34·14	− ·05	65	12·62	12·64	− ·02
34	33·36	33·42	− ·06	66	12·03	12·10	− ·07
35	32·63	32·69	− ·06	67	11·55	11·57	− ·02
36	31·90	31·97	− ·07	68	11·03	11·05	− ·02
37	31·15	31·25	− ·10	69	10·51	10·55	− ·04
38	30·41	30·53	− ·12	70	10·08	10·06	+ ·02
39	29·70	29·81	− ·11	71	9·61	9·59	+ ·02
40	28·99	29·09	− ·10	72	9·18	9·12	+ ·06
41	28·32	28·37	− ·05	73	8·73	8·68	+ ·05
42	27·60	27·66	− ·06	74	8·25	8·25	·00
43	26·87	26·94	− ·07	75	7·82	7·83	− ·01
44	26·17	26·23	− ·06	76	7·40	7·43	− ·03
45	25·47	25·52	− ·05	77	7·06	7·04	+ ·02
46	24·77	24·82	− ·05	78	6·61	6·67	− ·06
47	24·06	24·11	− ·05	79	6·37	6·31	+ ·06
48	23·37	23·42	− ·05	80	5·97	5·97	·00
49	22·68	22·72	− ·04				

I commend the three tables last given to the attention of those who entertain the notion that by adhering to the actual figures, or the "raw material" of the observations, we necessarily attain the highest possible degree of accuracy, and who look upon the process of adjustment as one of the fine arts, which imparts a pleasing regularity to our tables at the expense of truth. In no tables that I am acquainted with are the original data so little altered by the process of adjustment as in these, a circumstance attributable chiefly (no doubt) to the great number at risk at each age, and the exclusion of disturbing influences by the peculiar character of the observations. Let it not, then, be too hastily assumed (as I fear it sometimes is) that the more considerable alterations which we are compelled to make, when dealing with observations made under less favourable circumstances, are necessarily departures from truth. It may be difficult to determine the best mode of making these alterations, but we cannot get rid of the difficulty by evading it; and I have no hesitation in saying that the very worst course that could possibly be adopted is to pin our faith upon the crude results of observation, just as accident may have chanced to present them to us, and hoodwink ourselves into the belief that in so doing we are following the path indicated by experience. Let us have the *facts* by all means, but unless we also possess the power to interpret their meaning—to evolve the hidden laws of which they are the rude exponents—we shall, I am afraid, turn them but to a very poor account.

It may not be without interest to compare the decrements of life obtained by the theory of geometrical partial forces with those of De Moivre's celebrated hypothesis. In the preceding table we find the series is nearly stationary for a few years (or rather it begins with a slight decrease), and afterwards gradually increases with an augmenting velocity for a considerable period. The rapidity of its increase then slackens, and ultimately changes to a decrease at the age of 75, from which age the series rapidly diminishes until it becomes insignificant at about the age of 105.

The decrement at the climacteric age (75) is 153·4, while the smallest decrement in the preceding portion of the table is 44—(see ages 19 and 20). In this case the climacteric age is exactly the same as in Mr. Brown's adjustment of the "Clergy" observations; but in tables showing a heavier mortality in the preceding years it occurs somewhat earlier. Nevertheless, the general characteristics above enumerated will be found in all the tables of any authority.

Although subsequent observations have shown that De Moivre's attempt in the discovery of the law of mortality was made in the wrong direction, yet the idea was a happy and ingenious one, and the hypothesis by no means deserves the contumelious terms in which the late Mr. Morgan permitted himself to speak of it. De Moivre, we may be quite sure, was as fully alive as Mr. Morgan could be to the shortcomings of his hypothesis; but he was also equally alive (which the other was not) to the defects of the tables formed with the means of observation then existing. We may also safely assume that such a man as De Moivre, had he been living, would not have sanctioned the use of the Northampton table for granting annuities after the experience of the Equitable Society had proved the unfitness of that table for such a purpose; and until such experience had been obtained, his own "wretched hypothesis" would certainly have been as trustworthy a guide as the table which Mr. Morgan relied upon so implicitly—because (as he imagined) it was based upon *facts* and not upon *theory*.

I have now concluded my examination of the facts which have appeared to me sufficient fully to entitle Mr. Gompertz's hypothesis to the favourable reception it has met with from the scientific authorities of the day; and I will conclude this paper with a few remarks upon the formula which I have deduced from it to represent the numbers living at every age from the period immediately following childhood to the utmost extremity of life.

1. Although no two sets of observations will yield precisely the same value of q, yet the coincidence is, generally, sufficiently close to lead to the inference that this important constant differs from the others in the formula in being independent of the conditions which determine the mortality in different classes of individuals. In the mortality tables which I have constructed by the use of this formula, I have found that an average value of q may be used in all without materially affecting the result. This being the case, there remain but three constants to be determined in the formula for L_x, for which, of course, three terms of the function are sufficient. In other words, if we determine three values of L_x—say L_{15}, L_{50}, and L_{85}, or L_{15}, L_{55}, and L_{95}—from the original data, we have the means of constructing the entire series, which shall exhibit all the essential characteristics of the mortality table, as above described. Let any one endeavour to do this (with the same terms) by the ordinary rules of interpolation, and it will be speedily shown what a power it is that this discovery of Mr. Gompertz's has conferred upon us. For instance, if between the values of L_{15}, L_{50}, and L_{85}, of the Carlisle table, we interpolate by means of the method of

finite differences, we shall have a continually increasing decrement, which shortly after the age of 85 exceeds the number of survivors, so that the latter for all higher ages is negative. It is true that this absurdity may be avoided by *logarithmic* interpolation; but in that case we shall find that at the commencement of the series the differences are *positive*, so that the successive values of L_x (in that portion of the table) form an increasing series.

2. In a paper which I had the honour of reading during the last session of the Institute, I endeavoured to show that the principal defect of our tables consists not, as some imagine, in faulty methods of adjustment, or rather in the fault of over-adjustment, but in the difficulty, if not the impossibility, of securing an adequate correspondence between the bases of the table and the conditions which are known to affect the individual lives to which the table is to be applied. And it is evident that the path to improvement in this respect must lie in the minute subdivision of our observations, a necessity which will often compel us to rely upon data much more limited than it is desirable they should be. Now the use of a suitable formula with only three constants will enable us materially to lessen this disadvantage by the long series of years which is brought to bear upon the determination of the three terms required for the computation of the constants, a circumstance which is equivalent to a corresponding increase in the number of the lives observed.

3. We have seen that the expression for the number living at any given age in a normally-constituted increasing population—in which the yearly births as well as the yearly deaths (and consequently also the excess of the former over the latter) are proportional to the existing population—is of the same form as that representing the numbers living at the given age in a stationary population, and also the numbers living in a table of mortality. But the function $L_x v^x$ is also of the same form as the latter, for L_x contains a factor s^x which combines with that introduced by the interest of money. Hence it follows that to determine the number living between given ages in a population normally constituted—whether increasing or stationary—as well as the expectation of life and the value of annuities, the summation of a function of one form only (viz., $dg^{q^x}s^x$) is required.

4. The following transformation of the integral in question, $\int dg^{q^x}s^x dx$, is worthy of attention. $F_x = a + bq^x = -\frac{1}{L_x}\cdot\frac{dL_x}{dx}$, $\therefore\ -\log L_x = c + ax + \frac{b}{\log q}\cdot q^x$, $\therefore\ L_x = \varepsilon^{-c-ax-\frac{b}{\log q}\cdot q^x}$. Put

$v = \frac{b}{\log q} q^x$ and $n = -\frac{a}{\log q}$, whence $\varepsilon^{-ax} = \varepsilon^{\log q . nx} = q^{nx}$. $\therefore L_x = \varepsilon^{-c} . q^{nx} . \varepsilon^{-v} = m\varepsilon^{-v} . q^{nx}$. Now the series L_x may be multiplied by any arbitrary quantity whatever. Multiply the last expression by $\left(\frac{b}{\log q}\right)^n . \frac{\log q}{m}$, by which the equation to the number living is changed to $L_x = \varepsilon^{-v}\left(\frac{b}{\log q} \cdot q^x\right)^n . \log q = \varepsilon^{-v} . v^n . \log q$. But $v \log q = \frac{dv}{dx}$. Hence the integral in question becomes

$$\int L_x dx = \int \varepsilon^{-v} . v^{n-1} dv,$$

the limits of the right-hand member being the values of v, corresponding to the limits of the left-hand member for values of x. The form thus obtained will be recognised as that of Euler's celebrated second integral—which, taken between the limits of 0 and ∞, is one of the most important of the definite integrals which have been tabulated; and we have seen that it is the form to which all summations connected with questions of population, the expectation of life, as well as the values of annuities, are reduced.

Appendix.

I.

Let the number of annual births (L_0) in a stationary population be equally distributed over the year, then will the number annually completing the age v (viz., L_v) be similarly distributed. Hence the number of completions taking place in any given fraction of a year Δx will be $L_v . \Delta x$. And during any given portion of time Δx there will be

$L_v \Delta x$	who complete the age		v,
$L_{v+\Delta x} \Delta x$	,,	,,	$v + \Delta x$,
$L_{v+2\Delta x} \Delta x$	,,	,,	$v + 2\Delta x$,
. . .	. . .	. . .	. .
$L_{v+\overline{p-1}.\Delta x} . \Delta x$	,,	,,	$v + \overline{p-1} . \Delta x$.

Taking the sum we have for the total number completing every Δxth of a year's age—between v and $v + p\Delta x$, or v and $v+n$—during every portion of time Δx—

$$(L_v + L_{v+\Delta x} + L_{v+2\Delta x} + \ldots + L_{v+\overline{p-1}.\Delta x})\Delta x.$$

If Δx be decreased without limit, this becomes $\int_0^n L_{v+x} dx$, which consequently denotes the total number constantly living between ages v and $v+n$.

Again, in any year $L_v - L_{v+1}$ will be the number dying between the ages v and $v+1$, $L_{v+1} - L_{v+2}$ between the ages $v+1$ and $v+2$, and so on. Hence the total number dying in a year between the ages v and $v+n$ is

$L_v - L_{v+n} = -\Delta L_v$. And the ratio of the deaths during one year between the given ages to the number constantly living between the same ages is

$$\frac{-\Delta L_v}{\int_0^n L_{v+x}dx}.$$

If the population, instead of being stationary, increase in a geometrical progression, and if in any given portion (Δx) of a year $L_v \Delta x$ complete the age v, then during the same period of time there will be

$$\begin{array}{lll} L_v\Delta x & \text{who complete the age} & v, \\ L_{v+\Delta x}\Delta x . r^{-\Delta x} & \text{,,} \qquad \text{,,} & v+\Delta x, \\ \cdot \quad \cdot \quad \cdot & \cdot \quad \cdot \quad \cdot & \cdot \\ L_{v+\overline{p-1}.\Delta x}\Delta x . r^{-(p-1)\Delta x} & \qquad\quad \text{,,} & v+\overline{p-1}.\Delta x; \end{array}$$

and summing, we have for the total number completing every Δxth of a year of age, between v and $v+n$, during the given time Δx,

$$(L_v + L_{v+\Delta x}r^{-\Delta x} + L_{v+2\Delta x}r^{\ 2\Delta x} + \ldots . + L_{v+\overline{p-1}.\Delta x}r^{-(p-1)\Delta x})\Delta x.$$

Hence decreasing Δx without limit, we have, for the number living between the ages v and $v+n$ at any given instant of time, $\int_0^n L_{v+x}r^{-x}dx$.

Suppose that the number completing each interval Δx of age do so altogether in the middle of the given period of time Δx. Then taking a succession of such intervals of time p in number, we shall have for the number living, in the middle of the

$$\begin{array}{lll} \text{1st interval,} & & (L_v + L_{v+\Delta x}r^{-\Delta x} + L_{v+2\Delta x}r^{-2\Delta x} + \ldots . + L_{v+\overline{p-1}\Delta x}r^{-(p-1).\Delta x})\Delta x; \\ \text{2nd} & \text{,,} & (L_v r^{\Delta x} + L_{v+\Delta x} \ + L_{v+2\Delta x}r^{-\Delta x} + \ldots . + L_{v+\overline{p-1}\Delta x}r^{-(p-2)\Delta x})\Delta x; \\ \text{3rd} & \text{,,} & (L_v r^{2\Delta x} + L_{v+\Delta x}r^{\Delta x} + L_{v+2\Delta x} \ + \ldots . + L_{v+\overline{p-1}\Delta x}r^{-(p-3)\Delta x})\Delta x; \\ \cdot & \cdot & \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ p\text{th} & \text{,,} & (L_v r^{(p-1)\Delta x} + L_{v+\Delta x}r^{(p-2)\Delta x} + L_{v+2\Delta x}r^{(p-3)\Delta x} + \ldots + L_{v+\overline{p-1}\Delta x})\Delta x. \end{array}$$

Summing and dividing by p, we get

$$\frac{1 + r^{\Delta x} + r^{2\Delta x} + \ldots . + r^{(p-1).\Delta x}}{p} \times$$
$$(L_v + L_{v+\Delta x}r^{-\Delta x} + L_{v+2\Delta x}r^{-2\Delta x} + \ldots + L_{v+\overline{p-1}.\Delta x}r^{-(p-1)\Delta x})\Delta x;$$

and decreasing Δx without limit, this becomes $\left(n=p.\Delta x \text{ or } \frac{1}{p} = \frac{\Delta x}{n}\right)$

$$\frac{1}{n}\int_0^n r^x dx \times \int_0^n L_{v+x}r^{-x}dx.$$

But $\int_0^n r^x dx = \frac{r^n - 1}{\log r}$. Hence we have, for the mean or average number of the population living between the ages v and $v+n$, during a period of n years, $\frac{r^n - 1}{\log r^n}\int_0^n L_{v+x}r^{-x}dx$.

Again, the number of deaths during the same period will be, in the

1st interval, $-(\Delta L_v + \Delta L_{v+\Delta x} r^{-\Delta x} + \ldots\ldots\ldots + \Delta L_{v+\overline{p-1}\cdot\Delta x} r^{-\overline{p-1}\cdot\Delta x})\Delta x$;

2nd ,, $-(\Delta L_v r^{\Delta x} + \Delta L_{v+2\Delta x} + \ldots\ldots\ldots + \Delta L_{v+\overline{p-1}\cdot\Delta x} r^{-\overline{p-2}\cdot\Delta x})\Delta x$;

3rd ,, $-(\Delta L_v r^{2\Delta x} + \Delta L_{v+2\Delta x} r^{\Delta x} + \ldots\ldots + \Delta L_{v+\overline{p-1}\cdot\Delta x} r^{-\overline{p-3}\cdot\Delta x})\Delta x$;

. . . .

pth ,, $-(\Delta L_v r^{\overline{p-1}\cdot\Delta x} + \Delta L_{v+2\Delta x} r^{(p-2)\Delta x} + \ldots + \Delta L_{v+\overline{p-1}\Delta x})\Delta x$.

Summing and dividing by $p\Delta x$ or n, we have

$$-\frac{1 + r^{\Delta x} + r^{2\Delta x} + \ldots + r^{(p-1)\Delta x}}{n} \times$$

$$(\Delta L_v + \Delta L_{v+\Delta x} r^{-\Delta x} + \ldots + \Delta L_{v+\overline{p-1}\cdot\Delta x} r^{-\overline{p-1}\Delta x})\Delta x;$$

and reducing Δx without limit, this becomes

$$-\frac{r^n - 1}{\log r^n} \int_0^n \frac{d L_{v+x}}{dx} r^{-x} dx.$$

Dividing by the mean population, we have

$$\frac{-\int_0^n d^0}{\int_0^n L_{v+x} r^{-x} dx}.$$

which is the ratio of the average annual number of deaths during n years, between the ages v and $v+n$, to the mean population.

Instead of supposing the number living at age $v+x$ to be $L_{v+x} r^{-x}$, let us suppose it to be $L_{v+x} B_x$, where B_x may be any quantity whatever. We shall then have for the number living in the

1st interval, $(L_v + L_{v+\Delta x}.B_{\Delta x} + \ldots\ldots\ldots\ldots + L_{v+\overline{p-1}\cdot\Delta x}.B_{\overline{p-1}\cdot\Delta x})\Delta x$;

2nd ,, $(L_v B_{-\Delta x} + L_{v+\Delta x} + \ldots\ldots\ldots\ldots + L_{v+\overline{p-1}\Delta x}.B_{\overline{p-2}\cdot\Delta x})\Delta x$;

. . . . + ...

pth ,, $(L_v B_{-\overline{p-1}\Delta x} + L_{v+\Delta x}.B_{-(p-2)\Delta x} + \ldots + L_{v+\overline{p-1}\cdot\Delta x})\Delta x$.

Hence, summing and dividing by p, and reducing Δx without limit, we have

$$\frac{1}{n}\int_0^n \left(L_{v+x} \int_{x-n}^x B_x dx \right) dx;$$

or, putting $\int_{x-n}^x B_x dx = C_x$, this becomes

$$\frac{1}{n}\int_0^n . L_{v+x} C_x dx,$$

which is the mean population, during n years, between the ages v and $v+n$.

If $B_x = r^{-x}$, $\int_{x-n}^x B_x dx$ (or C_x) $= \frac{r^n - 1}{\log r} \cdot r^{-x}$, and the expression for the mean population becomes

$$\frac{r^n - 1}{\log r^n} \int_0^n L_{v+x} r^{-x} dx,$$

which is the formula already arrived at.

The number of deaths, during the n years, in the preceding case will be, in the

1st interval, $-(\Delta L_v+\Delta L_{v+\Delta x}B_{\Delta x}+\dots\dots\dots+\Delta L_{v+\overline{p-1}.\Delta x}B_{\overline{p-1}.\Delta x})\Delta x$;
2nd ,, $-(\Delta L_v B_{-\Delta x}+\Delta L_{v+x}+\dots\dots\dots+\Delta L_{v+\overline{p-1}.\Delta x}B_{\overline{p-2}.\Delta x})\Delta x$;
3rd ,, $-(\Delta L_v B_{-2\Delta x}+\Delta L_{v+\Delta x}B_{-\Delta x}+\dots\dots+\Delta L_{v+\overline{p-1}.\Delta x}B_{\overline{p-3}.\Delta x})\Delta x$;
. . . .
pth ,, $-(\Delta L_v B_{-\overline{p-1}.\Delta x}+\Delta L_{v+\Delta x}B_{-\overline{p-2}.\Delta x}+\dots+\Delta L_{v+\overline{p-1}.\Delta x})\Delta x$.

Summing, dividing by n, and reducing Δx without limit, we have for the average annual deaths $-\frac{1}{n}\int_0^n \frac{dL_x}{dx}C_x dx$. And dividing by the expression for the mean population, we have

$$\frac{-\int_0^n \frac{dL_x}{dx}C_x dx}{\int_0^n L_{v+x}C_x dx}$$

for the ratio of the average annual deaths to the mean population during a period of n years between the ages v and $v+n$.

If $\Delta' L_x$ and $d' L_x$ be substituted for ΔL_x and dL_x respectively, we see that the preceding processes are also applicable to deaths resulting from one or more particular causes only.

II.

To expand the differential coefficient of y_x in terms of the central finite differences:—

$$\text{1.)}\quad n\frac{dy_x}{dx}=\Delta y_x-\frac{1}{2}\Delta^2 y_x+\frac{1}{3}\Delta^3 y_x-\frac{1}{4}\Delta^4 y_x+\dots$$

$$=\Delta y_x-\frac{1}{2}(\Delta^2 y_{x-n}+\Delta^3 y_{x-n})+\frac{1}{3}(\Delta^3 y_{x-n}+\Delta^4 y_{x-n})-\frac{1}{4}(\Delta^4 y_{x-n}+\Delta^5 y_{x-n})+\dots$$

$$=(\Delta y_x-\frac{1}{2}\Delta^2 y_{x-n})-\frac{1}{6}\Delta^3 y_{x-n}+\frac{1}{12}\Delta^4 y_{x-n}-\dots$$

$$=\frac{\Delta y_x+\Delta y_{x-n}}{2}-\frac{1}{6}\Delta^3 y_{x-n}+\frac{1}{12}(\Delta^4 y_{x-2n}+\Delta^5 y_{x-2n})-\dots$$

$$=\frac{1}{2}(\Delta y_x+\Delta y_{x-n})-\frac{1}{6}(\Delta^3 y_{x-n}-\tfrac{1}{2}\Delta^4 y_{x-2n})+\dots$$

$$=\frac{1}{2}(\Delta y_x+y_{x-n})-\frac{1}{12}(\Delta^3 y_{x-n}+\Delta^3 y_{x-2n})+\dots$$

$$\text{2.)}\quad n\frac{dy_x}{dx}=\Delta y_x-\frac{1}{2}\Delta^2 y_x+\frac{1}{3}\Delta^3 y_x-\frac{1}{4}\Delta^4 y_x+\frac{1}{5}\Delta^5 y_x-\frac{1}{6}\Delta^6 y_x+\dots$$

$$\Delta y_{x-n:2}=\Delta y_x-\frac{1}{2}\Delta^2 y_x+\frac{3}{8}\Delta^3 y_x-\frac{5}{16}\Delta^4 y_x+\frac{35}{128}\Delta^5 y_x-\frac{63}{256}\Delta^6 y_x+\dots$$

$$n\frac{dy_x}{dx}=\Delta y_{x-n:2}-\frac{1}{24}\Delta^3 y_x+\frac{1}{16}\Delta^4 y_x-\frac{47}{640}\Delta^5 y_x+\frac{61}{768}\Delta^6 y_x-\dots\dots$$

$$-\frac{1}{24}\Delta^3 y_{x-3n:2}=\qquad -\frac{1}{24}\Delta^3 y_x+\frac{1}{16}\Delta^4 y_x-\frac{5}{64}\Delta^5 y_x+\frac{35}{384}\Delta^6 y_x-\dots\dots$$

$$n\frac{dy_x}{dx}=\Delta y_{x-n.2}-\frac{1}{24}\Delta^3 y_{x-3n\cdot 2}+\frac{3}{640}\Delta^5 y_x-\frac{3}{256}\Delta^6 y_x+\dots\dots$$

$$\frac{3}{640}\Delta^5 y_{x-5n:2}=\qquad +\frac{3}{640}\Delta^5 y_x-\frac{3}{256}\Delta^6 y_x+\dots\dots$$

$$\therefore\ n\frac{dy_x}{dx}=\Delta y_{x-n:2}-\frac{1}{24}\Delta^3 y_{x-3n:2}+\frac{3}{640}\Delta^5 y_{x-5n:2}-\dots\dots$$

By a process similar to the last, the following development of a definite integral may be obtained, viz.,

$$\int_x^{x+n} y_x dx=n\left\{y_{x+n:2}+\frac{1}{24}\Delta^2 y_{x-n:2}-\frac{17}{5760}\Delta^4 y_{x-3n:2}+\dots\dots\right\}$$

III.

Problem.—To find an expression for the number living at any age when the law of mortality is such that the chances of living a year,* multiplied by a constant factor, form a series whose logarithms are in geometrical progression.

The above is the form in which the law of mortality, which I have endeavoured to establish in the preceding pages, was first propounded in a paper published in vol. viii. of this *Journal*. Shortly after its publication I was favoured by Mr. Sprague with the following elegant solution of the above problem, which I doubt not will be received with the interest which the communications of so excellent a writer invariably command. I may add that I believe it was the first attempt to give an analytical expression for the function in question.

"Let L_x be the number living at age x. Then the chances of living a "year at the ages x, $x+1$, $x+2\dots$ are $\frac{L_{x+1}}{L_x}$, $\frac{L_{x+2}}{L_{x+1}}$, ... And by the "hypothesis, whatever the value of x, $\log\left(a\frac{L_{x+2}}{L_{x+1}}\right)=q\log\left(a\frac{L_{x+1}}{L_x}\right)$. Hence "$\log a+\log L_{x+2}-\log L_{x+1}=q\log a+q\log L_{x+1}-q\log L_x$. And $\log L_{x+2}$ "$-(q+1)\log L_{x+1}+q\log L_x=(q-1)\log a$.

"Now let $\log L_x=u_x$; then the last equation becomes

$$u_{x+2}-(q+1)u_{x+1}+q\,.\,u_x=(q-1)\log a.$$

"Separating the symbols of operation,

$$\{D^2-(q+1)D+q\}u_x=(q-1)\log a,$$

"or
$$(D-1)(D-q)u_x=(q-1)\log a.$$

* Of course any period whatever may be taken as the unit of time.

" Integrating this equation of finite differences,

$$u_x = \log L_x = c_1 + c_2.q^x - x \log a,$$

" and $$L_x = \varepsilon^{c_1 + c_2 q^x - x \log a} = \varepsilon^{c_1}.\varepsilon^{c_2 q^x}\varepsilon^{-x \log a}.$$

" Let now $\varepsilon^{c_1} = d$, $\varepsilon^{c_2} = g$, then we have

$$L_x = \frac{dg^{q^x}}{a^x},$$

" the expression required.

" For the solution of the equation $(D-1)(D-q)u_x = (q-1)\log a$, we " proceed as follows:—

" Since $D = 1 + \Delta$, we have

$$\Delta(\Delta + \overline{1-q})u_x = (q-1)\log a$$

$$\therefore\ u_x = c_1 + c_2 q^x + \frac{(q-1)\log a}{\Delta\{\Delta - (q-1\}}$$

$$= c_1 + c_2 q^x - \frac{\log a}{\Delta}\left(1 - \frac{\Delta}{q-1}\right)^{-1}$$

$$= c_1 + c_2 q^x - \frac{1}{\Delta}\left(1 + \frac{\Delta}{q-1} + \frac{\Delta^2}{(q-1)^2} + \ldots\right)\log a$$

$$= c_1 + c_2 q^x - \left(\Sigma + \frac{1}{q-1} + \frac{\Delta}{(q-1)^2} + \ldots\right)\log a$$

$$= c_1 + c_2 q^x - x \log a - \frac{\log a}{q-1}.$$

" And changing the constants,

$$u_x = c_1 + c_2 q^x - x \textit{ log } a.$$

" The expression for L_x may be got by a more gradual process, thus:—

" Let p_x denote the chance of living a year at the age x, then by hypo- " thesis, $\log(ap_{x+1}) = q.\log(ap_x)$, whence

$$\log a + \log p_{x+1} = q \log a + q \log p_x,$$

" and $$\log p_{x+1} - q \log p_x = (q-1)\log a.$$

" Then, putting $\log p_x = v_x$,

$$v_{x+1} - qv_x = (q-1)\log a$$

" or $$(D-q)v_x = (q-1)\log a,$$

" whence $v_x = \log p_x = cq^x - \log a$; and $p_x = \varepsilon^{cq^x - \log a} = \frac{\varepsilon^{cq^x}}{a}$; or, putting " $\varepsilon^c = g$, $p_x = \frac{g^{q^x}}{a}$.

" Then $\frac{L_{x+1}}{L_x} = \frac{g^{q^x}}{a}$, $\log L_{x+1} - \log L_x = q^x.\log g - \log a$,

$$u_{x+1} - u_x = q^x.\log g - \log a, \text{ or } \Delta u_x = q^x \log g - \log a.$$

"And, integrating,

$$u_x = \log L_x = \frac{q^x \log g}{q-1} - x \log a + c$$

$$L_x = \varepsilon^{\frac{q^x \log g}{q-1} - x \log a + c}$$

$$= \varepsilon^c . \varepsilon^{\frac{q^x \log g}{q-1}} . \varepsilon^{-x \log a}$$

$$\left.\begin{aligned} &= d\left(g^{\frac{1}{q-1}}\right)^{q^x} . \frac{1}{a^x} \\ &= \frac{d\gamma^{q^x}}{a^x} \end{aligned}\right\} \text{(putting } \varepsilon^c = d,\ \gamma = g^{\frac{1}{q-1}}\text{)}.$$

Mr. Sprague concludes with the following concise demonstration of the property of "equal lives":

$$p_{x,n} = \frac{L_{x+n}}{L_x} = \frac{dg^{q^{x+n}}}{a^{x+n}} \div \frac{dg^{q^x}}{a^x} = \frac{g^{q^x(q^n-1)}}{a^n}.$$

"Similarly, $p_{y,n} = \dfrac{g^{q^y(q^n-1)}}{a^n}$;

$$\therefore \quad p_{(xy)n} = \frac{1}{a^{2n}} g^{(q^x+q^y)(q^n-1)}.$$

"So also $\quad p_{(zz)n} = \dfrac{1}{a^{2n}} g^{2q^z(q^n-1)}.$

"Now, *whatever the value of n*, we shall have $p_{(xy)n} = p_{(zz)n}$, if z is found "from the equation $2q^z = q^x + q^y$. This being the case, it is clear that any "annuity, whether immediate, deferred, or temporary, on the joint lives x "and y, is equal to the similar annuity on the joint lives z and z."

IV.

The formulæ for the corrections to be applied to a series in order that the second order of differences may form a geometrical progression, are found as follows:—

Let the given series (B_x) consist of five terms; and, in order to produce the greatest effect upon the second differences with the least possible alteration of the original series, let the terms of the original series be alternately increased and diminished by the quantity p, the value of which is to be determined by the conditions of the problem.

$$\begin{array}{lll} B_0 + p & & \\ & \Delta B_0 - 2p & \\ B_n - p & & \Delta^2 B_0 + 4p \\ & \Delta B_n + 2p & \\ B_{2n} + p & & \Delta^2 B_n - 4p \\ & \Delta B_{2n} - 2p & \\ B_{3n} - p & & \Delta^2 B_{2n} + 4p \\ & \Delta B_{3n} + 2p & \\ B_{4n} + p & & \end{array}$$

The three terms of the third series are to form a geometrical progression: therefore,

$$(\Delta^2B_0+4p)(\Delta^2B_{2n}+4p)=(\Delta^2B_n-4p)^2;$$

or $\Delta^2B_0\times\Delta^2B_{2n}+4p(\Delta^2B_0+\Delta^2B_{2n})+(4p)^2=(\Delta^2B_n)^2-8p.\Delta^2B_n+(4p)^2$

$$4p(\Delta^2B_0+2\Delta^2B_n+\Delta^2B_{2n})=(\Delta^2B_n)^2-\Delta^2B_0\times\Delta^2B_{2n}$$

$$\therefore\quad 4p=\frac{(\Delta^2B_n)^2-\Delta^2B_0\times\Delta^2B_{2n}}{\Delta^2B_0+2\Delta^2B_n+\Delta^2B_{2n}}.$$

Let the given series now consist of six terms. We shall, in this case, require two unknown quantities, which we shall find it convenient to introduce as follows:—

$$\begin{array}{lll}
B_0\ +(v-w) & & \\
 & \Delta B_0\ -2(v-w) & \\
B_n\ -(v-w) & & \Delta^2B_0\ +(4v-3w) \\
 & \Delta B_n\ +(2v-w) & \\
B_{2n}+v & & \Delta^2B_n\ -(4v-w) \\
 & \Delta B_{2n}-2v & \\
B_{3n}-v & & \Delta^2B_{2n}+(4v+w) \\
 & \Delta B_{3n}+(2v+w) & \\
B_{4n}+(v+w) & & \Delta^2B_{3n}-(4v+3w) \\
 & \Delta B_{4n}-2(v+w) & \\
B_{5n}-(v+w) & &
\end{array}$$

Before proceeding to determine the values of v and w, it will be necessary to prove the following theorem, viz.:—If we have any four quantities, a, b, c, and d, and if $b-a$, $c-b$, and $d-c$ are in geometrical progression, and likewise $b+a$, $c+b$, and $d+c$, then the four given terms a, b, c, and d, shall also form a geometrical progression.

By hypothesis, $(b-a)(d-c)=(c-b)^2$ and $(b+a)(d+c)=(c+b)^2$; or, multiplying out,

$$bd-ad-bc+ac=c^2-2bc+b^2,$$

and
$$bd+ad+bc+ac=c^2+2bc+b^2,$$

whence, by adding and subtracting,

$$2bd+2ac=2c^2+2b^2;\ \text{or}\ bd+ac=b^2+c^2.$$

And
$$2ad+2bc=4bc;\ \text{or}\ ad=bc.$$

From $bd+ac=b^2+c^2$ we get $\frac{a-c}{b}=\frac{b-d}{c}$; and from $ad=bc$ we have $\frac{a-c}{a}=\frac{b-d}{b}$. Dividing the first of these equations by the second, we get the equation $\frac{a}{b}=\frac{b}{c}$; $\therefore\ a:b::b:c$.

But from $ad=bc$, we see that $a:b::c:d$; consequently,

$$a:b::b:c::c:d,$$

and the four terms are in geometrical progression.

Applying this to the four terms of the second order of differences of the series

$$\begin{array}{l|l|l} \begin{array}{l} \Delta^2B_0 + (4v-3w) \\ \Delta^2B_n - (4v-w) \\ \Delta^2B_{2n} + (4v+w) \\ \Delta^2B_{3n} - (4v+3w) \end{array} & \begin{array}{l} \Delta^3B_0 - 8v+4w \\ \Delta^3B_n + 8v \\ \Delta^3B_{2n} - 8v - 4w \end{array} & \begin{array}{l} \Delta^2B_0 + \Delta^2B_n - 2w \\ \Delta^2B_n + \Delta^2B_{2n} + 2w \\ \Delta^2B_{2n} + \Delta^2B_{3n} - 2w \end{array} \end{array}$$

We have to determine v and w, so that the three terms of each of the last two series shall be in geometrical progression. Putting $\Delta^2B_0 + \Delta^2B_n = A$, $\Delta^2B_n + \Delta^2B_{2n} = B$, $\Delta^2B_{2n} + \Delta^2B_{3n} = C$; $\Delta^3B_0 + 4w = A'$, $\Delta^3B_n = B'$, and $\Delta^3B_{2n} - 4w = C'$, and proceeding as in the determination of the value of $4p$, we shall find

$$2w = \frac{AC - B^2}{A + 2B + C}, \text{ and } 8v = \frac{A'C' - B'^2}{A' + 2B' + C'}.$$

V.

The following is a method of interpolating intermediate values in the series $\log L_0$, $\log L_n$, &c., which will be found convenient in constructing a mortality table according to the theory of geometrical partial forces.

Let Δ_x, $\Delta^2{}_x$, &c., represent the first and second differences of $\log L_x$ when the increment of x is unity, and ${}_n\Delta_x$, ${}_n\Delta^2{}_x$, &c., when the increment in question is n.

Then $${}_n\Delta_0 = \Delta_0 + (\Delta_0 + \Delta^2_0) + (\Delta_0 + \Delta^2_0 + \Delta^2_1) + .. + (\Delta_0 + \Delta^2_0 + \Delta^2_1 + .. + \Delta^2_{n-2})$$
$$= n\Delta_0 + (n-1)\Delta^2_0 + (n-2)\Delta^2_1 + \ldots + \Delta^2_{n-2},$$
and
$${}_n\Delta_0 - n\Delta_0 = (n-1)\Delta^2_0 + (n-2)\Delta^2_1 + \ldots\ldots\ldots + \Delta^2_{n-2}$$
$$= \Delta^2_0\{(n-1) + (n-2)q + \ldots\ldots\ldots + 2q^{n-3} + q^{n-2}\}. \qquad [1]$$

Put $$S = (n-1) + (n-2)q + (n-3)q^2 + \ldots + 2q^{n-3} + q^{n-2},$$
then $$Sq = \qquad (n-1)q + (n-2)q^2 + \ldots + 3q^{n-3} + 2q^{n-2} + q^{n-1},$$
and $$S(q-1) = -n + 1 + q + q^2 + \ldots\ldots + q^{n-3} + q^{n-2} + q^{n-1} = \frac{q^n - 1}{q-1} - n.$$
$$\therefore \qquad S = \frac{q^n - 1}{(q-1)^2} - \frac{n}{q-1}.$$

Hence, substituting in [1], we have

$${}_n\Delta_0 - n\Delta_0 = \Delta^2_0 \cdot \left\{\frac{q^n - 1}{(q-1)^2} - \frac{n}{q-1}\right\} = \Delta^2_0 \cdot \frac{(q^n - 1) - n(q-1)}{(q-1)^2},$$

and $$\frac{{}_n\Delta_0 - n\Delta_0}{(q^n - 1) - n(q-1)} = \frac{\Delta^2_0}{(q-1)^2}.$$

Again,

$${}_n\Delta^2_0 = {}_n\Delta_n - {}_n\Delta_0 = n\Delta_n + \Delta^2_n \cdot \frac{(q^n - 1) - n(q-1)}{(q-1)^2} - n\Delta_0 - \Delta^2_0 \cdot \frac{(q^n - 1) - n(q-1)}{(q-1)^2}$$

$$= n(\Delta_n - \Delta_0) + (\Delta^2_n - \Delta^2_0)\frac{(q^n - 1) - n(q-1)}{(q-1)^2}. \qquad [2]$$

But $\Delta_n - \Delta_0 = \Delta_0^2 + \Delta_1^2 + \ldots + \Delta_{n-1}^2 = \Delta_0^2(1+q+q^2+\ldots+q^{n-1})$

$$= \Delta_0^2 \frac{q^n - 1}{q-1} = \frac{\Delta_0^2}{(q-1)^2} \cdot (q^n - 1)(q-1),$$

and $\Delta_n^2 - \Delta_0^2 = \Delta_0^2(q^n - 1)$.

Hence, substituting in [2],

$$_n\Delta_0^2 = \frac{\Delta_0^2}{(q-1)^2}\{n.(q^n-1)(q-1)+(q^n-1)^2-n.(q^n-1)(q-1)\}$$

$$= \frac{\Delta_0^2}{(q-1)^2}(q^n-1)^2.$$

$$\therefore \quad \frac{_n\Delta_0^2}{(q^n-1)^2} = \frac{\Delta_0^2}{(q-1)^2} = \frac{_n\Delta_0 - n.\Delta_0}{(q^n-1)-n.(q-1)}.$$

By means of these equations the values of Δ_0 and Δ_0^2 may be determined from the values of $_n\Delta_0$ and $_n\Delta_0^2$. The successive terms of Δ_x^2 may then be calculated from the equation $\Delta_x^2 = \Delta_0^2 . q^x$.