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A LINEAR APPROACH TO LOAN AND VALUATION PROBLEMS

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An approach to loan and valuation problems through linear algebra is used to cast commonly applied rules into a general form suitable for routine use under complex conditions. An illustrative example is included.

1 INTRODUCTION

A loan repaid by *nominal instalments* $n_1, n_2, \ldots n_k$ over k years is said to be of *nominal amount* $N=n_1+n_2+\ldots+n_k$. Define the vector $\boldsymbol{n}=(n_1, n_2, \ldots n_k)$ to be the *nominal repayment vector*.

Suppose that in year i interest g_i (or $100g_i\%$) is charged, capital is redeemed at λ_i (or $100\lambda_i\%$), and the *loan repayment*, to cover both interest and capital, is r_i. Let Λ be the diagonal $k\times k$ *capital redemption matrix* with λ_i in the (i, i) position for $i=1, 2, \ldots, k$ and zeros elsewhere, so that the *capital repayment vector* is $\boldsymbol{c}^T=\Lambda\boldsymbol{n}^T$ ($\boldsymbol{n}$ and $\boldsymbol{n}^T$ are identical vectors, but written in row form and column form respectively). Write G for the upper triangular $k\times k$ *interest matrix* with g_i in positions (i, β) for which $i\leqslant\beta$ and zeros elsewhere, and $\boldsymbol{r}$ for the *repayment vector* $(r_1, r_2, \ldots r_k)$.

Then a *loan* can be formally defined as the system $\langle\boldsymbol{n}, \Lambda, G\rangle$, while its structural variables $\boldsymbol{n}$, Λ, G and $\boldsymbol{r}$ are connected by the equation

$$(\Lambda+G)\boldsymbol{n}^T=\boldsymbol{r}^T. \tag{1}$$

Multiplication of (1) by Λ^{-1} casts it in the *normal form*

$$(I+F)\boldsymbol{n}^T=\boldsymbol{q}^T, \tag{2}$$

where I is the $k\times k$ identity matrix, $F=\Lambda^{-1}G$ is the *normal interest matrix* (with entries $f_i=g_i/\lambda_i$ called *normal interest rates*), and $\boldsymbol{q}^T=\Lambda^{-1}\boldsymbol{r}^T$ is the *normal repayment vector* (with components $q_i=r_i/\lambda_i$ called *normal repayments*).

Many structural problems with loans revolve around processing (1) for information about $\boldsymbol{n}$ and/or $\boldsymbol{r}$. Thus one might, for instance, find $\boldsymbol{r}$ given $\boldsymbol{n}$, or, if $\boldsymbol{r}$ were given, attempt to invert $\Lambda+G$ to find $\boldsymbol{n}$ (in the case of constant interest g and redemption at par set $v=(1+g)^{-1}$; then $(\Lambda+G)^{-1}$ is the upper triangular $k\times k$ matrix with entries in the (α, β) position of v if $\alpha=\beta$, $-gv^{\beta-\alpha+1}$ if $\alpha<\beta$, and 0 if $\alpha>\beta$). Once the structure of a loan is known, valuation problems often require breaking repayments into total capital and total interest components.

The Accumulation and following two theorems, proved at the beginning of the next section, conveniently generate much of the information one needs to know about the structure of loans; subsequent results help find its value. Together they enable quite complicated problems to be solved and checked in a routine manner. Moreover, the techniques can be applied to any problem, such as valuation of securities, where the structural variables are connected by an equation like (1). An illustrative example that uses most of the results obtained, appears in the last section.

2 RESULTS

For $i=1,2,\ldots,k$ set $u_i=1+f_i$ [and $v_i=u_i^{-1}=(1+f_i)^{-1}$], and define the *amount* $S_{i=1}^{j}\, q_i$ of j consecutive payments $q_1, q_2, \ldots, q_j$ (some of which may be negative) to be

$$S_{i=1}^{j} q_i = \begin{cases} q_1u_2u_3\ldots u_j + q_2u_3\ldots u_j + \ \ldots\ + q_j & (j \geqslant 1), \\ 0 & (j<1). \end{cases}$$

'Amount' is a generalization of the concept of 'sum', because if $f_i=f$ a constant, and $q_i=1$ for $i=1,2,\ldots,j$, then $S_{i=1}^{j}\, q_i=s_{\overline{j}|}^{f}$.

Our main result is

Theorem 1 (Accumulation Theorem)

$$\sum_{i=1}^{j} n_i = S_{i=1}^{j}\,(q_i-f_iN) \text{ for } j=1,2,\ldots,k. \qquad \square$$

In the case of constant interest g and repayment at par, the theorem reduces to

$$\sum_{i=1}^{j} n_i = S_{i=1}^{j}\,(r_i-gN) \text{ for } j=1,2,\ldots,k$$

which says that if repayments less interest on the full nominal amount are accumulated at interest g, then at the end of each consecutive year the amount accumulated equals capital repayments to date. In other words a loan can be regarded either as reducing, or as a constant interest payment on the whole loan with balance accumulated: at any time the capital repaid under the first formulation equals the amount accumulated under the latter.

For convenience we introduce the notation D, D_L and D_U for the $k\times k$ matrices with, respectively: 1s in all positions; 1s in positions (α,β) for which $\alpha\geqslant\beta$ and 0s elsewhere, making a lower triangular matrix; 1s in positions (α,β) for which $\alpha\leqslant\beta$ and 0s elsewhere, making an upper triangular matrix. Thus, if interest were a constant g, we could write $G=gD_U$.

Proof of Theorem 1

Define the $k\times k$ *accumulation matrix* U to be a lower triangular matrix with entries in the (α,β) position of 0 if $\alpha<\beta$, 1 if $\alpha=\beta$, and $u_{\beta+1}u_{\beta+2}\ldots u_\alpha$ if $\alpha>\beta$. In addition, let F_D be the $k\times k$ diagnonal matrix with f_i in the (i,i) position and 0s elsewhere.

The statement of the theorem in matrix form is

$$D_L\boldsymbol{n}^T=U(\boldsymbol{q}^T-F_DD\boldsymbol{n}^T),$$

and we now prove that. From (2)

$$U(\boldsymbol{q}^T-F_D\,D\,\boldsymbol{n}^T)=U(I+F-F_D\,D)\,\boldsymbol{n}^T.$$

The entry in the (α, β) position in $U(I+F-F_D\,D)$ is the inner product

$(u_2u_3 \ldots u_\alpha, u_3u_4 \ldots u_\alpha, \ldots, u_\alpha, 1, 0, \ldots, 0)(0, \ldots 0, 1, -f_{\beta+1}, -f_{\beta+2}, \ldots, -f_k).^T$

When $\alpha<\beta$ that is 0, when $\alpha=\beta$ it is 1, and when $\alpha>\beta$ it is

$$u_{\beta+1} \ldots u_\alpha - f_{\beta+1}\, u_{\beta+2} \ldots u_\alpha - \ldots - f_{\alpha-1}\, u_\alpha - f_\alpha$$

which, on repeated use of $u_i - f_i = 1$ for $i=\beta+1, \ldots \alpha$, is found to be 1. Hence $U(I+F-F_D\,D)=D_L$ and the result follows. ∎

In the sense that annuities $a_{\overline{n}|}$ and sums $s_{\overline{n}|}$ are dual, Theorem 1 has a dual in Theorem 2 below.

Define the *value* $A^k_{i=j}$ of $(k-j+1)$ consecutive payments $q_j, q_{j+1}, \ldots, q_k$ to be

$$\mathrm{A}^k_{i=j}\, q_i = \begin{cases} v_jq_j + v_jv_{j+1}q_{j+1} + \ldots + v_jv_{j+1} \ldots v_kq_k & (1 \leqslant j \leqslant k), \\ 0 & (j<1 \text{ or } j>k). \end{cases}$$

'Value' is a generalisation of the concept of 'annuity', because if $f_i=f$ a constant, and $q_i=1$ for $i=j, \ldots, k$, then $\mathrm{A}^k_{i=j}\, q_i = a^{f}_{\overline{k-j+1}|}$. So $\mathrm{A}^k_{i=j}\, q_i$ can be regarded as the value at the beginning of the jth period of all subsequent payments (if valuation is at f_i in year i).

The next result states that at any time the nominal amount outstanding equals the present value of all subsequent normal repayments.

Theorem 2 (Amount Outstanding Theorem)

$$\sum_{i=j}^{k} n_i = \mathrm{A}^k_{i=j}\, q_i \text{ for } j = 1, 2, \ldots, k. \qquad \square$$

Proof: Define the upper triangular $k \times k$ *valuation matrix* V to have entries $v_\alpha v_{\alpha+1} \ldots v_\beta$ in the (α, β) position when $\alpha \leqslant \beta$, and 0s elsewhere. The statement of the theorem in matrix form is

$$D_U \boldsymbol{n}^T = V\boldsymbol{q}^T,$$

and we now prove that from (2)

$$V\boldsymbol{q}^T = V(I+F)\boldsymbol{n}^T.$$

The entry in the (α, β) position in $V(I+F)$ is the inner product $(0, \ldots, 0, v_\alpha, v_\alpha, v_{\alpha+1}, \ldots, v_\alpha v_{\alpha+1} \ldots v_k)(f_1, f_2, \ldots, f_{\beta-1}, u_\beta, 0, \ldots, 0)^T$. When $\alpha>\beta$ that is 0, when $\alpha=\beta$ it is 1, and when $\alpha<\beta$ it is $f_\alpha v_\alpha + f_{\alpha+1}v_\alpha v_{\alpha+1} + \ldots + f_{\beta-1}v_\alpha v_{\alpha+1} \ldots v_{\beta-1} + v_\alpha v_{\alpha+1} \ldots v_\alpha u_\beta$ which, on repeated use of $(1+f_i)v_i=1$ for descending $i=\beta-1, \ldots, \alpha$, is found to be 1. Hence $V(I+F)=D_U$, and the result follows. ∎

From Theorems 1 and 2

$$N = \sum_{i=1}^{k} n_i = \mathrm{S}^j_{i=1}\,(q_i - f_i N) + \mathrm{A}^k_{i=j+1}\, q_i \; (j=0, \ldots k), \tag{3}$$

which is a useful result for checking purposes.

For a cumulative sinking fund at constant interest g, redemption at par, and initial rate of sinking z, the amount of the loan is $N = 1$ and repayments are a constant $g + z$. Equation (3) then gives

$$z\, s^{g}_{\overline{j|}} + (g+z)\, a^{g}_{\overline{k-j|}} = 1 \; (j = 1, \ldots, k-1),$$

and, for the endpoints $j = 0$ and k, the two well-known formulae

$$(g+z) a^{g}_{\overline{k|}} = 1 = z\, s^{g}_{\overline{k|}}.$$

The value of any individual n_j can be found from Theorems 1 and 2 in a number of ways, but the most useful formula is

$$n_j = \mathrm{S}^{j}_{i=1}\,(q_i - f_i N) - \mathrm{S}^{j-1}_{i=1}\,(q_i - f_i N) = (q_i - f_j N) + f_j\, \mathrm{S}^{j-1}_{i=1}\,(q_i - f_i N). \quad (4)$$

In many practical problems normal interest rates f_i and repayments q_i are both constant over a period of years, that is, $f_j = f_\alpha$ and $q_j = q_\alpha$ for $j = \alpha, \alpha+1, \ldots, \beta$. Denote *this* sort of useful *period by the semi-open interval* $(\alpha-1, \beta]$ to show that the time under consideration runs from $(\alpha-1)$ exclusive to β inclusive, and let $\gamma = \beta - (\alpha - 1)$ be the *length of the interval.* In these circumstances we can talk about f_j and q_j (or anything else) being constant on the interval $(\alpha-1, \beta]$ with values f_α and q_α respectively.

The next result shows that under these conditions nominal repayments increase geometrically over the interval.

Theorem 3: *If* $\mathrm{f_j}$ *and* $\mathrm{q_j}$ *are constant on* $(\alpha-1, \beta]$, *then* $\mathrm{n_j} = \mathrm{n}_\alpha u^{\mathrm{j}-\alpha}$ *on* $(\alpha-1, \beta]$, *where* $u = 1 + \mathrm{f}_\alpha$ □

Proof: From Theorem 1

$$n_\alpha = \mathrm{S}^{\alpha}_{i=1}\,(q_i - f_i N) - \mathrm{S}^{\alpha-1}_{i=1}\,(q_i - f_i N)$$

while for $j = \alpha + 1, \alpha + 2, \ldots, \beta$

$$n_j = \mathrm{S}^{j}_{i=1}\,(q_i - f_i N) - \mathrm{S}^{j-1}_{i=1}\,(q_i - f_i N)$$

$$= \left[\mathrm{S}^{\alpha}_{i=1}\,(q_i - f_i N) - \mathrm{S}^{\alpha-1}_{i=1}\,(q_i - f_i N) \right] u^{j-\alpha} = n_\alpha u^{j-\alpha}$$ ■

A corollary to Theorem 3 is that the *total nominal amount* $N\,(\alpha-1, \beta]$ *repaid during* $(\alpha-1, \beta]$ is

$$N(\alpha-1, \beta] = n_\alpha (1 + u + u^2 + \ldots + u^{\beta-\alpha}) = n_\alpha (1 - u^\gamma)/(1-u) = n_\alpha\, s^{f_\alpha}_{\overline{\gamma|}} \quad (5)$$

Equation (4), which stems from the Accumulation Theorem, and Theorem 3 conveniently generate all n_j, while equation (5) provides a check because the $N(\alpha-1, \beta]$ must sum to N.

The remaining results of this section dovetail with the results already obtained, which are about the structure of loans, to yield methods of obtaining values of loans.

Suppose valuation is to be at interest ι (iota). Set $(1+\iota)^{-1}=\nu$ (nu), and let (vector nu)

$$\boldsymbol{\nu}=(\nu, \nu^2, \ldots, \nu^k)$$

be the *valuation vector*. (In future, care should be taken not to confuse i (aye) with ι (iota) and v (vee) with ν (nu).) We define the *present value* of a series of payments $\boldsymbol{r}$ to be $\boldsymbol{\nu r}^T$.

A further corollary to Theorem 3 is that the *present value* $M(\alpha-1, \beta]$ of *the nominal amounts* repaid during $(\alpha-1, \beta]$ is

$$M(\alpha-1, \beta]=n_\alpha \nu^\alpha [1+(u\nu)+(u\nu)^2+ \ldots +(u\nu)^{\beta-\alpha}]$$

$$= \left(\text{the present value of } n_\alpha\right) \times \begin{cases}[1-(u\nu)^\gamma]/[1-(u\nu)] & \text{if } f_\alpha \neq \iota, \\ \gamma & \text{if } f_\alpha = \iota\end{cases} \tag{6}$$

where $u = 1 + f_\alpha$.

Similarly, using (1), we define the *present value* A *at interest rate* ι *of a loan* $\langle \boldsymbol{n}, \Lambda, G\rangle$ to be

$$A=\boldsymbol{\nu}\, \boldsymbol{r}^T=\boldsymbol{\nu}(\Lambda \boldsymbol{n}^T)+\boldsymbol{\nu}(G\boldsymbol{n}^T)=K+J, \tag{7}$$

where $K=\boldsymbol{\nu}(\Lambda\boldsymbol{n}^T)=\boldsymbol{\nu}\,\boldsymbol{c}^T$ is the *present value of capital repayments*, and $J=\boldsymbol{\nu}(G\boldsymbol{n}^T)$ is the *present value of the interest payments*. Let $M=\boldsymbol{\nu n}^T$ denote the *present value of the nominal repayments.*

Our first theorem on the value of loans is a restatement of Makeham's Theorem for the case when interest is constant, but the redemption rates may perhaps vary.

Theorem 4 (Makeham's Theorem): The present value A at rate ι of a loan at constant interest g, is

$$A = K+\frac{g}{\iota}(N-M) \qquad \square$$

Proof: From (7) we have merely to show

$$J = \boldsymbol{\nu}(G\boldsymbol{n}^T) = (\boldsymbol{\nu}G)\boldsymbol{n}^T = \frac{g}{\iota}(\mathrm{N}-\mathrm{M}).$$

The β component of the row vector $\boldsymbol{\nu}G$ is

$$g(\nu + \nu^2 + \ \ldots \ + \nu^\beta) = \frac{g}{\iota}(1-\nu^\beta)$$

so that the vector itself is

$$\boldsymbol{\nu}G = \frac{g}{\iota}(\boldsymbol{e}-\boldsymbol{\nu}),$$

where $\boldsymbol{e}$ is the constant k-vector $(1,1, \ldots, 1)$.

Hence $$J = \frac{g}{\imath}(e\,\boldsymbol{n}^T - \boldsymbol{v}\boldsymbol{n}^T) = \frac{g}{\imath}(N - M).$$ ▨

In the case where interest rates g_i vary, however, the interest term in Theorem 4 must be modified. Suppose, as often occurs in practice, that the interest rate g_i is a constant g_α on the interval $(\alpha-1, \beta]$. Let G_α be the interest matrix G with $g_i = g_\alpha$ for $i = \alpha, \alpha+1, \ldots, \beta$ and $g_i = 0$ for $i < \alpha$ or $i > \beta$. The contribution $J(\alpha-1, \beta]$ *from this interval to the present value of the interest payments* J is

$$J(\alpha-1,\beta] = \boldsymbol{v}(G_\alpha \boldsymbol{n}^T) = (\boldsymbol{v}\, G_\alpha)\boldsymbol{n}^T.$$

Inserting a semi-colon after the $(\alpha-1)$ and β components, the row vector $\boldsymbol{v}\, G_\alpha$ can be written as

$$g_\alpha(0, \ldots, 0;\ v^\alpha,\ v^\alpha + v^{\alpha+1}, \ldots, v^\alpha + \ldots + v^\beta;\ v^\alpha + \ldots + v^\beta, \ldots, v^\alpha + \ldots + v^\beta)$$

$$= \frac{v^{\alpha-1} g_\alpha}{\imath}(0, \ldots, 0;\ 1-v,\ 1-v^2, \ldots 1-v^{\beta-\alpha+1};\ 1-v^{\beta-\alpha+1}, \ldots, 1-v^{\beta-\alpha+1})$$

$$= \frac{g_\alpha}{\imath}\left\{\begin{array}{l} v^{\alpha-1}(0, \ldots, 0;\ 1, \ldots 1;\ 1, \ldots 1) \\ \quad -(0, \ldots 0;\ v^\alpha,\ v^{\alpha+1}, \ldots, v^\beta;\ 0, \ldots 0) \\ \qquad - v^\beta(0, \ldots, 0;\ 0, \ldots, 0;\ 1, \ldots, 1) \end{array}\right\}.$$

Hence

$$J(\alpha-1,\beta] = (\boldsymbol{v}\, G_\alpha)\boldsymbol{n}^T = \frac{g_\alpha}{\imath}\left\{v^{\alpha-1}N(\alpha-1,k] - M(\alpha-1,\beta] - v^\beta N(\beta,k]\right\}. \tag{8}$$

The terms in the bracket can generally be calculated easily with equations (5) and (6); they are worth stating in words. The term $v^{\alpha-1} N(\alpha-1,k]$ is the present value of the nominal amount outstanding at the beginning of the period $(\alpha-1,\beta]$; $M(\alpha-1,\beta]$ is the present value of nominal amounts repaid during $(\alpha-1, \beta]$; $v^\beta N(\beta,k]$ is the present value of the nominal amount outstanding at the end of $(\alpha-1,\beta]$.

Thus, if interest and redemption rates vary, Makeham's Theorem becomes

Theorem 5 (Makeham's General Theorem)

$$A = K + \Sigma J(\alpha-1, \beta],$$

where summation is over intervals $(\alpha-1, \beta]$ *on which* g_i *is constant* □

Often each of the individual terms in the statement of Theorem 5 can be calculated directly; seeing if they satisfy Makeham's General Theorem is a useful check.

Loans subject to tax can be valued with the aid of a final theorem

Theorem 6 (Tax Theorem): Suppose a loan is purchased at price pN, and is subject to income tax at rate t (*or* 100t%) *and capital gains tax at rate* τ (*or* 100τ%). *The loan is worth*

$$A = (1-\tau)K + \tau p M + (1-t)J$$ □

Proof: The capital gains tax on capital repayment c_j is

$$\tau\left(c_j - \frac{n_j}{N}pN\right) = \tau c_j - \tau p n_j$$

Hence the loan is worth

$$A = v(c^T - \tau c^T + \tau p n^T) + J - tJ$$

and the result follows. ▨

If the purchase price is not known, a loan can be valued by setting $p = A/N$ in the Tax Theorem. That gives

$$A\left(1 - \tau\frac{M}{N}\right) = (1-\tau)K + (1-t)J. \tag{9}$$

3 AN ILLUSTRATIVE EXAMPLE

The following problem is based on Example 9.7 of Donald's 'Compound Interest and Annuities Certain' where it is described as complicated. Some refinements have been added to make it more complicated, yet still capable of being *solved and checked in a routine manner* with the results of the previous section. Conventional solutions to this type of problem generally seem to involve calculating various interdependent quantities sequentially through the period of the loan. That can increase the chances of a mistake, may be inconvenient to check, and introduces the possibility of roundoff errors.

Problem: A loan of $500,000 is subject to the following conditions:

(i) interest is at 5% per annum convertible quarterly for 15 years, and 4% per annum convertible half-yearly thereafter,
(ii) redemption is at par for the first 10 years, $111\frac{1}{9}$% for the next 5 years, and $114\frac{2}{7}$% thereafter,
(iii) up to $30,000 is available annually to service the loan,
(iv) a special extra payment of $100,000 is required at the end of the 10th year,
(v) income tax is at 40%, and capital gains tax is at 30%.

Find the price to yield 5% per annum effective. □

Solution: Until the last stage of the valuation process we can suppose that the interests g_i are convertible yearly. We first find the period k of the loan and the final repayment r_k. Construct Table 1 listing g_i, λ_i, r_i, f_i, q_i and $q_i - f_i N$ for intervals on which f_i and q_i are constant; note that entries with asterisks anticipate results to be obtained.

Set $k = 15 + z$ and apply the Accumulation Theorem (No. 1), initially assuming that $q_k - f_k N = 8750$. We get

Table 1

$(\alpha-1,\beta]$	(0,9]	(9,10]	(10,15]	(15,28*]	(28*,29*]
g_i	·05	·05	·05	·04	·04
λ_i	1	1	10/9	8/7	8/7
r_i	30,000	130,000	30,000	30,000	3,598·16*
f_i	·05	·05	·045	·035	·035
q_i	30,000	130,000	27,000	26,250	3,148·39*
q_i-f_iN	5,000	105,000	4,500	8,750	—

(* entries with asterisks anticipate results to be obtained.)

$$\begin{aligned}5{,}000\, s^{5}_{\overline{9}|}\,(1{\cdot}05)(1{\cdot}045)^5(1{\cdot}035)^z &+ 105{,}000(1{\cdot}045)^5(1{\cdot}035)^z \\ &+ 4{,}500\, s^{4\cdot5}_{\overline{5}|}\,(1{\cdot}035)^z + 8{,}750\, s^{3\cdot5}_{\overline{z}|} \\ &= 500{,}000\end{aligned}$$

or

$$(1{\cdot}035)^z\left\{\begin{matrix}5{,}000\, s^{5}_{\overline{9}|}(1{\cdot}05)(1{\cdot}045)^5 + 105{,}000(1{\cdot}045)^5 \\ + 4{,}500\, s^{4\cdot5}_{\overline{5}|} + 8{,}750/{\cdot}035\end{matrix}\right\} = 500{,}000 + \frac{8{,}750}{{\cdot}035}.$$

Hence $z=[13{\cdot}12]=14$, and the term of the loan is $k=29$ years.

Now apply equation (4) to find n_1, n_{10}, n_{11} and n_{16}.
Also find n_{29} with the aid of the Accumulation Theorem (No. 1), and then use the last component of equation (1) to get $r_{29}=(8/7+{\cdot}04)n_{29}$, which in turn also gives $q_{29}=\frac{7}{8}r_{29}$. Expressions for these n_i appear below, and their values are displayed in the first row of Table 2; together they give the structure of the loan.

$$\begin{aligned}n_1 &= 5{,}000 \\ n_{10} &= 105{,}000 + {\cdot}05 \times 5000\, s^{5}_{\overline{9}|} \\ n_{11} &= 4{,}500 + {\cdot}045[5{,}000\, s^{5}_{\overline{9}|}\,(1{\cdot}05) + 105{,}000] \\ n_{16} &= 8{,}750 + {\cdot}035[5{,}000\, s^{5}_{\overline{9}|}(1{\cdot}05)(1{\cdot}045)^5 + 105{,}000(1{\cdot}045)^5 \\ &\qquad + 4{,}500\, S^{4\cdot5}_{\overline{5}|}] \\ n_{29} &= N - \sum_{i=1}^{28} n_i = N - s^{28}_{i=1}\,(q_i - f_iN) \\ &= 500{,}000 - [5{,}000\, s^{5}_{\overline{9}|}(1{\cdot}05)(1{\cdot}045)^5(1{\cdot}035)^{13} \\ &\qquad + 105{,}000(1{\cdot}045)^5(1{\cdot}035)^{13} + 4{,}500 s^{4\cdot5}_{\overline{5}|}(1{\cdot}035)^{13} \\ &\qquad + 8750\, s^{3\cdot5}_{\overline{13}|}]\end{aligned}$$

The values of n_1, n_{10}, n_{11}, n_{16} and n_{29} can be checked by calculating $N(\alpha-1,\beta]$ with (5) for each of the five intervals and seeing if their sum is 500,000. On the third interval, for example,

$$N(10{,}15] = n_{11}[1-(1{\cdot}045)^5]/[1-1{\cdot}045] = 64{,}718{\cdot}66.$$

The values of $N(\alpha-1,\beta]$ are displayed in the second row of Table 2, with their (row) sum

$$N = 499{,}999{\cdot}97$$

Table 2

$(\alpha-1, \beta]$	(0,9]	(9,10]	(10,15]	(15,28]	(28,29]	Row Sums
n_α	5,000·00	107,756·64	11,830·03	16,716·28	3,041·92	—
$N(\alpha-1,\beta]$	55,132·82	107,756·64	64,718·66	269,349·93	3,041·92	$N=499{,}999{\cdot}97$
$M(\alpha-1,\beta]$	42,857·14	66,153·23	34,256·06	91,451·27	739·02	$M=235{,}456{\cdot}72$
$K(\alpha-1,\beta]$	42,857·14	66,153·23	38,062·29	104,515·74	844·59	$K=252{,}432{\cdot}99$
$N(\alpha-1,29]$	500,000·00	444,867·18	337,110·54	272,391·88	3,041·92	—
$v^{\alpha-1}N(\alpha-1,29]$	500,000·00	286,765·35	206,956·63	131,025·15	775·97	—
$J(\alpha-1,\beta]$	170,377·51	13,655·49	41,675·42	31,038·33	29·56	$J=256{,}776{\cdot}31$

in the final column. The last and last normal repayments are

$$r_{29}=3598{\cdot}16 \text{ and } q_{29}=3148{\cdot}39$$

respectively.

A check on the value of q_{29} can be made through the Amount Outstanding Theorem (No. 2). The expression

$$\begin{aligned}&30{,}000\, a_{\overline{9}|}^{5}+130{,}000(1{\cdot}05)^{-10}+27{,}000\, a_{\overline{5}|}^{4\cdot5}(1{\cdot}05)^{-10}\\&+26{,}250\, a_{\overline{13}|}^{3\cdot5}(1{\cdot}045)^{-5}(1{\cdot}05)^{-10}\\&+3148{\cdot}39(1{\cdot}035)^{-14}(1{\cdot}045)^{-5}(1{\cdot}05)^{-10}\end{aligned}$$

should equal 500,000, and it does (to two decimal places).

Knowing r_{29} we can, with(7), calculate $A_0=\boldsymbol{v}\boldsymbol{r}^T$, the value of the loan if interests g_j were paid yearly and there were no tax. That will be useful for checking, through Makeham's General Theorem (No. 5), the values, which we will calculate directly, of K and the $J(\alpha-1,\beta]$.

$$\begin{aligned}A_0&=30{,}000\, a_{\overline{28}|}^{5}+100{,}000(1{\cdot}05)^{-10}+3{,}598{\cdot}16(1{\cdot}05)^{-29}\\&=509{,}209{\cdot}30\end{aligned}$$

The value of M, needed when we apply equation (9), is the (row) sum of the $M(\alpha-1,\beta]$ which are displayed in the third row of Table 2, and which are calculated with (6). For example

$$M(10,15]=n_{11}(1{\cdot}05)^{-11}[1-(1{\cdot}045/1{\cdot}05)^5]/[1-1{\cdot}045/1{\cdot}05].$$

That gives $M\,(10,15]=34{,}256{\cdot}06$, and, summing the row

$$M=235{,}456{\cdot}72.$$

The value of K is found by multiplying $M(\alpha-1,\beta]$ by the appropriate λ_j to get the $K(\alpha-1,\beta]$ displayed in the fourth row of Table 2, and adding. Hence $K(10,15]=10/9\; M(10,15]=38{,}062{\cdot}29$, and $K=252{,}432{\cdot}99$.

To find the $J(\alpha-1,\beta]$ (seventh row of Table 2), first calculate the $N(\alpha-1,29]$

(fifth row of Table 2) and the $v^{\alpha-1}\ N(\alpha-1,29]$ (sixth row of Table 2) from the $N(\alpha-1,\beta]$ in the obvious way, and then apply equation (8). For example

$$\begin{aligned} N(10,29] &= N - N(0,9] - N(9,10) = 337{,}110{\cdot}54, \\ v^{10}\ N(10,29] &= (1{\cdot}05)^{-10}\ N(10,29] = 206{,}956{\cdot}63, \text{ and} \\ J(10,15] &= ({\cdot}05/{\cdot}05)\{v^{10}\ N(10,29] - M(10,15] - v^{15}\ N(15,29]\} \\ &= 41{,}675{\cdot}42. \end{aligned}$$

The row sum of the $J(\alpha-1,\ \beta]$ is

$$J = 256{,}776{\cdot}31.$$

As outlined above, the directly calculated values of J and K ought to add up to A_0, and they do (to two decimal places).

The value A of the loan can now be obtained by applying equation (9), but with a slight modification to allow for the interests g_i on the loan not being convertible yearly.

$$A(1-\tau M/N) = (1-\tau)K + (1-t)\{s^{(4)}_{\overline{1}|}{}^{5}\, J(0,15] + s^{(2)}_{\overline{1}|}{}^{5}\, J(15,29]\}$$

or

$$A\left(1 - \frac{{\cdot}3 \times 235{,}456{\cdot}72}{500{,}000}\right) = {\cdot}7 \times 252{,}432{\cdot}99 \ + {\cdot}6\left\{\begin{array}{l} 1{\cdot}01856 \times 225{,}708{\cdot}42 \\ +\,1{\cdot}01235 \times 31{,}067{\cdot}89 \end{array}\right\},$$

giving $$A = 388{,}380{\cdot}69 \text{ (or } 77{\cdot}68\%)$$ ■