LIDSTONE'S Z-METHOD WITHOUT MAKEHAM'S LAW

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INTRODUCTION

LIDSTONE showed that when a mortality table follows Makeham's law a Z-function from which to determine 'mean' maturity age in his bulk method of valuing endowment assurances may be suitably chosen in the form

$$Z(M)=A+Bc^M,$$

M being the variable maturity age. When Makeham's law does not apply, however, it is somewhat artificial to constrain the function $Z(M)$ to the form $A+Bc^M$, although Lidstone has shown (*J.I.A.* **38**, 1 and **64**, 478) that suitable functions of this form may be found for applying his method with the O^M and A 1924–29 (ultimate) tables within the range of maturity ages then most popular.

2. A method of deriving a more general Z-function is given below. This method is then used to derive a new Z-function for use with the recently published A 1949–52 (ultimate) table. Some examples are given with this mortality table to illustrate that although the Z-method may still be used with the same Z-function found suitable for A 1924–29 (ultimate) mortality, the new Z-function not of the form $A+Bc^M$ would be more satisfactory.

BASIS FOR DERIVING A Z-FUNCTION

3. If W_M is a set of positive weights such that $\Sigma W_M=1$, $Z(M)$ and $F(M)$ are functions of the maturity age M, and mean maturity ages M_Z and M_F are defined from the equations

$$Z(M_Z)=\Sigma W_M\, Z(M),$$

$$F(M_F)=\Sigma W_M\, F(M),$$

then it is well known that the necessary and sufficient condition that $M_Z=M_F$ for all possible weights W_M is $Z(M)=A+BF(M)$, where A and B are constants.

4. For integral values of M this condition will be satisfied if

$$\Delta Z(M)=B\Delta F(M),$$

i.e. if $$\Delta Z(M)/\Delta Z(M-1)=\Delta F(M)/\Delta F(M-1)$$

assuming $\Delta Z(M)$ and $\Delta F(M)$ are not zero at any of the values of M concerned.

For non-integral values of M it is assumed that $Z(M)$ and $F(M)$ may be obtained by the same linear interpolation formula

$$Z(M)=c_0\, Z(M_0)+c_1 Z(M_1)+\dots,$$

$$F(M)=c_0\, F(M_0)+c_1 F(M_1)+\dots,$$

where M_0, M_1, ..., are integral values of M and $c_0+c_1+\dots=1$.

It then follows that

$$\begin{aligned} Z(M) &= c_0 Z(M_0) + c_1 Z(M_1) + \ldots \\ &= c_0\{A + BF(M_0)\} + c_1\{A + BF(M_1)\} + \ldots \\ &= A + B\{c_0 F(M_0) + c_1 F(M_1) + \ldots\} \\ &= A + BF(M). \end{aligned}$$

The relation $\Delta Z(M)/\Delta Z(M-1) = \Delta F(M)/\Delta F(M-1)$ for integral M is then sufficient to establish $Z(M) = A + BF(M)$ for all M and thereby to ensure that $Z(M)$ and $F(M)$ will always produce the same mean ages.

5. The present problem is to find, if possible, one function $Z(M)$ that will produce a mean age close enough to the mean ages produced by each of a given series of functions $F_1(M)$, $F_2(M)$, ... A solution should be possible if $\Delta F(M)/\Delta F(M-1)$ does not vary too much from one function to another, even though for each function it might vary a lot with M; this solution is found by choosing $\Delta Z(M)/\Delta Z(M-1)$ close to the values of $\Delta F(M)/\Delta F(M-1)$. It is shown in the Appendix that a bias in

$$\Delta Z(M)/\Delta Z(M-1)$$

a little on the high side will in practice bring out higher mean ages M_z.

6. For the particular case when $F_n(M)$ is the temporary annuity $a_{M-n-1:\overline{n}|}$ with A 1949–52 (ultimate) mortality at 4% some values of $\Delta F(M)/\Delta F(M-1)$ are shown in Table A. (See §§12 and 13 for other rates of interest.)

In the preparation of this table annuity values were calculated to more than the customary three decimal places and the rates of mortality from which annuities were derived to more than five decimal places in order to enhance the smoothness of the figures. Whilst such refinement is not essential a cruder table would have been more difficult to interpret. This table also shows suitable values of $\Delta Z(M)/\Delta Z(M-1)$ obtained by reference to the body of the table alongside corresponding values from the function $Z(M) = c^{M-55}$ suggested by Lidstone for the A 1924–29 (ultimate) table, in which $c = 1{\cdot}117$. It will be noted that at each term n, $\Delta a_{M-n-1:\overline{n}|}/\Delta a_{M-n-2:\overline{n}|}$ tends to decrease as M increases.

A Z-FUNCTION FOR A 1949–52 (ULTIMATE) MORTALITY

7. The values of $\Delta Z(M)/\Delta Z(M-1)$ in the penultimate column of Table A were chosen to be slightly higher than the average value of $\Delta a_{M-n-1:\overline{n}|}/\Delta a_{M-n-2:\overline{n}|}$ in Table A with M constant. This should have the effect of over-estimating mean ages 'on the average' when mean ages are to be obtained for all outstanding terms, and thereby tend to a safe total reserve. This margin is important where bonus is valued with the same mean ages as sum assured because for a given outstanding term higher maturity ages are likely to be associated with relatively higher bonus. Intermediate values of $\Delta Z(M)/\Delta Z(M-1)$ were then chosen by smooth interpolation, a Z-function finally being calculated with the arbitrary constraints suggested by Cooksey* by first finding $\Delta Z(50)$ by trial to make $Z(60) - Z(50) = 300$. The resulting Z-function, tabulated in Table B, could be related to a sum assured of, say, £1000.

* *The Actuarial Society of Australasia*, Bulletin, no. 5, 251.

Table A

M \ n	$\Delta a_{M-n-1:\overline{n}\rvert}/\Delta a_{M-n-2:\overline{n}\rvert}$ by A 1949–52 (ultimate) 4 %												$\frac{\Delta Z(M)}{\Delta Z(M-1)}$	A 1924–29 $\frac{\Delta Z(M)}{\Delta Z(M-1)}$
	1	6	11	16	21	26	31	36	41	46	51	56		
25	1·352	1·354	1·356	—	—	—	—	—	—	—	—	—	1·354	—
30	1·339	1·347	1·349	1·350	—	—	—	—	—	—	—	—	1·347	—
35	1·308	1·326	1·334	1·337	1·337	—	—	—	—	—	—	—	1·329	1·117
40	1·246	1·282	1·299	1·306	1·308	1·309	—	—	—	—	—	—	1·292	1·117
45	1·165	1·212	1·242	1·256	1·261	1·263	1·263	—	—	—	—	—	1·238	1·117
50	1·109	1·138	1·175	1·198	1·208	1·212	1·213	1·213	—	—	—	—	1·184	1·117
55	1·094	1·098	1·119	1·145	1·163	1·170	1·173	1·173	1·174	—	—	—	1·146	1·117
60	1·096	1·090	1·092	1·106	1·125	1·138	1·143	1·145	1·146	1·146	—	—	1·123	1·117
65	1·098	1·091	1·086	1·086	1·096	1·110	1·120	1·124	1·125	1·126	1·126	—	1·108	1·117
70	1·096	1·087	1·082	1·078	1·079	1·086	1·097	1·105	1·108	1·109	1·109	1·109	1·096	1·117
75	1·091	1·078	1·072	1·069	1·067	1·068	1·075	1·084	1·090	1·093	1·093	1·094	1·082	1·117
80	1·081	1·063	1·056	1·053	1·052	1·052	1·054	1·061	1·069	1·074	1·076	1·077	1·064	—

8. Table B has not been extended below maturity age 30. For $M<30$, $a_{M-n-1:\overline{n}|}$ hardly differs from $a_{30-n-1:\overline{n}|}$ so $a_{M-n-1:\overline{n}|}$ may be replaced by $a_{30-n-1:\overline{n}|}$ without error. The rating up of maturity ages to a minimum of 30 may be given effect to by entering Table B at age 30 whenever the maturity age is 30 or less.

Table B. A 1949–52 (ultimate) $Z(M)$ table

M	$Z(M)$	M	$Z(M)$	M	$Z(M)$	M	$Z(M)$	M	$Z(M)$
30	135	40	141	50	200	60	500	70	1448
31	136	41	143	51	215	61	556	71	1604
32	136	42	145	52	232	62	619	72	1775
33	136	43	148	53	251	63	690	73	1961
34	136	44	152	54	274	64	768	74	2163
35	137	45	156	55	301	65	856	75	2382
36	137	46	162	56	331	66	952	76	2620
37	138	47	169	57	366	67	1059	77	2876
38	138	48	177	58	405	68	1177	78	3152
39	140	49	188	59	450	69	1306	79	3447
								80	3763

9. For the purpose of calculating the mean value of M by inverse entry Table C gives values of $Z(M-1/20)$ for a sum assured of £1000 for each tenth of a year from $M=45\cdot0$ to $M=74\cdot9$. This table follows the convenient plan proposed by E. H. Brown (*J.I.A.* **42**, 211). It has been constructed by linear interpolation; Lidstone has pointed out (*J.I.A.* **64**, 487) that the error thereby introduced is compensated when the mean ages derived from the table are used to determine annuity values etc. by linear interpolation. The mean value of M correct to the nearer ·1 of a year is obtained by taking from the table the age corresponding to the tabular value of Z next lower than the mean Z.

COMPARISON OF MEAN VALUES BY USE OF A 1924–29 (ULTIMATE) AND A 1949–52 (ULTIMATE) Z-FUNCTIONS

10. A comparison is made in Table D between true mean annuity values with A 1949–52 (ultimate) mortality at 4% (and true mean maturity ages) and the corresponding means obtained from Lidstone's A 1924–29 Z-function and those obtained from Tables B and C. The weights W_M are proportional to the numbers 1, 2, 3, 3, 2, 1, respectively, at maturity ages 35, 40, 45, 50, 55, 60 in example (*a*), at maturity ages 45, 50, 55, 60, 65, 70 in example (*b*), and at maturity ages 55, 60, 65, 70, 75, 80 in example (*c*).

In each example the weights W_M have been made zero, however, when the age attained $M-n-1$ is less than 18.

11. Example (*a*) calls for little comment, although A 1924–29 mean ages are on the whole perhaps a little on the low side. Example (*b*) demonstrates that near the 'mean' maturity ages assumed when deriving the A 1924–29 Z-function it would be reasonable to keep the same Z-function for use with A 1949–52 mortality, even though it would be more comforting to rely on the results of the new Z-function. Example (*c*) shows that an increase as much as

Table C. Table for finding mean ages

$Z(M - 1/20)$ for sum assured 1000. A 1949–52 (ultimate) mortality

M	·0	·1	·2	·3	·4	·5	·6	·7	·8	·9	M
45	156·0	156·5	157·1	157·7	158·2	158·8	159·4	160·0	160·5	161·1	45
46	161·7	162·3	163·0	163·7	164·4	165·1	165·8	166·5	167·2	167·9	46
47	168·6	169·4	170·2	171·1	171·9	172·8	173·6	174·5	175·4	176·2	47
48	177·1	178·0	179·0	180·0	181·1	182·1	183·1	184·2	185·2	186·2	48
49	187·2	188·4	189·6	190·8	192·0	193·3	194·5	195·7	196·9	198·2	49
50	199·4	200·7	202·2	203·6	205·1	206·5	208·0	209·4	210·9	212·3	50
51	213·8	215·4	217·1	218·8	220·5	222·2	223·9	225·6	227·3	229·0	51
52	230·7	232·6	234·5	236·5	238·5	240·5	242·5	244·5	246·5	248·5	52
53	250·4	252·6	254·9	257·2	259·5	261·8	264·1	266·4	268·7	271·0	53
54	273·3	275·8	278·4	281·1	283·7	286·4	289·0	291·7	294·3	297·0	54
55	299·6	302·5	305·5	308·5	311·6	314·6	317·7	320·7	323·7	326·8	55
56	329·8	333·1	336·5	340·0	343·4	346·9	350·4	353·8	357·3	360·8	56
57	364·2	367·9	371·9	375·8	379·7	383·7	387·6	391·5	395·5	399·4	57
58	403·4	407·5	412·0	416·4	420·9	425·4	429·8	434·3	438·7	443·2	58
59	447·6	452·3	457·4	462·4	467·4	472·4	477·4	482·4	487·5	492·5	59
60	497·5	502·8	508·4	514·1	519·7	525·4	531·0	536·6	542·3	547·9	60
61	553·5	559·5	565·8	572·1	578·4	584·7	591·0	597·3	603·7	610·0	61
62	616·3	623·0	630·0	637·1	644·1	651·1	658·2	665·2	672·3	679·3	62
63	686·4	693·8	701·7	709·5	717·4	725·2	733·1	740·9	748·8	756·6	63
64	764·5	772·8	781·5	790·2	799·0	807·7	816·4	825·1	833·8	842·6	64
65	851·3	860·5	870·1	879·8	889·5	899·1	908·8	918·5	928·1	937·8	65
66	947·5	957·6	968·3	979·0	989·7	1000	1011	1022	1032	1043	66
67	1054	1065	1077	1089	1100	1112	1124	1136	1147	1159	67
68	1171	1183	1196	1209	1222	1235	1248	1261	1274	1287	68
69	1300	1313	1328	1342	1356	1370	1384	1399	1413	1427	69
70	1441	1456	1472	1487	1503	1518	1534	1550	1565	1581	70
71	1596	1613	1630	1647	1664	1681	1698	1715	1732	1749	71
72	1766	1784	1803	1821	1840	1858	1877	1895	1914	1933	72
73	1951	1971	1991	2011	2031	2052	2072	2092	2112	2133	73
74	2153	2174	2196	2218	2240	2262	2284	2306	2327	2349	74

Table D

Example (*a*)

Term *n*	4% Annuities $a_{M-n-1:\overline{n}\mid}$			Maturity ages *M*		
	True mean	Deviation in excess of true mean		True mean	Deviation in excess of true mean	
		With A 1924–29 *Z*'s	With A 1949–52 *Z*'s		With A 1924–29 *Z*'s	With A 1949–52 *Z*'s
1	·957	·000	−·001	50·0	·0	+1·1
6	5·183	+·003	−·006	50·4	− ·4	+ ·7
11	8·640	+·010	−·003	50·8	− ·8	+ ·3
16	11·481	+·017	+·001	51·2	−1·2	− ·1
21	13·809	+·015	+·004	51·8	−1·1	− ·3
26	15·709	+·011	+·004	52·8	− ·8	− ·3
31	17·252	+·005	+·003	54·5	− ·3	− ·2
Totals	73·031	+·061	+·002	—	—	—

Example (*b*)

Term *n*	4% Annuities $a_{M-n-1:\overline{n}\mid}$			Maturity ages *M*		
	True mean	Deviation in excess of true mean		True mean	Deviation in excess of true mean	
		With A 1924–29 *Z*'s	With A 1949–52 *Z*'s		With A 1924–29 *Z*'s	With A 1949–52 *Z*'s
1	·948	·000	·000	60·0	+·1	+·2
6	5·074	−·009	−·011	59·6	+·5	+·6
11	8·431	−·015	−·019	59·7	+·4	+·5
16	11·222	−·009	−·014	59·9	+·2	+·3
21	13·552	+·004	−·002	60·2	−·1	·0
26	15·489	+·013	+·008	60·4	−·3	−·2
31	17·067	+·013	+·013	61·0	−·3	−·3
36	18·341	+·008	+·008	62·2	−·2	−·2
41	19·350	·000	+·004	64·2	·0	−·1
Totals	109·474	+·005	−·013	—	—	—

Example (*c*)

Term *n*	4% Annuities $a_{M-n-1:\overline{n}\mid}$			Maturity ages *M*		
	True mean	Deviation in excess of true mean		True mean	Deviation in excess of true mean	
		With A 1924–29 *Z*'s	With A 1949–52 *Z*'s		With A 1924–29 *Z*'s	With A 1949–52 *Z*'s
1	·927	−·002	·000	69·5	+ ·6	+·1
6	4·803	−·029	−·008	69·4	+ ·7	+·2
11	7·900	−·064	−·023	69·3	+ ·8	+·3
16	10·523	−·089	−·034	69·3	+ ·8	+·3
21	12·789	−·094	−·032	69·3	+ ·8	+·3
26	14·747	−·076	−·014	69·5	+ ·6	+·1
31	16·421	−·050	+·007	69·7	+ ·4	−·1
36	17·831	−·031	+·018	69·8	+ ·3	−·2
41	18·949	−·020	+·016	70·5	+ ·2	−·2
46	19·823	−·016	+·008	71·8	+ ·2	−·1
51	20·475	−·017	+·008	74·0	+ ·2	−·1
Totals	145·188	−·488	−·054	—	—	—

10 years in the over-all level of maturity ages could necessitate a complete new Z-function if this is constrained to the form $A+Bc^M$, a new valuation constant then being needed for every policy. A general change in the level of maturity ages as much as 10 years seems unlikely to occur, however, before the mortality table becomes out of date anyway.

USE OF OTHER RATES OF INTEREST

12. Table A was derived from annuities at 4%. At 2% the values of $\Delta a_{M-n-1:\overline{n}|}/\Delta a_{M-n-2:\overline{n}|}$ are a trifle smaller, so mean ages obtained with 2% annuities would be a little less than corresponding mean ages obtained with 4% annuities. The Z-function in Table C should, therefore, still be on the safe side when used to estimate mean 2% annuities. That the margin of safety is still reasonable may be seen from Table E, which is an extract from the full table corresponding to example (*b*) of Table D.

Table E

Term n	2% annuities		Maturity ages M	
	True mean	Deviation in excess of true mean with A 1949–52 Z's	True mean	Deviation in excess of true mean with A 1949–52 Z's
6	5·418	−·011	59·6	+·6
21	16·372	−·004	60·1	+·1
36	24·555	+·009	62·1	−·1

13. It seems clear that a change in the valuation rate of interest from 4% to 2% would not require any change in the Z-function for A 1949–52 (ultimate) mortality, and the Z-method would still be practicable. It is likely that with any mortality table if the Z-method were practicable at one rate of interest it would be practicable at another rate, and the same Z-function would do, but this would be readily verifiable in any particular case.

LIDSTONE'S GROUP-CHECK FOR PURE PREMIUMS

14. For a mortality table following Makeham's Law, Lidstone has shown (*J.I.A.* 52, 488) that the pure premiums on a group of endowment assurances all having the same original term can be checked in bulk because the mean maturity age obtained with pure premiums (i.e. with the pure premium rate per cent for the particular term replacing the Z-function as a function of maturity age) should differ only little from the mean maturity age obtained with the Z-function, the difference being an amount which can be expected to vary smoothly as different groups of policies are taken with successive terms.

15. Lidstone was clearly of the opinion that whether a mortality table follows Makeham's law or not, if the Z-method is practicable then his group-check for pure premiums would also be practicable. The correctness of this opinion can be readily demonstrated, as follows:

$$P_{\quad -n-1:\overline{n+1}|}=\frac{1}{\ddot{a}_{M-n-1:\overline{n+1}|}}-d.$$

Therefore $$\Delta P_{M-n-1:\overline{n+1}|} = \frac{1}{\ddot{a}_{M-n:\overline{n+1}|}} - \frac{1}{\ddot{a}_{M-n-1:\overline{n+1}|}}$$

$$= \frac{-\Delta a_{M-n-1:\overline{n}|}}{\ddot{a}_{M-n:\overline{n+1}|} \cdot \ddot{a}_{M-n-1:\overline{n+1}|}}.$$

Similarly, $$\Delta P_{M-n-2:\overline{n+1}|} = \frac{-\Delta a_{M-n-2:\overline{n}|}}{\ddot{a}_{M-n-1:\overline{n+1}|} \cdot \ddot{a}_{M-n-2:\overline{n+1}|}}.$$

Therefore $$\frac{\Delta P_{M-n-1:\overline{n+1}|}}{\Delta P_{M-n-2:\overline{n+1}|}} = \frac{\Delta a_{M-n-1:\overline{n}|}}{\Delta a_{M-n-2:\overline{n}|}} \cdot \frac{\ddot{a}_{M-n-2:\overline{n+1}|}}{\ddot{a}_{M-n:\overline{n+1}|}}.$$

$\Delta P_{M-n-1:\overline{n+1}|}/\Delta P_{M-n-2:\overline{n+1}|}$ is thus usually slightly greater than $\Delta a_{M-n-1:\overline{n}|}/\Delta a_{M-n-2:\overline{n}|}$, but the difference is quite small (for example $\Delta P_{39:\overline{21}|}/\Delta P_{38:\overline{21}|}$ exceeds $\Delta a_{39:\overline{20}|}/\Delta a_{38:\overline{20}|}$ by ·008). Since $\Delta Z(M)/\Delta Z(M-1)$ is on the average also slightly greater than $\Delta a_{M-n-1:\overline{n}|}/\Delta a_{M-n-2:\overline{n}|}$ it is not surprising that the true 'mean' age for pure premiums is close to the mean age obtained with the Z-function.

APPENDIX

A bias in $\Delta Z(M)/\Delta Z(M-1)$ a little on the high side will in practice bring out higher mean ages M_Z. This can be seen as follows by comparing the mean ages M_Z and $M_{Z'}$ produced from two functions $Z(M)$ and $Z'(M)$ given that, for all M,

$$\frac{\Delta Z(M)}{\Delta Z(M-1)} > \frac{\Delta Z'(M)}{\Delta Z'(M-1)}.$$

It is assumed for convenience that $Z(M)$ and $Z'(M)$ are increasing functions of M. A given value of $Z'(M)$ will then determine a unique value of M which in turn will determine a unique value of $Z(M)$. Z can therefore be regarded as a function of Z'.

The inequality can be written

$$\frac{\Delta Z(M)}{\Delta Z'(M)} > \frac{\Delta Z(M-1)}{\Delta Z'(M-1)}.$$

$\Delta Z(M)/\Delta Z'(M)$ is a divided first difference of Z as a function of Z' for the values of Z' corresponding to ages M and $M+1$. These divided first differences therefore increase as successive increasing values of Z' are taken corresponding to successive integral values of M.

Assuming that in practice Z would be a smooth function of Z' it follows that the slope dZ/dZ' would increase with Z'. The curve of Z plotted as a function of Z' would therefore lie above the tangent to the curve at any point, and in particular the tangent at the point $Z'(M_{Z'})$. Let the slope of this particular tangent be

$$\left[\frac{dZ}{dZ'}\right]_{Z'=Z'(M_{Z'})} = k \text{ say.}$$

Then $$[Z(M) - Z(M_{Z'})] > k[Z'(M) - Z'(M_{Z'})], \qquad (1)$$

whether M is greater or less than $M_{Z'}$.

As in §3 let the mean ages be defined with a set of positive weights W_M, such that $\Sigma W_M = 1$, from the equations

$$Z(M_z) = \Sigma W_M Z(M),$$
$$Z'(M_{z'}) = \Sigma W_M Z'(M).$$

Then
$$\begin{aligned} Z(M_z) - Z(M_{z'}) &= \Sigma W_M Z(M) - Z(M_{z'})\Sigma W_M \\ &= \Sigma W_M [Z(M) - Z(M_{z'})] \\ &> k\Sigma W_M [Z'(M) - Z'(M_{z'})] \text{ from (1)} \\ &= k[\Sigma W_M Z'(M) - Z'(M_{z'})\Sigma W_M] \\ &= k[Z'(M_{z'}) - Z'(M_{z'})] \\ &= 0, \end{aligned}$$

and since $Z(M)$ is an increasing function of M, M_z exceeds $M_{z'}$.