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MATCHING

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1. ACTUARIAL PRELIMINARIES

1.1 The basic task of actuarial valuation is to compare the quantity of assets with the quantity of liabilities. A refinement is to compare qualities as well as quantities.

1.2 The qualities of assets and liabilities are their characteristics of cash flow, duration, growth, price volatility, etc. This note considers a conceptual and mathematical framework for matching in the most general terms.

1.3 Insurance work in the United Kingdom uses the notion of reserves for mismatching. Pension fund valuations in the U.K. also tend to use a notion of matching when considering the actuarial value placed on the fund.

2. MATHEMATICAL PRELIMINARIES

2.1 Vector notation will be used to represent expected future cash flows arising from assets and liabilities. Thus vector $\mathbf{l}$ denotes a particular set of liabilities giving rise to net payments l_i in year i. Vector $\mathbf{a}$ represents a particular portfolio of assets giving rise to payments of interest, dividends, rent and capital returns totalling a_i in year i.

2.2 All vectors are taken as length m, e.g. $\mathbf{a}=(a_1 \ldots, a_m)$ meaning cash flow projections for m future years. The rules for handling vectors are summarized as follows.

2.3 *Equality*: $\mathbf{a}=\mathbf{l}$ means $a_i=l_i$ for every i.

If $\mathbf{a}$ is an asset vector and $\mathbf{l}$ a liability vector, $\mathbf{a}=\mathbf{l}$ therefore means that the assets are absolutely matched to the liabilities in their cash flows year by year. Vector equality is equivalent to absolute matching.

2.4 *Addition*: $\mathbf{a}+\mathbf{b}=\mathbf{c}$, where $c_i=a_i+b_i$.

If $\mathbf{a}$ and $\mathbf{b}$ represent two portfolios of assets, then putting the portfolios together produces portfolio $\mathbf{c}$. Similarly liabilities can be added together.

2.5 *Subtraction*: $\mathbf{l}-\mathbf{m}=\mathbf{n}$, where $n_i=l_i-m_i$.

For example if $\mathbf{l}$ represents benefit payments, $\mathbf{m}$ represents future contribution or premium income, then $\mathbf{n}$ represents net liabilities.

2.6 *Multiplication*: $p\mathbf{l}=\mathbf{m}$ where $pl_i=m_i$.

For example multiplying liability vector $\mathbf{l}$ by $p=1{\cdot}10$ is equivalent to raising the liabilities by a uniform 10% on all future cash flows.

2.7 *Scalar multiplication*: $\mathbf{xy} = p$ where:

$$p = x_1 y_1 + \ldots + x_m y_m.$$

For example actuarial valuation can be represented as the scalar multiplication of liability vector $\mathbf{l}$ by a valuation vector

$$\mathbf{v} = (v, v^2, \ldots, v^m).$$

This values payments made annually in arrears:

$$\mathbf{vl} = vl_1 + v^2 l_2 + \ldots + v^m l_m.$$

3. PENSION FUND VALUATIONS

3.1 U.K. practice in pension fund valuation commonly treats the assets in the same way as the liabilities, for consistency,

$$\mathbf{va} = va_1 + v^2 a_2 + \ldots + v^m a_m$$

Thus $\mathbf{va} = \mathbf{vl}$ means that the assets and liabilities are in balance. The quantity:

$\mathbf{va} - \mathbf{vl}$ is the value of any surplus.

Generally, $\mathbf{va}$ differs from market value.

3.2 Note that surplus must be zero with absolute matching because $\mathbf{a} = \mathbf{l}$ implies $\mathbf{va} = \mathbf{vl}$.

This is a justification for valuing the assets at $\mathbf{va}$ instead of market value, which is generalized to the case where there is no absolute match.

3.3 Day-to-day management of a pension fund's investments can change the nature of the portfolio, which might be of significance in the actuarial calculation of $\mathbf{a}$. If portfolio $\mathbf{a}$ is switched in the market to a different portfolio $\mathbf{b}$ at the date of actuarial valuation, $\mathbf{vb}$ may not equal $\mathbf{va}$.

3.4 There is therefore a question of whether to value the actual portfolio or some alternative switched portfolio which is well matched to the liabilities. For example, an absolute matching portfolio might be an excellent benchmark for actuarial valuation, though it is seldom attainable in practice. This suggests the desirability of a criterion for matching the liabilities which is attainable and which points to the absolute matching portfolio when there is one.

4. INVESTMENT POLICY FOR PENSION FUNDS

4.1 The basic problem of investment policy for U.K. pension funds is asset allocation—the proportions of the fund to be invested in the major asset classes. The most common investment system is balanced management, where the fund manager ensures a reasonable spread across the asset classes.

4.2 Asset allocation tends to be similar for most funds because:

(1) liability profiles have been similar for most funds;
(2) the investment performance statistics of pension funds are generally

compared with median fund returns; an unusual asset allocation produces volatile returns relative to the median;

(3) volatility of investment returns, whether measured absolutely or relative to the median benchmark, is widely equated to investment risk;

(4) investment policy should seek maximum returns consistent with an acceptable level of risk, and is therefore directed in part to reducing risk.

4.3 On the other hand:

(1) there is now greater variation in the liability profiles of U.K. pension funds, so it is less appropriate to measure investment performance relative to the median of all such funds;

(2) the pursuit of investment policies directed towards the policies of other major participants in the market may possibly be responsible for instabilities in the market;

(3) there are different interpretations of investment risk; volatility of returns either absolutely or relative to median statistics may not be the appropriate sort of risk to be minimized.

4.4 A natural response to these criticisms is to decide a long-term asset allocation having regard to the nature of the liabilities, and to use this instead of the median fund as the benchmark for assessing investment performance.

4.5 Relating the long term investment strategy to the liabilities therefore suggests the idea of a matching portfolio—perhaps similar to that used in the actuarial valuation.

5. GENERAL CRITERIA FOR MATCHING

5.1 We now postulate five basic criteria as to what might be meant in the most general terms by matching assets to liabilities, with the foregoing applications in mind.

(1) *Completeness*: for any given liabilities a matching portfolio can be found.

(2) *Uniqueness*: for any given liabilities there is only one matching portfolio.

(3) *Absolute matching*: if the given liability cash flows can be matched by a portfolio absolutely year by year, then the matching portfolio is that portfolio.

(4) *Scaling*: if the liability cash flows are scaled up by a common factor, then the matching portfolio is scaled up by the same factor.

(5) *Combination*: if two sets of liabilities are combined, then the matching portfolios add together at least in volume terms.

5.2 These five criteria translate into mathematics as follows.

(1) *Completeness and* (2) *Uniqueness*: for any $\mathbf{l}$ there is a unique portfolio $\mathbf{a}$. This relationship defines a mapping $\mathbf{M}$ from the liability vectors to the asset vectors: $\mathbf{lM} = \mathbf{a}$.

(3) *Absolute matching*: for any $\mathbf{l}$ if $\mathbf{a}$ is an asset vector such that $\mathbf{a}=\mathbf{l}$, then $\mathbf{lM}=\mathbf{a}$. Equivalently, $\mathbf{aM}=\mathbf{a}$ for every asset vector.

(4) *Scaling*: for any $\mathbf{l}$ and any scaling factor p:

$$(p\mathbf{l})\,\mathbf{M}=p\,(\mathbf{lM})$$

(5) *Strict combination*: for any two sets of liabilities $\mathbf{l}$ and $\mathbf{m}$:

$$(\mathbf{l}+\mathbf{m})\mathbf{M}=\mathbf{lM}+\mathbf{mM}$$

Note that this rule of combination is stronger than saying that the matching portfolios add just in volume terms. The strict rule is relaxed later when considering portfolios in which negative asset holdings are disallowed.

5.3 Criteria (4) and (5) specify $\mathbf{M}$ as a linear mapping which can be represented as an $m \times m$ matrix.

$$\text{e.g. } (\mathbf{lM})_i = \sum_{j=1}^{m} l_j M_{ji}$$

So all five criteria reduce to a general notion of matching represented by a linear mapping $\mathbf{M}$ such that $\mathbf{aM}=\mathbf{a}$ for all asset vectors.

6. GENERAL SOLUTION

6.1 This mathematical formulation of the five postulates for matching yields a general solution. Suppose that S_1, S_2 .... S_n are distinct basic assets for constructing portfolios. Any portfolio allowed in the actuarial model can be regarded as a combination of the base assets thus:

$$x_1\,S_1+x_2\,S_2+\ldots+x_n\,S_n.$$

6.2 Define the $n \times m$ matrix $\mathbf{E}$ such that E_{ij} is the expected cash flow from asset S_i in year j. The general solution is as follows.

6.3 If $\mathbf{M}$ is any matching of assets to liabilities as in § 5.3 then $\mathbf{M}=\mathbf{D}\,(\mathbf{ED})^{-1}\,\mathbf{E}$ for some $m \times n$ matrix $\mathbf{D}$.

(Note that $\mathbf{ED}$ is an $n \times n$ matrix of which $(\mathbf{ED})^{-1}$ is the matrix inverse.)

6.4 The general solution involves only the two matrices $\mathbf{D}$ and $\mathbf{E}$, where $\mathbf{E}$ is just the matrix of projected cash flows from the base assets. $\mathbf{D}$ is as yet unspecified, because in § 5.1 we postulated only the most general criteria for matching.

6.5 Consider the $m \times n$ matrix $\mathbf{D}$ in terms of n column vectors $\mathbf{d}_i$ thus:

$$\mathbf{D}=[\mathbf{d}_1,\,\mathbf{d}_2,\,\ldots,\,\mathbf{d}_n]$$

From § 6.3 it is seen that $\mathbf{MD}=\mathbf{D}$ and so

$$(\mathbf{lM})\,\mathbf{d}_i=\mathbf{l}\,\mathbf{d}_i \text{ for every } i.$$

So **D** may be thought of as n distinct vectors **d**, each of which has the property that for any liabilities **l** and matching portfolio **a**:

$$\mathbf{ad} = \mathbf{ld}$$

6.6 The general solution is therefore that any matching of the type specified in Section 5 is equivalent to n equalities of scalar products: $\mathbf{ad} = \mathbf{ld}$. Different values of **d** are equivalent to different sorts of matching. Many sorts of matching are therefore theoretically possible, subject to the consideration of risk.

7. RISK

7.1 The purpose of matching is to reduce investment risk relative to the liabilities. Therefore the nature of the matching must depend on the nature of the risk which is to be reduced.

7.2 Risk in uncertain economic conditions is to do with the emergence of a deficiency of assets relative to the liabilities, measured over a suitable period. The key variable is therefore the surplus (i.e. deficiency if negative) as calculated either at a future actuarial valuation, or after expiry of all liabilities. Risk is a function of this future surplus, and the minimization of this risk will define a particular sort of matching.

7.3 Much of modern portfolio theory is based on consideration of the expected return and the variance of returns on a portfolio of assets. This approach can be extended to consideration of the expected surplus and variance of surplus relative to liabilities. Liability-related investment risk can be equated to variance of surplus: minimizing the variance of surplus subject to expected surplus being zero gives a workable definition of matching.

7.4 In practice it is usual to impose the extra constraint that a matching portfolio cannot include 'negative holdings' of any assets. At this point the strict rule of combination in § 5.2 must be relaxed. The practical solution of a mean/variance problem with non-negativity constraint may be achieved by quadratic programming techniques. Sample calculations yield consistent results which demonstrate matching so far as possible of all the relevant characteristics: notably cash flow, duration and growth factors.

7.5 The variance of surplus is an easy risk measure to minimize, but other functions of future surplus may provide a more suitable measure of risk for the purpose of matching. Examples are:

(a) the probability of there being a deficiency of at least X; or
(b) the size of deficiency which, except with probability p, will not be exceeded.

7.6 Over long periods the probability distribution of investment returns tends to be skewed in the direction of surplus because the potential investment loss is limited to the capital invested. If risk is equated with variance of surplus, an investment strategy could be labelled as risky just because of the 'risk of large surplus'. The alternative measures of risk mentioned before may be more appropriate for this reason, but in practice they seem unlikely to yield to computation except for the simplest economic models.

8. MIS-MATCHING

8.1 Whatever the definition of risk, investment policy is normally directed to maximizing returns for an acceptable level of risk, not minimizing risk irrespective of the expected returns. Since the latter objective leads to the matching portfolio, this may in fact be an inappropriate benchmark for the purposes set out in Sections 3 and 4.

8.2 Instead it is reasonable to consider the 'efficient frontier' of alternative portfolios, each of which minimizes risk subject to a given rate of expected return.

8.3 The efficient frontier is notably well defined when (a) risk is equated to variance of surplus and (b) expected return is seen in terms of the price of the portfolio which is expected to meet the liabilities. In this situation the proportions of any efficient portfolio in the base assets are given by the simple relationship:

$$x_i = x_i^0 + v\, z_i$$

where x_i^0 is the holding of base asset S_i in the matching portfolio, and $-v$ is the gradient of variance of surplus with respect to the price of the portfolio.

8.4 The parameter v can be regarded as an alternative measure of risk. When $v=0$ the result is the matching portfolio. When v is increased the portfolio departs from the position of minimum risk but the price of the resulting portfolio reduces.

8.5 When negative holdings are disallowed, the calculation of asset proportions again becomes a problem of quadratic programming. In this situation there is a portfolio of maximum risk and minimum price, namely a holding of that base asset S_j which yields the highest expected return. The efficient frontier can be traced out along a line of alternative portfolios between the matching portfolio at one extreme and the asset which offers maximum risk, maximum expected return at the other.

8.6 Between the twin extremes of matching the liabilities and of maximum mis-matching, there are likely to be portfolios which offer a reasonable balance between risk and return. In practice it is found that the calculation of these intermediate portfolios is sensitive to the assumptions adopted as to factors of risk and return. It follows that in seeking to employ these ideas, for example in helping to decide investment policy, care is needed in setting the assumptions, in considering the effects of changes in the assumptions, as well as in considering the relevant meaning of investment risk as noted in Section 7.

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