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# LAWS OF MORTALITY WHICH SATISFY A UNIFORM SENIORITY PRINCIPLE

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## GENERAL IDEAS

IT is well known that a principle of uniform seniority applies to mortality tables which follow Gompertz's or Makeham's law, and, in a modified form, in the case of certain other laws, such as Makeham's second law and the double geometric law. It is natural to inquire what is the most general class of mortality laws to which such a principle applies.* In its most general form the uniform seniority principle implies that the value of a joint-life annuity on $m$ lives of different ages is equal to that of a joint-life annuity (possibly computed at a different interest rate) on $k$ lives of equal age, in such a way that the new interest rate $i'$, the number of substituted lives $k$, and the difference between the youngest age and the substituted equal age, depend only on the various differences in age between the youngest life and the other original lives, and not on the actual age of the youngest life.

Let $x_1$, $x_1+h_2$, $x_1+h_3$, ..., $x_1+h_m$ be the ages of the original lives, $x_1$ being the youngest. We seek values of $i'$, $k$ and $g$ such that

$$\bar{a}_{x_1, x_1+h_2, \ldots, x_1+h_m} = \bar{a}'_{x_1+g, x_1+g, \ldots (k)}, \tag{1}$$

where the accented symbol denotes a value computed at the rate $i'$. If a principle of uniform seniority applies, we can determine $i'$, $k$ and $g$ so that all three are functions of $h_2$, $h_3$, ..., $h_m$ only, and do not depend on $x_1$. Differentiating on both sides with respect to $x_1$, treating the $h$'s as constants (and omitting the subscripts of the annuity symbols), we obtain

$$\bar{a}(\mu_{x_1}+\mu_{x_1+h_2}+ \ldots +\mu_{x_1+h_m}+\delta)-1=\bar{a}'(k\mu_{x_1+g}+\delta')-1.$$

Upon simplifying and making use of equation (1), this gives

$$\delta+\mu_{x_1}+\mu_{x_1+h_2}+ \ldots +\mu_{x_1+h_m}=\delta'+k\mu_{x_1+g}. \tag{2}$$

On the other hand, if equation (2) holds for all values of $x_1$, integration of both members with respect to $x_1$ gives

$$\text{colog } D_{x_1, x_1+h_2, \ldots, x_1+h_m}=\text{colog } D'_{x_1+g, x_1+g, \ldots (k)}+\text{colog } C,$$

where $C$ is an arbitrary constant. In other words, we have (omitting subscripts), $D=CD'$ for all values of $x_1$, from which equation (1) follows easily. Therefore, condition (2) is both necessary and sufficient for a uniform seniority principle. If we write $x_j$ for $x_1+h_j$ ($j=2, 3, \ldots, m$) and $w$ for $x_1+g$, it becomes

$$\delta+\mu_{x_1}+\mu_{x_2}+\ldots+\mu_{x_m}=\delta'+k\mu_w,$$

* Several of the results obtained here were previously given by Walter Borland, Jr. in *T.F.A.* 7, 138. The approach used here has been suggested by that of Aúthos Pagano (*Lições de Estatística*, São Paulo, Brazil, 1953, 3, 313–14). He considers, however, only the case of two lives, assumes that $i'=i$ and $k$ is constant, and obtains only Gompertz's and Makeham's laws as solutions.

which, for $\delta'=\delta$, evidently reduces to the well-known relation between the forces of mortality for a table which follows Gompertz's or Makeham's law.

Partial differentiation of equation (2) with respect to one of the $h$'s yields

$$\mu'_{x_1+h_j}=\frac{\partial\delta'}{\partial h_j}+k\mu'_{x_1+g}\frac{\partial g}{\partial h_j}+\mu_{x_1+g}\frac{\partial k}{\partial h_j},$$

where the accented symbols other than $\delta'$ indicate derivatives. Taking partial derivatives in this equation with respect to both $x_1$ and $h_j$, equating the right members of the resulting equations, and simplifying, we obtain, finally,

$$k\frac{\partial g}{\partial h_j}\left(1-\frac{\partial g}{\partial h_j}\right)\mu''_{x_1+g}+\left(\frac{\partial k}{\partial h_j}-2\frac{\partial g}{\partial h_j}\frac{\partial k}{\partial h_j}-k\frac{\partial^2 g}{\partial h_j^2}\right)\mu'_{x_1+g}-\frac{\partial^2 k}{\partial h_j^2}\mu_{x_1+g}=\frac{\partial^2\delta'}{\partial h_j^2}. \quad (3)$$

Since $x_1$ can have any value, $x_1+g$ can have any value, and we may replace $x_1+g$ by $x$ in equation (3). Moreover, by assigning fixed values to the $h$'s we shall obtain a second-order differential equation for $\mu_x$ with constant coefficients. By investigating the various possible solutions of such an equation, we shall ascertain the various possible expressions for the force of mortality which satisfy a uniform seniority principle.

## SOLUTION OF THE DIFFERENTIAL EQUATION

The second-order differential equation obtained from equation (3) by assigning fixed values to the $h$'s is of the form

$$py''+qy'+ry=s. \quad (4)$$

Three essentially different situations arise as regards the possible solutions of this equation. These are (i) the case in which $r\neq 0$, (ii) the 'degenerate' case in which $r=0$ (but $q\neq 0$) and (iii) the 'doubly degenerate' case in which $q=r=0$ (but $p\neq 0$). All possible solutions (including those arising in other special situations as regards the values of the coefficients $p$, $q$, and $r$) are included in the solutions of these three cases. Under case (i) there are three sub-cases according as the roots of the characteristic equation

$$px^2+qx+r=0 \quad (5)$$

are (*a*) real and unequal, (*b*) real and equal or (*c*) complex. The expressions for the force of mortality obtained in these five situations are, respectively,

$$\mu_x=A+Bc^x+Mn^x, \quad (6)$$

$$\mu_x=A+(B+Hx)\,c^x, \quad (7)$$

$$\mu_x=A+Bc^x\sin(dx+f), \quad (8)$$

$$\mu_x=A+Hx+Bc^x, \quad (9)$$

$$\mu_x=A+Hx+Bx^2. \quad (10)$$

The expressions (6) and (9) are well known, and the simple quadratic expression (10) has been mentioned by Borland in the actuarial note previously referred to. The form (9) is of course Makeham's second law; and the form (6), which was mentioned (without the constant term $A$) by Sir George Hardy, we shall call the double geometric law. For convenience, we shall refer to the expressions (7) and (8), respectively, as the arithmetic-geometric law and the trigonometric law. The latter seems unlikely to be useful in connexion with mortality tables, though it must be admitted that ungraduated mortality rates derived from some population data present an appearance not unlike what this equation might be expected to produce. On the other hand, it reduces to Makeham's law when $d=0$, and by taking $d$ quite small the span of human life can be made to correspond to a small portion of the oscillation of the sine function.

## DETERMINATION OF $i'$, $k$ AND $w$

The procedure to be followed in determining the adjusted interest rate $i'$, the number of substituted lives $k$, and the equivalent equal age $w$ is somewhat different for each of the five expressions (6)–(10). In those cases where $i'$ differs from $i$, the formulae will give directly the adjusted force of interest $\delta'$; this is, of course, readily converted to an interest rate $i'$.

It is convenient to consider each of the three expressions obtained in case (i) as of the form

$$\mu_x = A + \phi(x), \tag{11}$$

where, in each case, $\phi(x)$ is an expression containing an exponential factor, or a sum of such expressions. Substitution of the right member of equation (11) for $\mu_x$ in equation (2) gives

$$\delta + mA + \phi(x_1) + \phi(x_2) + \ldots + \phi(x_m) = \delta' + kA + k\phi(w), \tag{12}$$

where, for convenience, we have written $x_j$ for $x_1 + h_j$ $(j = 2, 3, \ldots, m)$ and $w$ for $x_1 + g$. Successive partial differentiation of this equation with respect to $x_1$ gives

$$\phi'(x_1) + \phi'(x_2) + \ldots + \phi'(x_m) = k\phi'(w), \tag{13}$$

$$\phi''(x_1) + \phi''(x_2) + \ldots + \phi''(x_m) = k\phi''(w). \tag{14}$$

We now observe that in all three sub-cases of case (i), $\phi(x)$ satisfies the homogeneous differential equation

$$p\phi''(x) + q\phi'(x) + r\phi(x) = 0.$$

Thus, on multiplying equations (14), (13) and (12) by $p$, $q$ and $r$, respectively, adding, and dividing by $r$ (which, for case (i), is different from zero), we obtain

$$\delta' = \delta + (m-k)\,A. \tag{15}$$

This result, taken in conjunction with equation (12), gives

$$\phi(x_1) + \phi(x_2) + \ldots + \phi(x_m) = k\phi(w). \tag{16}$$

From equations (13) and (16), we could, in theory, determine $k$ and $w$. However, in all three sub-cases it turns out that simpler equations are obtained by suitable combinations of these two equations. Let us suppose that

$$\alpha_1\phi(x) + \beta_1\phi'(x) = \gamma_1\psi_1(x) \tag{17}$$

and

$$\alpha_2\phi(x) + \beta_2\phi'(x) = \gamma_2\psi_2(x), \tag{18}$$

where the $\alpha$'s, $\beta$'s and $\gamma$'s are suitable constants (both $\gamma_1$ and $\gamma_2$ being different from zero) and $\psi_1(x)$ and $\psi_2(x)$ are simpler expressions than $\phi(x)$ and $\phi'(x)$. Then, on multiplying equations (16) and (13) by $\alpha_1$ and $\beta_1$, respectively, adding and dividing by $\gamma_1$, we obtain

$$\psi_1(x_1) + \psi_1(x_2) + \ldots + \psi_1(x_m) = k\psi_1(w). \tag{19}$$

In a similar manner, we find that

$$\psi_2(x_1)+\psi_2(x_2)+\ldots+\psi_2(x_m)=k\psi_2(w). \tag{20}$$

Dividing equation (20) by equation (19) gives

$$\frac{\psi_2(w)}{\psi_1(w)}=\frac{\psi_2(x_1)+\psi_2(x_2)+\ldots+\psi_2(x_m)}{\psi_1(x_1)+\psi_1(x_2)+\ldots+\psi_1(x_m)}. \tag{21}$$

In all three sub-cases, it turns out that the $\alpha$'s, $\beta$'s and $\gamma$'s can be so chosen that equation (21) can be solved for $w$. The value of $k$ is then determined from equation (19) or (20).

The $\alpha$'s, $\beta$'s, $\gamma$'s and $\psi$'s to be used for the three sub-cases are as follows:

| Quantity | Double geometric law | Arithmetic-geometric law | Trigonometric law* |
|---|---|---|---|
| $\alpha_1$ | $-\log n$ | $-\log c$ | $d\cos(a-f)-\log c\sin(a-f)$ |
| $\beta_1$ | 1 | 1 | $\sin(a-f)$ |
| $\gamma_1$ | $B(\log c-\log n)$ | $H$ | $Bd$ |
| $\psi_1(x)$ | $c^x$ | $c^x$ | $c^x\sin(dx+a)$ |
| $\alpha_2$ | $\log c$ | $H+B\log c$ | $-d\sin(a-f)-\log c\cos(a-f)$ |
| $\beta_2$ | $-1$ | $-B$ | $\cos(a-f)$ |
| $\gamma_2$ | $M(\log c-\log n)$ | $H^2$ | $Bd$ |
| $\psi_2(x)$ | $n^x$ | $xc^x$ | $c^x\cos(dx+a)$ |

* $a$ denotes any convenient constant value.

It is easily verified that the use of the $\alpha$'s and $\beta$'s given in the table will result in the indicated expressions for the $\gamma$'s and $\psi$'s.

The requirement that both $\gamma_1$ and $\gamma_2$ be different from zero calls for some discussion. Referring to the expressions in the table for the $\gamma$'s, we note that in sub-case ($a$) the expression (6) reduces to Makeham's first law if $B=0$ or $M=0$. This special case will be discussed later in the note. The same result would follow from $\log c-\log n=0$ (which implies $n=c$), but this relation cannot hold in any event, since it would imply equal roots of the characteristic equation (5), and sub-case ($a$) is the case of unequal roots.

In sub-case ($b$), the expression (7) likewise reduces to Makeham's first law if $H=0$. In sub-case ($c$), $d=0$ would imply real roots of equation (5), whereas for this sub-case the roots are complex. On the other hand, if $B=0$, the expression (8) would reduce to the trivial case of a constant force of mortality $\mu_x=A$, under which

$$\bar{a}_{xyz\ldots(m)}=\frac{1}{mA+\delta},$$

whatever the ages of the lives involved.

The specific equations for determining $k$ and $w$ in each of the three sub-cases are given in an Appendix to the paper. When the value of $k$ is known, equation (15) gives $\delta'$.

In discussing case (ii), it will be assumed that $B$ and $H$ in expression (9) are both different from zero. (If $H=0$, the expression reduces to Makeham's first law, while, for $B=0$, it reduces to the simple arithmetic law, $\mu_x=A+Hx$, which also will be taken up later.) Substitution of the expression (9) in equation (2) gives

$$\delta'=\delta+(m-k)\,A+H\theta(x_1)+B\lambda(x_1), \tag{22}$$

where

$$\theta(x_1)=x_1+x_2+\ldots+x_m-kw,$$

and

$$\lambda(x_1)=c^{x_1}+c^{x_2}+\ldots+c^{x_m}-kc^{w}.$$

Successive partial differentiation of equation (22) with respect to $x_1$ gives

$$(m-k)\,H+B\lambda(x_1)\log c=0 \tag{23}$$

and

$$B\,(\log c)^2\,\lambda(x_1)=0. \tag{24}$$

Equation (24) gives $\lambda(x_1)=0$, or, in other words,

$$c^{x_1}+c^{x_2}+\ldots+c^{x_m}=kc^{w}; \tag{25}$$

and substitution in equation (23) yields $k=m$. Finally, substitution of these two results in equation (22) gives

$$\delta'=\delta+H\theta(x_1). \tag{26}$$

In discussing case (iii), it will be assumed that $B\neq 0$; otherwise it reduces to the arithmetic case to be discussed later. Substitution of the expression (10) in equation (2) gives

$$\delta'=\delta+(m-k)\,A+H\theta(x_1)+B\rho(x_1), \tag{27}$$

where

$$\rho(x_1)=x_1^2+x_2^2+\ldots+x_m^2-kw^2.$$

Successive partial differentiation with respect to $x_1$ yields

$$(m-k)\,H+2B\theta(x_1)=0 \tag{28}$$

and

$$2B(m-k)=0,$$

whence $k=m$ and $\theta(x_1)=0$, or, in other words,

$$x_1+x_2+\ldots+x_m=mw. \tag{29}$$

Substitution in equation (27) then gives

$$\delta'=\delta+B\rho(x_1). \tag{30}$$

It may not be immediately evident from the preceding equations that, in every case, $i'$, $k$ and $g=w-x_1$ are functions of the $h$'s only and independent of $x_1$, as demanded in the introductory part of the paper. This can, however, be verified in each case by suitable algebraic manipulation. The actual expressions in terms of the $h$'s can be obtained from those given in the appendix by setting $x_1=0$, so that $w$ is replaced by $g$ and each $x_j\,(j=2, 3, \ldots, m)$ by the corresponding $h_j$.

It should be pointed out also that the equations for determining $i'$, $k$ and $w$ are not only necessary but also sufficient for the uniform seniority property. As regards case (i), we note first that equations (17) and (18) will give a solution for $\phi(x)$ as a linear combination of the $\psi$'s provided $\alpha_1\beta_2-\beta_1\alpha_2$, the determinant of the coefficients in the left members, does not vanish. The values of this expression in the three sub-cases are, respectively, $\log n-\log c$, $-H$ and $d$. It appears, therefore, from the table previously given that this determinant does not vanish if the necessary condition as to the non-vanishing of the $\gamma$'s is met. We may conclude, then, that if equation (21) holds, together with either equation (19) or (20), equation (16) necessarily follows. This, in conjunction with equation (15), leads back to equation (12), which is the form taken in case (i) by the general condition (2).

Likewise, it is evident that equation (22), which is the corresponding condition for case (ii), follows at once from equations (25) and (26) and the equality of $k$ and $m$. Finally, it is clear that the same equality, together with equations (29) and (30), leads immediately to equation (27), which is the necessary and sufficient condition for uniform seniority in case (iii).

## THE INDETERMINATE CASES

In the preceding section, frequent reference has been made to two special cases: Makeham's first law, under which

$$\mu_x = A + Bc^x, \tag{31}$$

and the arithmetic law represented by

$$\mu_x = A + Hx. \tag{32}$$

In these cases, the values of $\delta'$, $k$ and $w$ are not completely determined. The indeterminacy arises from the fact that these two expressions satisfy a differential equation of the *first* order. The former case is conveniently treated as a special case of Makeham's second law by taking $H=0$ in expression (9). Equation (22) then reduces to

$$\delta' = \delta + (m-k)\,A + B\lambda(x_1), \tag{33}$$

and equation (23) becomes $\quad B\lambda(x_1)\log c = 0.$

From these two equations we obtain at once equations (15) and (25), and further partial differentiation does not yield any new equations. Conversely, the necessary and sufficient condition (33) follows immediately from equations (15) and (25). Since we have three quantities, $i'$, $k$ and $w$, to be chosen and only two conditions to be satisfied, we may select any one of the three quite arbitrarily.

The traditional method is, of course, to take $i'=i$, so that $k=m$. However, another expedient, also well known, is to take $k=1$, so that $\delta'=\delta+(m-1)\,A$. This avoids the necessity of tabulating multiple-life annuity values, but requires interpolation between single-life annuity values at different interest rates. A third possibility would be to take a fixed interest rate: for example, $i'=\delta'=0$, so that

$$k = m + \frac{\delta}{A}.$$

This method would require the tabulation of expectations of life for various numbers of joint lives of equal age, and interpolation between the values for different numbers of lives to obtain that corresponding to the value of $k$ computed by the formula.

The expression (32) may be considered as the special case of expression (10) when $B=0$, and, accordingly, equation (27) becomes

$$\delta' = \delta + (m-k)A + H\theta(x_1), \tag{34}$$

while equation (28) becomes

$$(m-k)\,H=0,$$

and further partial differentiation yields only the trivial identity $0=0$. Thus we arrive at the equality of $k$ and $m$ and the equation (26). On the other hand, the two latter assumptions take us immediately back to the condition (34).

In this case, $k$ is determined (as equal to $m$), but either $i'$ or $w$ may be chosen arbitrarily, the other being then determined by equation (26). The natural choice is, of course, to take $i'=i$, in which case $w$ is the simple arithmetic average of the ages $x_1, x_2, \ldots, x_m$.

## RESTRICTIONS ON THE VALUES OF $k$ AND $i'$

Exception may be taken to the fact that, in case (i), the value of $k$ will usually not be an integer. In theory, at least, this is not a serious difficulty, because the number of substituted lives $k$, like the equal age $w$, is merely a computational device, and the annuity value

$$\bar{a}_{ww\ldots(k)}=\int_0^\infty v^t({}_tp_w)^k\,dt$$

is defined mathematically even when $k$ is not an integer. If, however, it is desired to fix the value of $k$, equation (3) shows that we are limited to cases (ii) and (iii). We must have $k=m$ if the expression for $\mu_x$ contains a first or second degree term in $x$, while $k$ may be chosen arbitrarily if such terms are absent.

If it is thought inconvenient to tabulate values at many different interest rates, and it is therefore desired to fix the value of $i'$, the right-hand member of equation (3) vanishes. In case (i), the general solution of the resulting homogeneous equation is $\phi(x)$, and, from equation (11), we have, therefore, $A=0$. It follows then from equation (15) that $i'=i$. In cases (ii) and (iii), equation (3) is transformed by direct integration to the form

$$P\mu'_x+Q\mu_x=R,$$

of which the general solution is of the form (32) if $Q$ vanishes, and otherwise of the form (31). In both these cases we have seen that it is possible to choose $i'$ quite arbitrarily.

If $i'$ and $k$ are both to have fixed values, we are limited to the two special cases last mentioned. In the former case we must have $k=m$; in the latter $i'$ and $k$ must be chosen so that equation (15) is satisfied. This shows, incidentally, the well-known fact that in the case of Gompertz's law, when $A=0$, we must have $i'=i$, but $k$ can be chosen quite arbitrarily.

## COMBINATIONS OF MORTALITY TABLES

Under the various laws discussed, the principle of uniform seniority can be applied to a set of lives for which different mortality tables are assumed, provided the same differential equation is satisfied by each of the different expressions for the force of mortality. In other words, those constants which depend on the roots of the characteristic equation must be the same for all the tables employed, while those depending only on the constants of integration may differ. The constants which must have identical values for all the tables used are $c$ and $n$ in the case of the double geometric law, $c$ for the arithmetic-geometric law and for Makeham's first and second laws, and $c$ and $d$ in the case of the trigonometric law. For the quadratic and arithmetic laws, all three constants may differ.

Thus, in the case of the double geometric law, if we wish to use for the $j$th life a mortality table in which

$$\mu_x = A_j + B_j c^x + M_j n^x,$$

while constants without subscripts denote those of the mortality table to be used for the $k$ substituted lives, we have

$$\frac{B}{M}\left(\frac{c}{n}\right)^w = \frac{B_1 c^{x_1} + B_2 c^{x_2} + \ldots + B_m c^{x_m}}{M_1 n^{x_1} + M_2 n^{x_2} + \ldots + M_m n^{x_m}},$$

$$kBc^w = B_1 c^{x_1} + B_2 c^{x_2} + \ldots + B_m c^{x_m},$$

and

$$\delta' = \delta + A_1 + A_2 + \ldots + A_m - kA.$$

In cases (ii) and (iii), it will generally be most convenient to take $k = m$ and to use the same combination of mortality tables for the substituted lives as for the original lives. Thus, for Makeham's second law,

$$c^w = \frac{B_1 c^{x_1} + B_2 c^{x_2} + \ldots + B_m c^{x_m}}{B_1 + B_2 + \ldots + B_m}$$

and

$$\delta' = \delta + H_1 x_1 + H_2 x_2 + \ldots + H_m x_m - w(H_1 + H_2 + \ldots + H_m).$$

## SUMMARY

We have shown that there are five distinct mathematical forms which admit of a uniform seniority principle—distinct in the sense that no one of them is merely a special case of another. These are the double geometric law, the trigonometric law, the arithmetic-geometric law, Makeham's second law and the quadratic law. All but the last of these include Makeham's first law as a special case. The last three include as a special case the simple arithmetic law. In the case of the three forms first mentioned, both the number of substituted lives and the adjusted interest rate (in general) depend on the differences between the various ages. In the last two cases, the number of substituted lives is, in general, the same as the number of original lives, but it can be chosen arbitrarily in the special case of Makeham's first law. The dependence of the adjusted interest rate on the differences between the ages is eliminated only (i) when the constant term is lacking from any of the first three forms and (ii) in the case of Makeham's first law and the simple arithmetic law. Only in the last two cases can both the adjusted interest rate and the number of substituted lives be chosen independently of the differences between the ages.

## APPENDIX

*Formulae for determining $\delta'$, $k$, and $w$ in the various cases*

I. Double geometric law:

$$\left(\frac{c}{n}\right)^w = \frac{c^{x_1}+c^{x_2}+\ldots+c^{x_m}}{n^{x_1}+n^{x_2}+\ldots+n^{x_m}},$$

$$kc^w = c^{x_1}+c^{x_2}+\ldots+c^{x_m},$$

$$\delta' = \delta+(m-k)\,A.$$

II. Arithmetic-geometric law:

$$w = \frac{x_1 c^{x_1}+x_2 c^{x_2}+\ldots+x_m c^{x_m}}{c^{x_1}+c^{x_2}+\ldots+c^{x_m}},$$

$$kc^w = c^{x_1}+c^{x_2}+\ldots+c^{x_m},$$

$$\delta' = \delta+(m-k)\,A.$$

III. Trigonometric law:

$$\tan(dw+a) = \frac{c^{x_1}\sin(dx_1+a)+c^{x_2}\sin(dx_2+a)+\ldots+c^{x_m}\sin(dx_m+a)}{c^{x_1}\cos(dx_1+a)+c^{x_2}\cos(dx_2+a)+\ldots+c^{x_m}\cos(dx_m+a)},$$

$$kc^w\sin(dw+a) = c^{x_1}\sin(dx_1+a)+c^{x_2}\sin(dx_2+a)+\ldots+c^{x_m}\sin(dx_m+a)$$

(where $a$ is any convenient constant),

$$\delta' = \delta+(m-k)\,A.$$

IV. Makeham's second law:

$$mc^w = c^{x_1}+c^{x_2}+\ldots+c^{x_m},$$

$$k = m,$$

$$\delta' = \delta+H(x_1+x_2+\ldots+x_m-mw).$$

V. Quadratic law:

$$mw = x_1+x_2+\ldots+x_m,$$

$$k = m,$$

$$\delta' = \delta+B(x_1^2+x_2^2+\ldots+x_m^2-mw^2).$$

VI. Makeham's first law:*

$$kc^w = c^{x_1}+c^{x_2}+\ldots+c^{x_m},$$

$$\delta' = \delta+(m-k)\,A.$$

VII. Arithmetic law:†

$$k = m,$$

$$\delta' = \delta+H(x_1+x_2+\ldots+x_m-mw).$$

* In this case, the equations do not fully determine $\delta'$, $k$ and $w$.

† In this case, the equations do not fully determine $\delta'$ and $w$.