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THE MEASUREMENT OF REPRODUCTIVITY

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THE measurement of the rate of population growth has attracted considerable attention in scientific literature of recent years. This is, no doubt, a result of the continual decline in the birth-rate which has formed the topic of innumerable articles in the popular press and elsewhere. Several attempts have been made to obtain a simple statistical measure of the reproductivity of a population at a particular time—that is, a measure of the extent to which a population will be replacing itself if current fertility and mortality continue indefinitely. It is the aim of this paper,

in section 1, to discuss the simple approximate formulae that have been suggested;
in section 2, to discuss some more elaborate and more efficient formulae;
in section 3, to analyse the effect on the formulae of section 2 of a change in the proportions married at a given age;
in section 4, to outline the male versus female rate anomaly;
in section 5, to suggest a formula which avoids the anomaly; and finally,
in section 6, to discuss the application of these formulae to Australian population statistics.

1. SIMPLE APPROXIMATE FORMULAE

1·1. *Crude birth- and death-rates*

Vital statisticians were at first satisfied to study the excess of the crude birth-rate over the crude death-rate. With the marked changes in age structure which resulted from decreasing mortality and fertility, it soon became apparent that this measure was not suitable for comparing the rates at which different populations (which includes the same population at different periods) were reproducing themselves. The crude birth-rates, for example, of two 'equally fertile' populations would be quite different if they had different proportions of women in the reproductive age-group.

1·2. *Standardized birth- and death-rates*

A more satisfactory measure of the same type which would make some allowance for the age distribution is the difference between the standardized birth- and death-rates. The standardized death-rate is the crude death-rate of a standard population experiencing at the various ages the rates of mortality of the population under consideration. This measure will depend partly on the standard population selected, but it is also subject to the more serious objection explained by C. D. Rich (1) as follows: Suppose the number of deaths above the reproductive age increased and those below that age reduced so that the standardized death-rate is not altered. Even though a larger number of children would then survive to the reproductive ages and eventually bear children, this measure of reproductivity is unchanged and yet there is no doubt that the rate of growth of such a population would, on our supposition, be increased.

Fairly extensive basic data—forces of mortality and fertility—are required to determine this (and for that matter any) measure of reproductivity. In addition to the two objections mentioned above, we may, in this connexion, add a third—this measure does not make optimum use of these basic data. For a measure which fulfils this requirement see paragraph 2·2.

1·3. *Replacement Index*

The replacement index was introduced by W. S. Thompson and later used by Lorimer and Osborne in their book *The dynamics of population growth.* It is the most useful of the simple reproductivity formulae, and requires only a knowledge of the population in age-groups and the corresponding life table. The replacement index has three useful forms. They are all particular cases of the general formula which is obtained by dividing the number of children in a given age-group in the actual population by the number of women in the actual population who would have been in the reproductive age-group when these children were born, and then dividing this quotient by the corresponding quotient in the life-table population.

The three useful forms of the replacement index are:

(I) J_1, obtained by using the children under age 5 (say) and the females aged 20–45 in the above formula;

(II) J_2, obtained by using the annual births and the females aged 20–45 in the formula; and

(III) J_3, obtained by using any group of children (say ages 10–14) and the corresponding females aged 30–55.

A. J. Lotka (2) has shown that J is related to other reproductivity formulae ρ, the true rate of natural increase (see paragraph 2·2) and R_0, the net reproduction rate (see paragraph 2·1) by the following approximate formulae:

$$\log_e J \doteqdot (\alpha_2 - \alpha_1)\rho \quad \text{and} \quad J \doteqdot R_0^{\frac{\alpha_2-\alpha_1}{\alpha}},$$

where α_1 and α_2 are the average ages of the junior and senior groups respectively and α is the mean age of net reproductivity of women.

The following remarks about this index (for details see Lotka (2)) should be noted:

(I) Even substantial changes in fertility and mortality cause little alteration in the means α_1, α_2 and α, and hence $(\alpha_2-\alpha_1)/\alpha$ is approximately constant for all conditions met with in practice. Furthermore, numerically it is approximately unity, and hence, as a rough first approximation, for all populations, we may take $J = R_0$. In any case, we may compare R_0 for various populations by comparing the corresponding J's.

(II) If we determine J_3 (above) for various age-groups from the data of a single year (e.g. Australia 1939 as in Table 8) we obtain replacement indices for earlier years, the value of J_3 obtained by using the junior age-group 10–14 giving the replacement index 10 years earlier, and so on. For more remote years the measure is only rough, the factor of immigration alone being sufficient to cause material discrepancies.

(III) It is simple to calculate and is the only useful index when age specific fertility is not available. If these rates are available a superior measure should be used.

(IV) The effect of altering the age limits of the junior group (say from under 5 to under 3) has an effect which, though not marked, is not negligible. The replacement index is not, therefore, a unique measure of net fertility.

(V) It is not naturally related to general population analysis and tells us little besides the rate of population increase.

2. SOME MORE EFFICIENT FORMULAE

2·1. *Net reproduction rate* R_0

This measure was originally introduced by R. Boeckh in his study of the 1879 Berlin population, and it has been extensively used since by R. R. Kuczynski and others. It consists simply of the ratio of births of a given sex (usually female) in two successive generations under constant conditions of forces of mortality $\mu(x)$ and fertility $f(x)$:

$$R_0 = \int_0^\infty \frac{l_x}{l_0} f(x)\,dx. \tag{1}$$

The population will increase, remain stationary, or decrease according as R_0 is greater than, equal to, or less than, unity.

Lotka (2) has shown that R_0 is approximately given, in a community growing slowly by natural increase, by the ratio of total annual births at two epochs of time t and $t-\alpha$, α years apart, where α is (as above) the mean age of net reproductivity of women.

R_0 is not an annual rate of increase, but a rate of increase per unit of time which here is the 'generation'. This varies slightly from one population to another, and hence a more satisfactory measure (ρ) converting this to an annual basis was introduced.

2·2. *True rate of natural increase*

In 1911 Sharpe and Lotka (3) proved that a population subject to a given age schedule of mortality and fertility, no matter what its original age distribution, will eventually approach a stable distribution with a fixed annual rate of increase ρ.* In 1925 Dublin and Lotka (4) showed that ρ, which satisfies the integral equation

$$\int_0^\infty e^{-\rho x} \frac{l_x}{l_0} f(x)\,dx = 1, \tag{2}$$

may be obtained with sufficient accuracy by solving the quadratic

$$\frac{1}{2}\left(\frac{R_1^2}{R_0^2} - \frac{R_2}{R_0}\right)\rho^2 + \frac{R_1}{R_0}\rho - \log_e R_0 = 0, \tag{3}$$

where
$$R_n = \int_0^\infty x^n \frac{l_x}{l_0} f(x)\,dx.$$

* See also C. D. Rich (1) pp. 44, 45 and 74–77, and p. 43 for formula (3).

By fitting a Pearson Type III curve to the net fertility function $(l_x/l_0)f(x)$, S. D. Wicksell (5) obtained the following formula, which gives identical results in practice:

$$\rho = \frac{R_0 R_1}{R_0 R_2 - R_1^2} \{ R_0^{(R_0 R_2 - R_1^2)/R_1^2} - 1 \}. \tag{4}$$

It has been shown (e.g. Rhodes (6)) that $\rho \gtreqless 0$ according as $R_0 \gtreqless 1$.

ρ is a true annual rate of increase which makes optimum use of the given data. It occupies a central position in general population analysis.

2·3. *Gross reproduction rate*

Mention only should be made of the gross reproduction rate which is the total of the age fertility schedule or $\int_0^\infty f(x)\,dx$. It provides an upper limit to R_0 when mortality has improved to such an extent as to be negligible.

2·4. *Properties of these reproduction rates*

It should be remembered that, other things being equal, if all persons were to die after passing the reproductive ages, although the expectation of life and the age distribution of the population would be altered, the above three measures of reproductivity would not change.

Each of these measures can be used to estimate the effect of different rates of mortality, fertility remaining constant, or of various fertility rates with the same mortality. They can therefore be used to measure separately the effect on reproductivity of the declining mortality and the declining fertility of the present century.

Given age schedules of mortality and fertility for one sex, we can determine not only R_0 and ρ, but also many other characteristics of the ultimate population which it is interesting to compare with the present population. We could thus determine the ultimate age distribution, the true birth- and death-rates, the distribution of daughters' ages for given mother's age, the average age of daughters for given age of mother, the proportion of daughters of a given age whose mothers are alive or the proportion of female maternal orphans. Given further data our knowledge of the ultimate population could be extended.

These measures of reproductivity, by ignoring duration of marriage, yield misleading results if, for any reason, marriage conditions are abnormal. A sudden temporary increase in the number of marriages would, because of the high fertility of early married life, result in increased births in the following few years if marriage fertility remained constant. If we use these increased births to determine age specific fertility without allowing for the abnormal marriages (as in the case of R_0 and ρ) we are overestimating the reproductivity because we are, in effect, assuming that the high marriage rates will continue indefinitely. In many cases this is theoretically impossible as it may lead to the assumption that more females are married than actually exist! Two measures suggested with a view to overcoming this difficulty, which has no doubt occurred in most civilized countries during the last 15 years because of the deferred depression marriages and accelerated war-time marriages, will be outlined in the next paragraphs. The effect of varying marriage rates on the several measures will be more closely examined in section 3.

2·5. *Karmel formula*

Allowance can be made for the inflated births resulting from an abnormal number of marriages of short duration by using a formula based on current birth-rates, as a function of duration of marriage, combined with normal marriage-rates to be expected in the future.

A practical difficulty arises here in that the necessary data to determine birth-rates in the required form may not be available. With Australian data we have to be content to relate the annual births for the year under investigation, divided according to marriage duration (r), to the annual marriages recorded in the returns of previous years. This ratio b_r (say), which in a modified form was first used by P. H. Karmel (7) in determining his 'index of current marriage fertility', makes no allowance for discontinuance of marriage due to divorce or death of either spouse, and is not a birth-rate in the true sense. While it does not give all the information we would like, it can be used to give the Karmel measure of reproductivity K_0 (say) corresponding to R_0 if we know the proportion m_y of females aged y who marry at that age. Thus

$$K_0 = \int_0^\infty \frac{l_y}{l_0} m_y \, dy \sum_0^\infty b_r = \sum_0^\infty b_r \frac{\int_0^\infty l_y m_y \, dy}{l_0}. \tag{5}$$

There are three obvious methods of determining m_y namely:

(I) from the marriages during the year being investigated;

(II) from the proportion of females married at age y at the end of the year being investigated; or

(III) by using typical average values of the proportions mentioned in (II), such, for example, as the proportions during a period of normal marriages or the proportions at a previous census.

Method (I) should not be used because of the large marriage fluctuations mentioned in paragraph 2·4 which could give values of m_y such that

$$\int_0^\infty l_y m_y \, dy > l_0,$$

which is impossible. If we use method (II) during a period of increased marriages (e.g. war-time) it may be unlikely that such high proportions married will be maintained. However, whether it will be maintained or not is beside the point; it certainly *can* be and as it is an indication of recent trends it probably should be used. For many countries the proportions never married will only be available at census dates and method (III) will have to be used. If, as is now the case in Australia, the previous census was taken at a time when these proportions were abnormally low, then, with this method, reproductivity would throughout be underestimated.

The following points regarding this measure should be noted:

(I) migration affects b_r, particularly for large values of r—births are included from marriages contracted outside and vice versa;

(II) illegitimate births, though small in proportion, have to be allowed for. A simple sufficiently accurate method is to increase the results obtained in the ratio of total births to legitimate births;

(III) changes in the proportion of remarriages would slightly alter the index because of their lower fertility, resulting probably from the higher marriage age;

and, most important of all,

(IV) no allowance is made for the age at marriage. The number of children per family depends to a large extent on age at marriage (see Table 1) and hence, if the average age at marriage is changing appreciably, fertility must be considered as a function of age at marriage.

Table 1. Marriage fertility rates in each quinquennial age-group based on all legitimate births registered in 1944—Queensland

(Data from Clark and Dyne (8), p. 32)

Calendar year of marriage	Births per 1000 marriages at age of mother at marriage							
	−19	20–24	25–29	30–34	35–39	40–44	45–	All ages
1944	140	76	53	52	50	21	—	78
1943	490	364	315	265	208	70	—	346
1942	278	250	224	186	126	23	6	229
1941	275	252	228	200	97	22	—	229
1940	281	236	211	177	93	10	—	215
1939	248	217	200	166	63	6	—	199
1938	213	186	183	119	27	—	—	170
1937	200	176	153	101	38	—	—	156
1936	174	160	135	85	32	6	—	139
1935	159	150	123	48	7	—	—	119
1934	153	129	92	39	7	—	—	106
1933	135	117	88	30	5	—	—	103
1932	108	98	64	23	—	—	—	85
1931	148	89	52	11	5	—	—	78
1930	123	77	43	7	—	—	—	72
1929	112	69	33	4	—	—	—	58
1928	96	58	20	4	—	—	—	44
1927	80	46	16	—	—	—	—	34
1926	76	36	5	—	—	—	—	29
1925	67	30	2	2	—	—	—	26
1924	68	25	3	—	—	—	—	20
1923	54	18	1	—	—	—	—	15
1922	55	9	—	—	—	—	—	10
1921	27	7	—	—	—	—	—	7
1920	21	4	—	—	—	—	—	5
1919	24	2	—	—	—	—	—	3
1918	8	2	—	—	—	—	—	2
1917	7	—	—	—	—	—	—	1
1916	5	—	—	—	—	—	—	1
1915	7	1	—	—	—	—	—	1
Total	3832	2884	2244	1519	758	158	6	2580

2·6. *Clark-Dyne formula*

Clark and Dyne (8) suggested a modification of the Karmel formula to take marriage age into consideration and hence to correct for (III) and (IV) above. They obtained b_r for various ages y (or age-groups) at marriage (say) ${}_yb_r$ and hence obtained a measure of reproductivity C_0, corresponding to R_0 and K_0, given by

$$C_0 = \int_0^\infty \frac{l_y}{l_0} m_y \sum_0^\infty {}_yb_r \, dy. \tag{6}$$

Table 2. Percentages of females married at a given age
(Data from Clark and Dyne (8), p. 33)

Age (x)	20	25	30	35	40	45
Queensland 1938 (p_x)	16·0	51·3	72·8	81·6	86·0	87·2
Queensland 'war-time' (P_x)	17·3	64·0	80·3	84·5	86·0	87·3

Table 3. Calculation of C_0 and γ using Queensland 1944 fertility and Australian 1933 census mortality

(1) Age at marriage (y)	(2) $1{,}000 \times \frac{l_y}{l_0}$	(3) Annual marriages	(4) Legitimate births from (3)	(5) Legitimate births (4) according to marriage duration: –4	5–9	10–14	15–19	20–24	25–29	(6) Age at confinement (x)	(7) Annual legitimate births	(8) $x \times (7)$	(9) $x^2 \times (7)$
–19	937·04	149·89	574·4	219·4	149·0	100·0	64·6	33·7	7·6	$19\frac{1}{2}$	219·4	4,279	83,441
20–24	928·88	332·58	959·2	391·8	295·7	169·6	79·5	21·0	1·7	$24\frac{1}{2}$	540·8	13,249	324,591
25–29	917·89	215·97	484·6	222·7	171·5	73·2	16·4	·9	—	$29\frac{1}{2}$	618·4	18,240	538,074
30–34	905·27	102·33	155·4	90·1	53·1	11·3	1·0	—	—	$34\frac{1}{2}$	495·8	17,104	590,081
35–39	890·31	61·47	46·6	35·3	10·3	1·0	—	—	—	$39\frac{1}{2}$	274·8	10,855	428,757
40–44	872·68	32·32	5·3	4·9	·4	—	—	—	—	$44\frac{1}{2}$	71·5	3,180	141,510
										$49\frac{1}{2}$	5·0	246	12,157
			2225·5								2225·7*	67,153	2,118,611

Total female births $= 2225{\cdot}5 \times 1{\cdot}077 \times {\cdot}487 = 1167{\cdot}26$,
$C_1/C_0 = 30{\cdot}174$, $C_2/C_0 = 951{\cdot}979$, $\log_e C_0 = {\cdot}154659$.
Therefore $-20{\cdot}754\,\gamma^2 + 30{\cdot}174\,\gamma - {\cdot}154659 = 0$.

Hence $\begin{cases} C_0 = 1{\cdot}167, \\ \gamma = {\cdot}514\,\%\ \text{p.a.} \end{cases}$

* The small discrepancy in the total is caused by rounding off.

This formula involves more calculation and requires data which are not published for many countries. Its undoubted superiority over R_0 and K_0 under abnormal marriage condition will be demonstrated in section 3.

2·7. *Corresponding annual rates*

C_0 and K_0 are rates of increase per 'generation' and not annual rates of increase. The corresponding annual rates, which are more useful, will be denoted, following the precedent of R_0 and ρ, by the corresponding small Greek letters γ and κ respectively. Let C_n and K_n be the moments corresponding to R_n. The births for each age at confinement x, which in the case of R_0 are given by $(l_x/l_0)f(x)$, are for C_0 given by

$$\sum_{y=0}^{x} \frac{l_y}{l_0} m_{y\,y} b_{x-y}.$$

Having determined these births, we obtain C_n by taking moments and then γ by substituting these values of C_n for R_n in equation (3) and solving.

2·8. *Numerical calculation*

The calculation of C_0 and γ is illustrated in Table 3.

Column 3 is obtained from the survivors in column (2) and the proportions p_x of Table 2 with a small addition for remarriages. Columns (4) and (5) result from applying the marriages of column (3) to the fertility rates of Table 1. The total of column (4) adjusted for illegitimate and female births gives C_0. Adding columns (5) diagonally gives column (7). The totals of columns (7), (8) and (9) enable us to write down the equation for γ. The grouping for ages and durations is broad, and, for more accurate results, narrower groups or individual durations should be used.

The calculation of K_0 and κ proceeds in the same way.

In the determination of R_0 and ρ (for several examples, see Dublin and Lotka (9)) we obtain column (7) immediately from column (2) by multiplying by $f(y)$ and then continuing in the same way, using y (column (1)) for the age at confinement.

3. THE EFFECT ON REPRODUCTIVITY FORMULAE OF CHANGES IN THE PROPORTION MARRIED AT AGE x

[*Note.* The reader interested only in results could proceed immediately from paragraph 3·1 to paragraph 3·8]

3·1. Periods of economic depression or years of war are examples of 'events' which either accelerate or postpone the marriages of a community. Even if exactly the same pairings are made at a later date and these pairings have the same number of children large variations will occur in the annual births. False impressions of the fertility trend and the reproductivity trend will result if these birth variations are not correctly interpreted. It is the aim of this section, therefore, to investigate the effect of a change in the proportion of females married at a given age on the reproductivity formulae of the previous section, assuming that the fertility, measured according to age at marriage and duration of marriage, remains constant throughout. This investigation will yield some indication of the efficiency of the various reproductivity formulae, and, with respect to a given formula, will at the same time indicate conditions under which it may be expected to underestimate or overestimate the reproductivity.

3·2. Let us assume that the population being considered has resulted from constant annual births throughout the past and is subject to fixed mortality; and let us denote the total females at age x by l_x. Let the proportion of females aged x who are married be represented at time t by $\pi(x, t)$, or more briefly π_x.

Then, for a given value of t, the marriages between ages x and $x+dx$ less the married deaths between these ages equals

$$l_{x+dx}\pi(x+dx, t+dx) - l_x\pi(x, t).$$

This may be written $$dx\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial t}\right) l_x\,\pi(x, t),$$

that is, $$dx\left(l_x\frac{\partial \pi_x}{\partial t}+l_x\frac{\partial \pi_x}{\partial x}+\pi_x\frac{\partial l_x}{\partial x}\right).$$

Since the last term, with its sign changed, equals the married deaths between ages x and $x+dx$, the number of marriages between x and $x+dx$, for a given value of t, must be given by

$$l_x\left(\frac{\partial \pi_x}{\partial t}+\frac{\partial \pi_x}{\partial x}\right)dx. \tag{7}$$

This result could have been immediately written down.

For a given t, then, the number marrying between x_1 and x_2 is

$$\int_{x_1}^{x_2} l_x\left(\frac{\partial \pi_x}{\partial t}+\frac{\partial \pi_x}{\partial x}\right)dx. \tag{8}$$

Integrating by parts, this becomes

$$\frac{\partial}{\partial t}\int_{x_1}^{x_2} l_x\pi_x\,dx + l_{x_2}\pi_{x_2} - l_{x_1}\pi_{x_1} - \int_{x_1}^{x_2}\pi_x\frac{\partial l_x}{\partial x}\,dx$$

$$= \frac{\partial}{\partial t}\int_{x_1}^{x_2} l_x\pi_x\,dx + l_{x_2}\pi_{x_2} - l_{x_1}\pi_{x_1} + \int_{x_1}^{x_2}\pi_x l_x \mu_x\,dx. \tag{9}$$

If we assumed π_x to change gradually with time in accordance with an inverse tangent function from a steady value of p_x to a steady value of P_x, a given percentage of the change (say 95 %) occurring over n years, then, by fitting polynomials to l_x, p_x, P_x and the marriage-age marriage-duration birth function, we could, making use of expression (9), obtain an analytic expression for the births at time t since $\int \tan^{-1} kt\,dt$, $\int t\tan^{-1} kt\,dt$, etc. are integrable. This method, however, has no advantages over the straightforward method discussed in the next paragraph, particularly when the effect of varying π_x on several formulae is required.

3·3. Let ${}_{x_1x_2}b_r$ be the chance that a female marrying between ages x_1 and x_2 has a child during the rth calendar year after marriage;

Σ indicate summation over all marriage age-groups;

S indicate summation for all values of r;

$B_{x_1x_2} = S_{x_1x_2}b_r$.

We will proceed to determine the annual births in a community subject throughout to Australian 1933 census female mortality ($A^{F^{33}}$) and, for $t>0$, to the marriage-age marriage-duration fertility rates given in Table 1 (i.e. Queensland 1944), and in which $\pi_x = p_x + (P_x - p_x)\phi_t$, where ϕ_t is a function of t only.

We assume that the fertility in the past was such that the population just replaced itself.

The values chosen for p_x and P_x in our examples are given in Table 2 and are respectively the Queensland 1938 and Queensland 'war-time' proportions. The figures are thus not exaggerated but are selected from actual experience.

From (9), for given t, the marriages between ages x_1 and x_2 are

$$A + B\phi_t + \frac{\partial \phi_t}{\partial t}\int_{x_1}^{x_2} l_x (P_x - p_x)\,dx = A + B\phi_t + C\phi_t' \quad \text{(say)},$$

where
$$A = l_{x_2} p_{x_2} - l_{x_1} p_{x_1} + \int_{x_1}^{x_2} p_x l_x \mu_x \,dx,$$

and
$$A + B = l_{x_2} P_{x_2} - l_{x_1} P_{x_1} + \int_{x_1}^{x_2} P_x l_x \mu_x \,dx.$$

Using values for single ages from the $A^{F^{33}}$ tables and the Newton-Cotes formula for approximate integration for 5 intervals we obtain the values for A, B and C given in Table 4.

Table 4. Marriages per age-group at time t

Age-group	$A+B\phi_t+C\phi_t'$
–19	$14{,}969 + 1{,}214\,\phi_t + 400\,\phi_t'$
20–24	$32{,}796 + 10{,}613\,\phi_t + 44{,}313\,\phi_t'$
25–29	$19{,}749 - 4{,}768\,\phi_t + 48{,}292\,\phi_t'$
30–34	$7{,}978 - 4{,}169\,\phi_t + 22{,}030\,\phi_t'$
35–39	$3{,}919 - 2{,}581\,\phi_t + 6{,}474\,\phi_t'$
40–44	$1{,}136$
Total	$80{,}547 + 309\,\phi_t + 121{,}509\,\phi_t'$

We obtain the total marriages during the calendar year t to $t+1$ by integrating the marriages at time t from t to $t+1$. If we write

$$Q_t = \int_t^{t+1} \phi_t\,dt \quad \text{and} \quad Q_t' = \int_t^{t+1} \phi_t'\,dt,$$

the total marriages during calendar year t to $t+1$ are given, for all integral values of t, by

$$\sum_{x_1 x_2} (A + BQ_t + CQ_t') \tag{10}$$

and the total annual births during year r by

$$\sum_{x_1 x_2} \underset{t}{S} (A + BQ_t + CQ_t')_{x_1 x_2} b_{r-t}. \tag{11}$$

3·4. We will now consider several forms for ϕ_t.

(I) *A permanent increase in* π_x. Let π_x increase steadily over n years from constant values of p_x to constant values of P_x following a cosine blending function, the time variation in π_x being given by

$$\left.\begin{array}{lll} t<0, & \phi_t=0, & \phi_t'=0, \\ 0\leqslant t\leqslant n, & \phi_t=\frac{1}{2}-\frac{1}{2}\cos\frac{t\pi}{n}, & \phi_t'=\frac{\pi}{2n}\sin\frac{t\pi}{n}, \\ t>n, & \phi_t=1, & \phi_t'=0. \end{array}\right\} \tag{12}$$

Writing q for the values of Q in (10) which apply to this example, we obtain

$$\begin{array}{lll} t<0, & q_t=0, & q_t'=0, \\ 0\leqslant t\leqslant n, & q_t=\frac{1}{2}-\frac{n}{2\pi}\left(\sin\frac{t+1}{n}\pi-\sin\frac{t}{n}\pi\right), & q_t'=\frac{1}{2}\cos\frac{t}{n}\pi-\frac{1}{2}\cos\frac{t+1}{n}\pi, \\ t>n, & q_t=1, & q_t'=0. \end{array}$$

Substituting these values in (10) we obtain the initial annual marriages ΣA in marriage age-groups, and the subsequent extra annual marriages

$$\Sigma\,(Bq_t+Cq_t')$$

shown for $n=6$ in Table 5.

Table 5. Annual marriages during a change in proportions married from p_x to P_x over 6 years

Age-group	Initial annual marriages	Extra annual marriages during year t to $t+1$ where t is						
		0	1	2	3	4	5	$\geqslant 6$
–19	14,969	54	255	551	861	1,103	1,212	1,214
20–24	32,796	3,202	9,710	15,030	17,748	17,140	13,348	10,613
25–29	19,749	3,131	8,121	10,298	9,078	4,785	−1,434	−4,768
30–34	7,978	1,383	3,402	3,958	2,890	485	−2,600	−4,169
35–39	3,919	377	797	658	—	−1,007	−2,090	−2,581
40–44	1,136	—	—	—	—	—	—	—
Total	80,547	8,147	22,285	30,495	30,577	22,506	8,436	309

Applying these marriages to the fertility rates of Table 1 we obtain the initial annual births and the subsequent extra annual births

$$\sum_{x_1 x_2}\mathop{\mathrm{S}}_{t}\,(Bq_t+Cq_t')_{x_1x_2}b_{r-t}.$$

The calculation of the latter function is shown for illustration for selected years in Table 6.

(II) *A permanent decrease in* π_x. Let π_x decrease steadily over n years from constant values P_x to constant values p_x with the same blending function. In this case, the values of ϕ_t are obtained, for all values of t, by subtracting those in (12) from unity and the values of ϕ_t' are equal in magnitude to those in (12) but are opposite in sign.

Hence, in this case,

$$Q_t=1-q_t \quad\text{and}\quad Q_t'=-q_t'.$$

Table 6. Entries (for selected years) in the working sheet for calculating the extra annual births following an increase over 6 years from p_x to P_x of Table 2

Marriage age-group	Marriage year	Extra births during year t to $t+1$ where t equals: 0	2	4	6	9	12	15	19	23	27
–19	0	8	15	15	12	9	6	6	4	1	—
	1	—	125	70	63	44	34	31	19	14	2
	2	—	77	153	155	[illegible]10	84	82	44	30	13
	3	—	—	422	237	[illegible]83	137	93	83	59	18
	4	—	—	154	307	[illegible]74	192	149	124	74	30
	5	—	—	—	594	[illegible]41	242	185	149	92	67
	⩾ 6	—	—	—	170	1,[illegible]36	2,337	2,983	3,642	4,141	4,463
20–24	0	243	801	756	596	[illegible]80	314	221	96	22	—
	1	—	3,534	2,447	2,107	1,[illegible]54	1,136	748	350	87	19
	2	—	1,142	3,758	3,547	2,[illegible]45	1,939	1,338	691	271	30
	3	—	—	6,460	4,472	3,[illegible]01	2,662	1,739	1,029	444	71
	4	—	—	1,303	4,285	3,719	2,742	2,005	1,183	515	120
	5	—	—	—	4,859	3,[illegible]50	2,349	1,722	1,028	481	120
	⩾ 6	—	—	—	807	9,[illegible]97	16,779	21,937	26,533	29,189	30,345
25–29	0	166	701	661	573	[illegible]85	200	103	6	—	—
	1	—	2,558	1,852	1,624	1,[illegible]96	715	349	41	—	—
	2	—	546	2,307	2,173	1,[illegible]76	947	535	165	10	—
	3	—	—	2,860	2,070	1,[illegible]61	1,117	581	182	27	—
	4	—	—	254	1,072	[illegible]57	646	421	158	10	—
	5	—	—	—	–452	–[illegible]03	–219	–132	–62	–7	—
	⩾ 6	—	—	—	–253	–3,[illegible]10	–6,743	–8,703	–10,115	–10,648	–10,701
30–34	0	72	257	245	165	66	32	6	3	—	—
	1	—	902	680	565	[illegible]89	102	24	—	—	—
	2	—	206	736	701	[illegible]00	154	44	—	—	—
	3	—	—	766	578	[illegible]44	139	66	12	—	—
	4	—	—	25	90	81	41	14	2	1	—
	5	—	—	—	–689	–[illegible]60	–263	–101	–18	—	—
	⩾ 6	—	—	—	–217	–2,[illegible]31	–4,857	–5,832	–6,262	–6,323	–6,331
35–39	0	19	48	35	10	3	—	—	—	—	—
	1	—	166	77	50	26	4	—	—	—	—
	2	—	33	83	61	25	5	3	—	—	—
	3	—	—	—	—	—	—	—	—	—	—
	4	—	—	–50	–127	–63	–32	–5	—	—	—
	5	—	—	—	–435	–194	–79	–15	—	—	—
	⩾ 6	—	—	—	–129	–1,241	–1,714	–1,913	–1,957	–1,957	–1,957
Total		508	11,111	26,069	29,641	25,[illegible]50	21,148	18,684	17,130	16,533	16,309

Substituting in (11), the total annual births during year r are

$$\sum_{x_1 x_2\, t} \mathrm{S}(\mathrm{A}+\mathrm{B}-\mathrm{B}q_t-\mathrm{C}q'_t)_{x_1 x_2} b_{r-t}.$$

We may therefore find the total annual births by subtracting from the initial births $\sum_{x_1 x_2} (\mathrm{A}+\mathrm{B})\,\mathrm{B}_{x_1 x_2}$ the subsequent extra annual births determined for case (I) above.

(III) *A temporary increase in* π_x. Let π_x increase in n years from p_x to P_x according to the same blending formula, remain constant at P_x for m years and then return symmetrically to p_x over the next n years.

The values of ϕ_t for this case are obtained, for all values of t, from those in (12) by subtracting the value there at time $t-m-n$ from the value at time t. The values of ϕ'_t are obtained from those in (12) in the same way.

Hence, in this case, for all values of t

$$\mathrm{Q}_t = q_t - q_{t-m-n} \quad \text{and} \quad \mathrm{Q}'_t = q'_t - q'_{t-m-n}.$$

Substituting in (11) the total annual births during year r are

$$\sum_{x_1 x_2\, t} \mathrm{S}(\mathrm{A}+\mathrm{B}[q_t-q_{t-m-n}]+\mathrm{C}[q'_t-q'_{t-m-n}])_{x_1 x_2} b_{r-t}$$
$$= \sum_{x_1 x_2\, t} \mathrm{S}(\mathrm{A}+\mathrm{B}q_t+\mathrm{C}q'_t)_{x_1 x_2} b_{r-t} - \sum_{x_1 x_2\, t} \mathrm{S}(\mathrm{B}q_{t-m-n}+\mathrm{C}q'_{t-m-n})_{x_1 x_2} b_{r-t}.$$

The total annual births during year r may therefore be obtained for this case by subtracting from the total annual births during year r of case (I) above the extra annual births there obtained for the year $m+n$ years earlier.

The result of these examples can be confirmed by general reasoning.

(IV) The effect of a temporary decrease in π_x can be determined in the same manner.

3·5. From (1) $\mathrm{R}_0 = \sum \frac{l_x}{l_0} \frac{\text{Female births to females aged } x}{\text{Total females aged } x}$.

Now in the particular case we are considering (defined at the beginning of paragraph 3·2) the births for $t<0$ have been constant and equal to l_0. Therefore for $t<15$ the total females aged x equals l_x for the reproductive ages, and therefore for $t<15$ R_0 equals the total annual female births divided by l_0.

The number of annual accessions at age 0 is, for $t>0$, different from l_0 because of the altered fertility and marriage intensity after $t=0$. If, however, in a particular year the annual entrants are kl_0 instead of l_0, and also if the marriage intensity and marriage fertility are constant, then, x years later the number of females aged x will be kl_x instead of l_x and the annual female births during that year will increase in the same proportion. We can see from the above formula therefore that, for $t>15$, when births will occur to the additional births in Table 6, we may determine R_0 by neglecting these second generation of births and at the same time assuming l_0 entrants throughout. The reason is, of course, that in this case the numerator and denominator in the above expression vary in proportion. We have therefore shown that R_0 for all values of t is equal to the total annual female births (excluding births to the additional births of Table 6) divided by l_0.

3·6. The Karmel index of current marriage fertility for a given year may be obtained by dividing the total births for given marriage durations (irrespective of marriage age) by the corresponding marriages and by summing for all durations. Since on our assumption the population we are considering is experiencing the fertility of Table 1, to determine the total births for a given marriage duration r it is necessary to multiply the marriages at the various ages that occurred r years ago by the marriage fertility for duration r for the various ages at marriage given in Table 1. However, since we know the proportion of the marriages in a given year that occurred in the various age-groups, we can determine the contribution to the Karmel index for a given year of marriages of duration r, by weighting the fertility of Table 1 for duration r by the proportion of marriages at the various ages that actually occurred r years ago. This will give us the marriage fertility for duration r that would have been obtained in our given population if we had ignored age at marriage altogether. Adding for all durations of marriage gives the Karmel index for the given year.

A change in the actual annual births will alter this index as it will alter, in subsequent years, the proportions marrying at various ages. (These proportions are used for weighting the fertility rates for a given duration of marriage.) It will be necessary therefore to make allowance, for $t > 20$, for the additional marriages resulting from the increased births following $t = 0$.

3·7. We obtain the Clark-Dyne formula at time t by weighting the values of $B_{x_1 x_2}$, not by the marriages at age x_1 to x_2 actually taking place at time t, but by the marriages that would be taking place in that age-group if the values of π_x had always been the same as at that particular time. That is, it is given by

$$\sum_{x_1 x_2} (A + B\phi_t)\, B_{x_1 x_2}.$$

3·8. Graphs have been drawn (see Figs. 1–3) showing the variations which occur in (I) the net reproduction rate, (II) the Karmel formula, and (III) the Clark-Dyne formula, in a community built up from constant annual births in the past and subject throughout to A^{F33} mortality and, for $t > 0$, to the fertility of Table 1. The large variations shown result from the one factor which was allowed to vary with time—the proportion of females married at a given age. The limits of the variation (given in Table 2) are selected from actual experience and are the Queensland 1938 and Queensland 'war-time' proportions. The graphs show the effect of (I) a permanent increase in the proportions married from p_x to P_x taking place gradually (see (12)) over n years ($n = 3$, 6 and 12), and (II) a temporary increase from p_x to P_x over n years ($n = 3$, 6 and 12) at the same rate, remaining constant at P_x for m years ($m = 0$ and 6) returning symmetrically to p_x over a further n years.

(I) *Permanent increase*. The following points may be noted:

(*a*) The number of annual marriages after the proportions have changed is practically the same as before the change, the increased reproductivity which occurs being due almost entirely to the higher fertility (see Table 1) associated with the younger marriage ages after the change.

(*b*) To build up the higher proportions married the number of marriages occurring each year during the change-over is considerably higher than before or after. For the case $n = 6$ the number of marriages rises to a value 37·5% above normal for the third year during the change. The net reproduction rate makes no allowance for these abnormal marriages. In effect, it assumes they will continue indefinitely. This may be an impossible assumption, as mentioned previously, leading to more females being married than actually exist. The

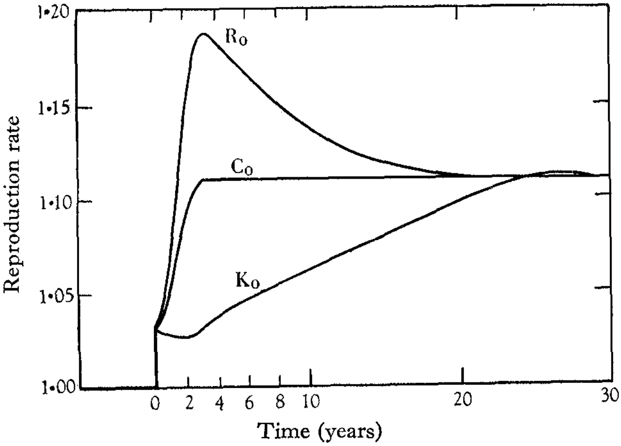

Fig. 1 *a*. The effect on R_0, C_0 and K_0 of an increase in the proportion of females married at a given age from p_x to P_x (Table 2) spread over 3 years.

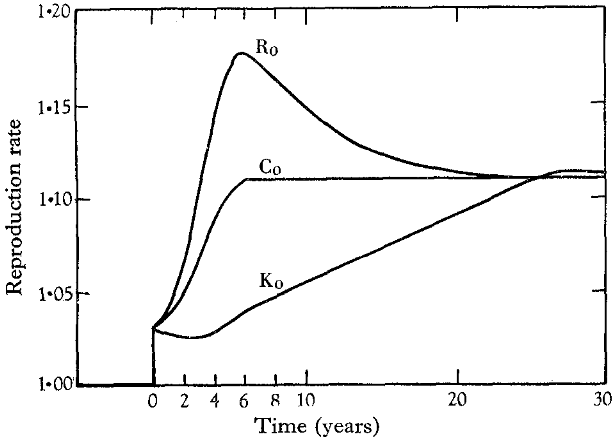

Fig. 1 *b*. The effect on R_0, C_0 and K_0 of the Fig. 1 *a* increase spread over 6 years.

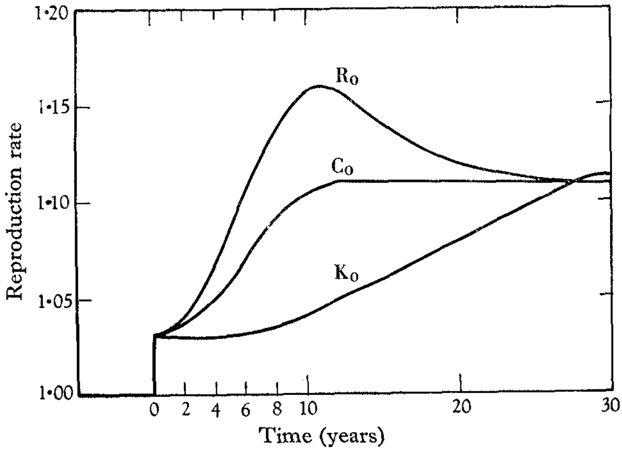

Fig. 1 *c*. The effect on R_0, C_0 and K_0 of the Fig. 1 *a* increase spread over 12 years.

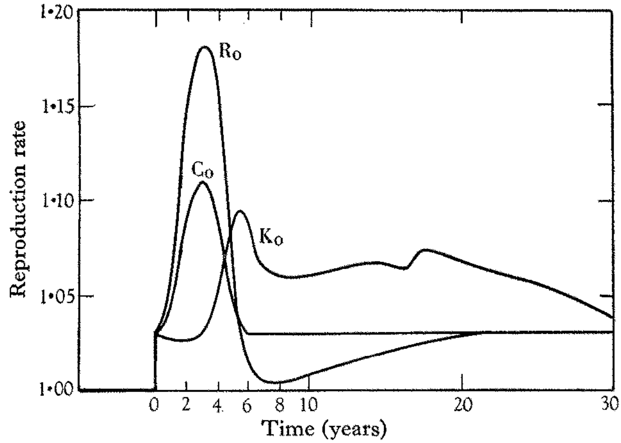

Fig. 2*a*. The effect on R_0, C_0 and K_0 of an increase in the proportion of females married at a given age from p_x to P_x (Table 2) spread over 3 years followed immediately by a symmetrical return to p_x over 3 years.

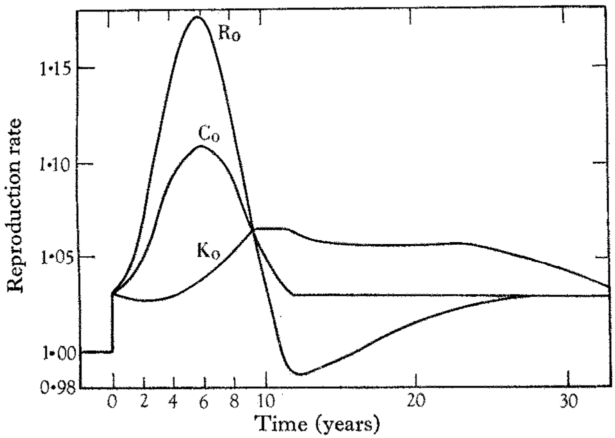

Fig. 2*b*. The effect on R_0, C_0 and K_0 of the Fig. 2*a* increase spread over 6 years followed immediately by a symmetrical return to p_x over 6 years.

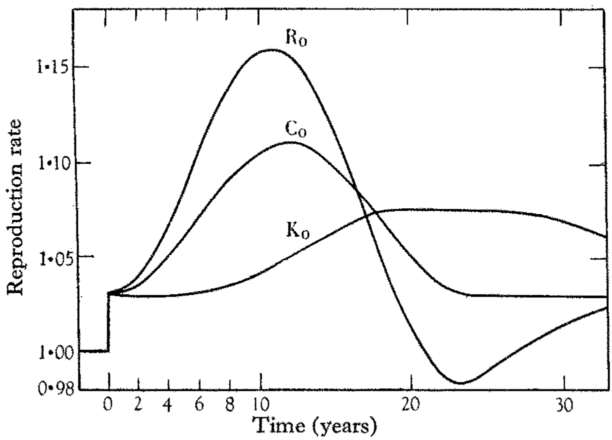

Fig. 2*c*. The effect on R_0, C_0 and K_0 of the Fig. 2*a* increase spread over 12 years followed immediately by a symmetrical return to p_x over 12 years.

births which result from these abnormal marriages, particularly in the following few years, cause the net reproduction rate grossly to overestimate the reproductivity. R_0 reaches its peak at the moment the community attains the final

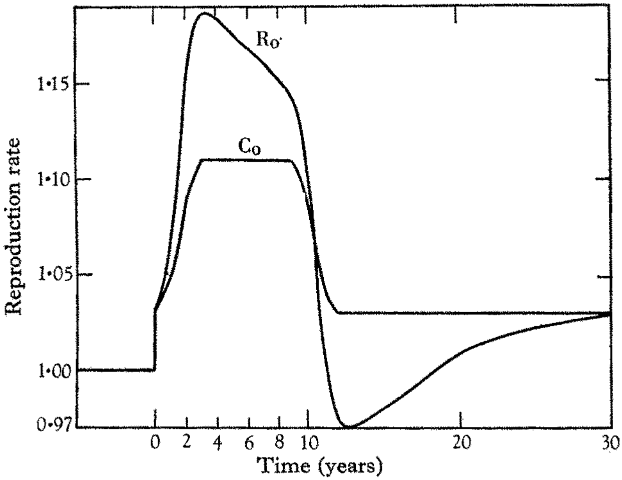

Fig. 3*a*. The effect on R_0 and C_0 of an increase in the proportion of females married at a given age from p_x to P_x (Table 2) spread over 3 years followed, after remaining constant at P_x for 6 years, by a symmetrical return to p_x over 3 years.

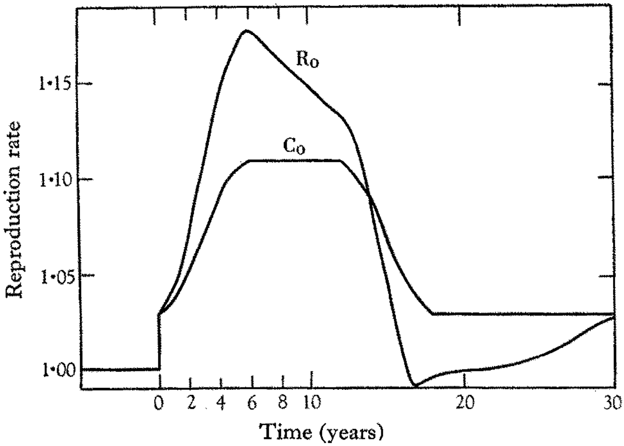

Fig. 3*b*. The effect on R_0 and C_0 of the Fig. 3*a* increase spread over 6 years followed, after remaining constant at P_x for 6 years, by a symmetrical return to p_x over 6 years.

proportions married, receding to the correct figure as the births from the abnormal marriages of the change-over become negligible. The extent of the error depends on the rapidity of the change.

(*c*) To build up the higher proportions married, the marriages in the first few years of the change-over occur relatively in the higher age-groups. As the proportions married are built up, fewer and fewer marriages are required in the older age-groups and more and more in the younger (see Table 5). During the first few years of the change-over the average age of marriage therefore actually increases, but soon falls rapidly back past the initial figure to the lower average age commensurate with the higher proportions married. The Karmel formula which depends on the sum of the average number of births to marriages of each previous year, will therefore fall during the first few years of the change-over and rise fairly sharply during the later years of the change-over. It thereafter increases almost linearly towards the correct figure as more and more of the years with the higher proportions married and therefore with lower marriage ages are included. After about 25 years marriages of the children born during the change-over begin to take place. At first they are only in the younger age-groups. The Karmel formula then slightly overestimates the reproductivity but gradually approaches the correct value.

(*d*) The Clark-Dyne formula is not unduly biased by the sudden increase in births and it gives an indication of the changed reproductivity immediately. The other formulae give a totally false picture for about 20 years.

(II) *Permanent decrease*. The reasoning here is the same as in the previous case. The Clark-Dyne formula is again accurate; the Karmel formula overestimates and the net reproduction rate underestimates for about 20 years.

(III) *Temporary increase*. The reasoning here can be deduced from the case above and will not therefore be given. The following features of the curves should be noted:

(*a*) R_0 overestimates the peak;

(*b*) even though the community is always more than replacing itself, R_0 passes through a dip with values less than unity;

(*c*) K_0 is too low during the change-over and then overestimates for many years after;

(*d*) C_0 is satisfactory.

(IV) *Temporary decrease*. The same discussion applies as in (III) with the curves roughly inverted.

(V) *General remarks*. It would appear from the above that more attention should be paid than has been done in the past to movements in the proportions married at a given age. Variations in R_0 cannot be correctly assessed unless viewed in the light of these additional data (see section 6). There seems to be a good case for publishing annual births according to mother's age at marriage and year of marriage so that C_0 may be determined. If this information were available, variations in the proportions married which are generally of a temporary nature could be ignored and typical values used throughout.

4. MALE *v*. FEMALE REPRODUCTIVITY FORMULAE

One of the most serious objections which can be levelled against all the formulae of section 2 has not been mentioned so far. It will be considered in some detail in the following paragraphs.

The formulae of section 2 are based on a determination of the rate at which a given sex is replacing itself. For various reasons (shorter reproductive period makes for shorter calculation; the required data are more often available for the female sex; illegitimate births are easily referable to the mother, etc.) the female sex is commonly used. There is no reason, of course, why the male sex should not be used as a basis. It is here, however, that the anomaly arises. In practice, for reasons discussed below, two quite different values are obtained for a given measure of reproductivity using the two sexes. If these two values of ρ for males and for females were to continue until stable conditions eventually emerged one sex would in course of time swamp the other. The conception of two stable populations, one for males and one for females, with different values of ρ is therefore untenable. The population as a whole, and both sexes in particular, must therefore ultimately increase at the same rate, which presumably would lie somewhere between the values obtained for the separate sexes. Unfortunately this frequently leaves a large range (for example, see section 6) anywhere within which the required value might fall and seriously detracts from the value of the method. Translated into other terms, the anomaly means that it is an impossible hypothesis to assume that the rates of fertility obtained for each sex can continue indefinitely in the future.

To assist us, in a particular case, in deciding where, within the range bounded by the values of ρ for the separate sexes, the true rate of increase lies, we will consider briefly some possible reasons for the difference between the male and female rates. R. J. Myers (10) elaborates these reasons with some actual figures.

(I) If there is a temporary excess of females (e.g. as a result of the ravages of war) at the reproductive ages, age specific fertility rates and the net reproduction rate for women will be relatively low as compared with those for men.

(II) Excess female over male immigration at the reproductive ages, if females are already well represented, is unlikely appreciably to increase the number of births. This would lower the female rate relative to the male.

(III) The tendency for females over 30 to underestimate, and females under 20 to overestimate, their age tends to lower the computed female net reproduction rate.

(IV) Because husbands are, on the average, about 5 years older than their wives the supply of husbands will tend to fall the more rapidly the population is increasing, and hence the female rate will become smaller relative to the male rate the more rapid the population increase.

Male rates have generally been found to be appreciably higher than female and hence, in view of the discussion above, it might well be that some of the pessimistic discussion in demographic literature, caused by the fall of the female net reproduction rate below unity, may not be well founded.

The serious theoretical difficulty of the male *v.* female rates discussed in this section, and the serious practical difficulty of having an inherent rate of increase only known to lie within two (perhaps) widely separated limits, amply justify considerable further investigation whether a unique index of reproductivity can be found. An index which can easily be calculated, for which data are readily available, which is theoretically unique and which lies between the male and female rates will now be discussed.

5. JOINT RATE OF INCREASE

5·1. *Aim*

The aim of this section is to outline the properties of an index of reproductivity which has all the advantages of those previously discussed and yet which does not suffer from their main weakness—the anomaly of section 4.

5·2. *Basic data*

The basic data required for the determination of this index consist of the probability at birth that a male will have a female child between ages x and $x+dx$ (written $\phi(x)\,dx$) and the probability at birth that a female will give birth to a male child between ages y and $y+dy$ (written $\xi(y)\,dy$). For almost all countries, the annual male and female births are published according to age of mother, and, for most countries, according to age of father also. If these data are not published they are of such a simple nature that they can readily be obtained from the birth records. Combining these fertility data with mortality gives the net fertility functions $\phi(x)$ and $\xi(y)$.

The theory which, in parts, resembles that applied by Rhodes (6) to R_0 and ρ will be developed in the next few paragraphs and the results obtained summarized in paragraph 5·13. Readers only interested in results could, therefore, turn immediately to that paragraph.

5·3. *Theory*

The female births $F(t)$ and the male births $M(t)$ at time t are given by

$$F(t)=\int_0^\infty M(t-x)\,\phi(x)\,dx, \tag{13}$$

and

$$M(t)=\int_0^\infty F(t-y)\,\xi(y)\,dy. \tag{14}$$

Hence, we have

$$F(t)=\int_0^\infty\int_0^\infty F(t-x-y)\,\phi(x)\,\xi(y)\,dx\,dy, \tag{15}$$

$$M(t)=\int_0^\infty\int_0^\infty M(t-x-y)\,\phi(x)\,\xi(y)\,dx\,dy, \tag{16}$$

and thus the total births

$$B(t)=\int_0^\infty\int_0^\infty B(t-x-y)\,\phi(x)\,\xi(y)\,dx\,dy. \tag{17}$$

The last three equations are of the same form and hence so also will be their solution. We can see at once that equation (17) will be satisfied by a function of the form

$$B(t)=\sum_n B_n e^{s_n t}.$$

Substituting in (17) we find that that equation is satisfied by this form if the values of s are given by

$$\int_0^\infty\int_0^\infty e^{-(x+y)s}\,\phi(x)\,\xi(y)\,dx\,dy=1. \tag{18}$$

This equation (18) is obtained whether we are solving (15), (16) or (17) and hence the values of s obtained apply to male, to female and to total births.

5·4. This equation has only one real solution; for if we suppose s to be real and denote the left-hand side of (18) by f then

$$\frac{df}{ds}=-\int_0^\infty\int_0^\infty (x+y)\,e^{-(x+y)s}\,\phi(x)\,\xi(y)\,dx\,dy.$$

Now since $\phi(x), \xi(y), e^{-(x+y)s}$ and $x+y$ are greater than or equal to zero, df/ds must always be negative for all values of s. Hence $f=1$ can have only one real solution σ (say).

5·5. If $\sigma=0$, then $$\int_0^\infty \int_0^\infty \phi(x)\,\xi(y)\,dx\,dy=1.$$

If $\sigma>0$, then $$e^{-\sigma(x+y)}<1,$$

and hence, from (18) $$\int_0^\infty \int_0^\infty \phi(x)\,\xi(y)\,dx\,dy>1.$$

Similarly, if $\sigma<0$, $$\int_0^\infty \int_0^\infty \phi(x)\,\xi(y)\,dx\,dy<1.$$

Therefore $\sigma \gtreqless 0$ according as

$$\int_0^\infty \int_0^\infty \phi(x)\,\xi(y)\,dx\,dy \gtreqless 1.$$

This latter expression which will be denoted by S_0 is analogous to the net reproduction rate and can be used as a measure of reproductivity which is independent of sex. It will be called the 'joint reproduction rate'. It is a rate of increase using as unit of time the total male and female 'generation'. It cannot, therefore, be compared directly with the net reproduction rate or other rates and is not therefore recommended.

5·6. If $s=u+iv$ is a complex root of (18), then, substituting and equating real and imaginary parts

$$\int_0^\infty \int_0^\infty e^{-(x+y)u}\cos\{(x+y)v\}\,\phi(x)\,\xi(y)\,dx\,dy=1,$$

and $$\int_0^\infty \int_0^\infty e^{-(x+y)u}\sin\{(x+y)v\}\,\phi(x)\,\xi(y)\,dx\,dy=0.$$

Therefore $u-iv$ is also a root.

Comparing the former of these equations with the real solution of (18) we have, since $\cos(x+y)v<1$,

$$e^{-(x+y)u}>e^{-(x+y)\sigma}.$$

Therefore u, the real part of any imaginary root, is less than the real root σ.

Combining conjugate complex roots we may express the solution of (17) as

$$B(t)=B_0e^{\sigma t}+\sum_n e^{u_n t}(\alpha_n \sin v_n t+\beta_n \cos v_n t),$$

which, since $u_n<\sigma$, tends to $B_0 e^{\sigma t}$ as t becomes large.

We have thus proved that the total births, and may similarly prove that the male and the female births, of a community subject to the net fertility of 5·2 all ultimately increase at an annual rate of σ. σ will be called the 'joint rate of natural increase'.

5·7. σ which from (18) is given by

$$\int_0^\infty e^{-\sigma x}\phi(x)\,dx \int_0^\infty e^{-\sigma y}\xi(y)\,dy=1 \tag{19}$$

may be determined by two methods, corresponding to the method of Lotka (4) and that of Wicksell (5) for determining the true rate of natural increase.

Denoting $\int_0^\infty e^{-\sigma x}\phi(x)\,dx$ by z, we have $dz/d\sigma = -Cz$,

$$\text{where}\quad C = \frac{\int_0^\infty xe^{-\sigma x}\phi(x)\,dx}{\int_0^\infty e^{-\sigma x}\phi(x)\,dx} \tag{20}$$

$$= \frac{M_1 - \sigma M_2 + \frac{\sigma^2}{2}M_3 - \dots}{M_0 - \sigma M_1 + \frac{\sigma^2}{2}M_2 - \dots}, \quad \text{where}\quad M_n = \int_0^\infty x^n \phi(x)\,dx$$

$$= a + b\sigma + c\sigma^2 + \dots, \quad \text{where}\quad a = \frac{M_1}{M_0},\quad b = \frac{M_1^2}{M_0^2} - \frac{M_2}{M_0},\quad \text{etc.}$$

This series has been shown by Lotka to converge very rapidly and only the first two terms need be considered. Hence, substituting in the differential equation, integrating and determining the constant thus introduced, we have

$$z = \int_0^\infty e^{-\sigma x}\phi(x)\,dx = M_0 e^{-a\sigma - \frac{1}{2}b\sigma^2}. \tag{21}$$

We may obtain a similar expression for $\int_0^\infty e^{-\sigma y}\xi(y)\,dy$, and, writing N_n, α, β, etc. for the functions of $\xi(y)$ corresponding to M_n, a, b, etc. for $\phi(x)$, we may obtain, by substituting in (19), the following equation for σ:

$$\tfrac{1}{2}(b+\beta)\sigma^2 + (a+\alpha)\sigma - \log_e M_0 N_0 = 0. \tag{22}$$

Alternatively, following Wicksell, we may use a Pearson Type III curve to represent $\phi(x)$, thus

$$\phi(x) = M_0 \frac{t^u}{\Gamma(u)} x^{u-1} e^{-tx},$$

$$\text{where}\qquad t = \frac{M_0 M_1}{M_0 M_2 - M_1^2} \quad\text{and}\quad u = \frac{M_1^2}{M_0 M_2 - M_1^2}.$$

Let the Type III curve representing $\xi(y)$ involve constants v and w corresponding to t and u for $\phi(x)$. Then, substituting in (19) and integrating, we have

$$\frac{M_0}{\left(1+\frac{\sigma}{t}\right)^u}\,\frac{N_0}{\left(1+\frac{\sigma}{v}\right)^w} = 1. \tag{23}$$

For all practical purposes (23) gives the same results as (22), but here (22) is easier to solve.

5·8. Having determined a value of s, say s_m, from equation (18) or a more accurate form of equation (22) the following method may be used to determine B_m.

Writing l and L for the limits of the reproductive period and substituting for B (t) we have

$$\int_l^{L+l} B(t)\,e^{-s_m t}\,dt = \sum^{m\neq n} \frac{B_n}{s_n - s_m}\{e^{(s_n - s_m)(L+l)} - e^{(s_n - s_m)l}\} + B_m L. \tag{24}$$

Also

$$\int_{2l}^{L+l}\left\{\int_{l}^{t-l} B(t-x)\,\phi(x)\,dx\right\} e^{-s_m t}\,dt = \int_{l}^{L}\left\{\Sigma B_n e^{-s_n x}\phi(x)\int_{x+l}^{L+l} e^{(s_n-s_m)t}\,dt\right\}dx$$
$$= \sum^{n\neq m}\frac{B_n}{s_n-s_m}\left\{e^{(s_n-s_m)(L+l)}\int_{l}^{L} e^{-s_n x}\phi(x)\,dx - e^{(s_n-s_m)l}\int_{l}^{L} e^{-s_m x}\phi(x)\,dx\right\}$$
$$+B_m L\int_{l}^{L} e^{-s_m x}\phi(x)\,dx - B_m\int_{l}^{L} x e^{-s_m x}\phi(x)\,dx. \tag{25}$$

From (24) and (25)

$$\frac{\displaystyle\int_{2l}^{L+l}\left\{\int_{l}^{t-l} B(t-x)\,\phi(x)\,dx\right\} e^{-s_m t}\,dt}{\displaystyle\int_{l}^{L} e^{-s_m x}\phi(x)\,dx} - \int_{l}^{L+l} B(t)\,e^{-s_m t}\,dt + B_m\frac{\displaystyle\int_{l}^{L} x\,e^{-s_m x}\phi(x)\,dx}{\displaystyle\int_{l}^{L} e^{-s_m x}\phi(x)\,dx}$$
$$= \sum^{n\neq m}\frac{B_n e^{(s_n-s_m)(L+l)}}{s_n-s_m}\left\{\frac{\displaystyle\int_{l}^{L} e^{-s_n x}\phi(x)\,dx}{\displaystyle\int_{l}^{L} e^{-s_m x}\phi(x)\,dx} - 1\right\}. \tag{26}$$

Again using a Pearson Type III curve to represent $\phi(x)$ we may in equation (26) put

$$\frac{\displaystyle\int_{l}^{L} e^{-s_n x}\phi(x)\,dx}{\displaystyle\int_{l}^{L} e^{-s_m x}\phi(x)\,dx} - 1 = \left(\frac{t+s_n}{t+s_m}\right)^{-u} - 1 = \left(1+\frac{s_n-s_m}{t+s_m}\right)^{-u} - 1$$
$$= -u\left(\frac{s_n-s_m}{t+s_m}\right) + \frac{u(u+1)}{2}\left(\frac{s_n-s_m}{t+s_m}\right)^2 - \dots \tag{27}$$

If in the thus modified equation (26) we substitute successively

$$x\phi(x),\quad x^2\phi(x),\quad \dots,$$

in place of $\phi(x)$ we obtain a series of equations of which all terms on the left-hand side (except B_m) are known and from which the unknown terms on the right-hand side may be eliminated. Writing $I_1 B_m$, $I_2 B_m$, etc. for the left-hand sides and eliminating unknowns, we have

$$\begin{vmatrix} I_1 B_m, & u, & \dfrac{u(u+1)}{2}, & \dots \\ I_2 B_m, & u+1, & \dfrac{(u+1)(u+2)}{2}, & \dots \\ I_3 B_m, & u+2, & \dfrac{(u+2)(u+3)}{2}, & \dots \\ \vdots & \vdots & \vdots & \end{vmatrix} = 0. \tag{28}$$

Determinant (28) with as many terms as are necessary may be used to determine B_m.

5·9. As $t \to \infty$ the ratio $M(t)/F(t)$ of male to female births tends to

$$M_0 e^{\sigma t} \div F_0 e^{\sigma} = M_0/F_0$$

which is constant.*

If the ratio of male to female births up to time t has been constant and equal to X, then, by expanding the determinant (28) to determine M_0 and F_0, it can be seen by inspection of the form of IB_0 that the ratio M_0/F_0 must be equal to X. If the masculinity ratio has not been constant in the past, M_0/F_0 is a weighted average of past ratios. Hence the ultimate age and sex distribution of the population is determined by the joint rate of natural increase, the male and female mortality and the past sex ratios at birth. This ultimate age and sex distribution is

$$f(x) = l_x^{(f)} e^{-\sigma x} \quad \text{and} \quad m(x) = X l_x^{(m)} e^{-\sigma x} \tag{29}$$

5·10. If we assume the ratio of male to female births to be constant, independent of age or sex of parent and equal to X, then the true rates of natural increase for males ρ_m and females ρ_f are given by

$$\int_0^\infty e^{-\rho_m x} \phi(x) X \, dx = 1 \quad \text{and} \quad \int_0^\infty e^{-\rho_f y} \xi(y) X^{-1} dy = 1.$$

Hence, from (19)

$$\int_0^\infty e^{-\sigma x} \phi(x) \, dx \int_0^\infty e^{-\sigma y} \xi(y) \, dy = \int_0^\infty e^{-\rho_m x} \phi(x) \, dx \int_0^\infty e^{-\rho_f y} \xi(y) \, dy. \tag{30}$$

Now if $\qquad \rho_m \geqslant \sigma, \quad \text{then} \quad e^{-\rho_m x} \leqslant e^{-\sigma x}.$

Hence $$\int_0^\infty e^{-\sigma y} \xi(y) \, dy \leqslant \int_0^\infty e^{-\rho_f y} \xi(y) \, dy.$$

That is, $\qquad e^{-\sigma y} \leqslant e^{-\rho_f y} \quad \text{or} \quad \sigma \geqslant \rho_f.$

Thus if $\qquad \rho_m \geqslant \sigma, \quad \rho_m \geqslant \sigma \geqslant \rho_f.$

Similarly if $\qquad \rho_f \geqslant \sigma, \quad \rho_f \geqslant \sigma \geqslant \rho_m.$

Therefore, if the sex ratio at birth is constant for parents of any age or sex, σ must lie between ρ_m and ρ_f.

5·11. If we continue to make the assumption that the sex ratio at birth is constant for parents of any age or sex, we can find approximately the relation between S_0 and the net reproduction rates for males and females R_0^m and R_0^f respectively, and also an approximate relation between σ and ρ_m and ρ_f.

Since

$$S_0 = \int_0^\infty \phi(x) \, dx \int_0^\infty \xi(y) \, dy, \quad R_0^m = \int_0^\infty \phi(x) X \, dx, \quad R_0^f = \int_0^\infty \xi(y) X^{-1} dy,$$

we have, on the above assumptions,

$$S_0 = R_0^m \cdot R_0^f. \tag{31}$$

If for the integrals in equation (30) we substitute Lotka's exponential form (21) we obtain, equating powers of e,

$$(a+\alpha)\sigma + \tfrac{1}{2}(b+\beta)\sigma^2 = a\rho_m + \tfrac{1}{2}b\rho_m^2 + \alpha\rho_f + \tfrac{1}{2}\beta\rho_f^2.$$

* Notice that M_0 in section 5·9 is not the same as M_0 in section 5·7.—Eds. *J.I.A.*

Table 7. Calculation of S_0 and σ using Australian 1944 fertility and 1933 census mortality

Age (x)	Male parent					Female parent				
	$1{,}000 \times \frac{l_x}{l_0}$	Probability of having female child	Female births $5 \times (2) \times (3)$	x (4)	x^2 (4)	$1{,}000 \times \frac{l_x}{l_0}$	Probability of having male child	Male births $5 \times (7) \times (8)$	x (9)	x^2 (9)
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)
–19	922·40	·00125	5·7	101	1,760	937·04	·01196	56·0	981	17,161
20–24	912·72	·02785	127·1	2,859	64,331	928·88	·06507	302·2	6,800	152,989
25–29	901·46	·06573	296·3	8,147	224,032	917·89	·08072	370·5	10,188	280,170
30–34	889·42	·06543	291·0	9,456	307,325	905·27	·06298	285·1	9,265	301,103
35–39	874·48	·04908	214·6	8,047	301,772	890·31	·03663	163·0	6,114	229,284
40–44	854·96	·02705	115·6	4,915	208,879	872·68	·01123	49·0	2,082	88,491
45–49	828·03	·01146	47·4	2,253	107,022	850·68	·00070	3·0	142	6,726
50–54	789·88	·00408	16·1	846	44,436	—	—	—	—	—
55–59	734·76	·00156	5·7	330	18,998	—	—	—	—	—
60–	659·05	·00088	2·9	181	11,316	—	—	—	—	—
Total			1122·4	37,135	1,289,871			1228·8	35,572	1,075,924

Correcting M_0 for illegitimate births, $M_0 = 1{\cdot}1224 \times 1{\cdot}04532 = 1{\cdot}1733$, $N_0 = 1{\cdot}2288$, $M_1/M_0 = 33{\cdot}084$, $M_2/M_0 = 1149{\cdot}154$, $\log_e M_0 = {\cdot}159837$, $N_1/N_0 = 28{\cdot}948$, $N_2/N_0 = 875{\cdot}593$, $\log_e N_0 = {\cdot}206033$. Equation (22) is $-46{\cdot}1045\,\sigma^2 + 62{\cdot}032\,\sigma - {\cdot}365870 = 0$. Therefore $\sigma = {\cdot}592\,\%$ p.a. and $S_0 = M_0 N_0 = 1{\cdot}442$.

Solving for σ,

$$\sigma = [-(a+\alpha) \pm \{(a+\alpha)^2 + 2(b+\beta)(a\rho_m + \alpha\rho_f + \tfrac{1}{2}b\rho_m^2 + \tfrac{1}{2}\beta\rho_f^2)\}^{\frac{1}{2}}](b+\beta)^{-1}.$$

But, on our assumption, σ must lie between ρ_f and ρ_m. Hence if $\rho_f = \rho_m = 0$, $\sigma = 0$. Substituting these values in the above expression for σ shows that, of the alternatives, we must select the positive sign.

Expanding by the binomial theorem and cancelling, we have

$$\sigma = \frac{a\rho_m + \alpha\rho_f}{a+\alpha} \text{ approximately.} \tag{32}$$

This simple formula reproduces to 2 decimal places the values of σ for Australia 1933–44 calculated from formula (22) and given in section 6.

5·12. It should be emphasized that the measures S_0 and σ in no way depend on the assumption of constant sex ratio at birth made in the preceding two paragraphs. One of their functions is to allow for variations in this ratio. The usefulness of S_0 and σ in building up a complete theory would be lost if we simply defined S_0 as the right-hand side of equation (31). Relations (31) and (32) while useful practical formulae are limited because of this assumption.

5·13. *Summary*

Given a population subject to the net fertility functions $\phi(x)$ and $\xi(y)$ we have thus shown, amongst other things, that:

(I) the male, female and total births all ultimately increase at the 'joint rate of natural increase' σ given by equation (19) or approximately by equations (22) and (23);

(II) S_0, the 'joint reproduction rate', defined in paragraph 5·5 is a unique reproduction rate corresponding to σ;

(III) $\sigma \gtreqless 0$ according as $S_0 \gtreqless 1$;

(IV) the ultimate age-sex distribution of the population is given by (29);

(V) if the sex ratio at birth has been constant in the past σ lies between ρ_f and ρ_m;

(VI) on the same assumption, S_0 and σ are related to R_0^m, R_0^f, ρ_f and ρ_m by the approximate relations (31) and (32).

A sample calculation of S_0 and σ is given in Table 7.

This suggestion could be applied to the Karmel or the Clark-Dyne formulae to avoid, in them, the male *v.* female anomaly.

6. APPLICATION TO AUSTRALIAN DATA

6·1. *Replacement index* J_3

Table 8 gives values of J_3 calculated from the estimated Australian female population in 1939 and A^{F33} mortality. The figures give a rough but clear indication of the downward trend in fertility during this century.

6·2. *Crude rate of increase*

The values of this index given in Table 10 show that, while it is a measure of the present rate of increase, it is useless as a measure of reproductivity or of ultimate rate of increase. During the period considered its magnitude is not of the same order as the measures of reproductivity and its time variations, relatively, are small. These discrepancies result mainly from the fact that this measure is based on the present age distribution which is inflated at the reproductive ages by the higher fertility of earlier decades (see Table 8) and hence yields a high crude birth-rate. As this crude birth-rate falls the crude death-rate will not fall correspondingly because of the ageing of the population.

Table 8. Replacement index J_3—Australia

Junior age-group (1)	Senior age-group (2)	Ratio of females in (1) to females in (2). June 1939 (3)	Ratio in A^{F33} (4)	$J_3=(3)\div(4)$ (5)	Average year of birth (6)
0– 4	20–44	·20844	·21296	·979	1937
5– 9	25–49	·21412	·21343	1·003	1932
10–14	30–54	·26487	·21717	1·220	1927
15–19	35–59	·29721	·22239	1·336	1922
20–24	40–64	·31712	·22944	1·382	1917
25–29	45–69	·35975	·24017	1·498	1912
30–34	50–74	·39213	·25811	1·519	1907
35–39	55–79	·46048	·28895	1·594	1902

6·3. *Net reproduction rates and rates of increase*

(I) *Male* v. *female rates.* The male and female rates follow one another remarkably closely and hence either may be used to measure the time trend of reproductivity. For actual magnitude the relative merits of the two indices would have to be considered. In the absence of any information favouring one particularly, the joint rate of natural increase has a good deal to commend it.

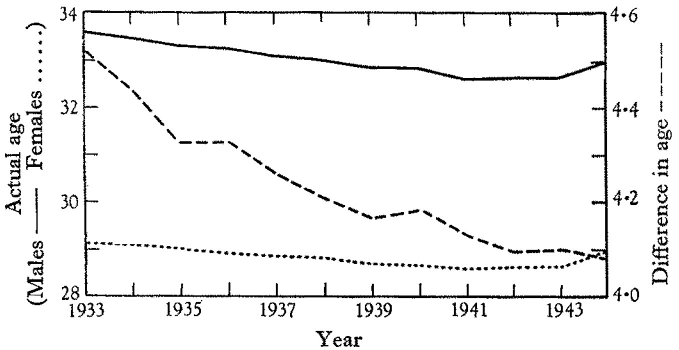

Fig. 4. Average age of net reproductivity (males and females) and difference in ages—Australia 1933–44.

(II) *Reproduction rates* v. *rates of increase.* The reproduction rates for male and female move almost parallel throughout the 12 years; the rates of natural increase, however, converge with time. This convergence is a direct result of the convergence shown in Fig. 4 of the average ages of the male and female net reproductivity schedules. The steady fall over 12 years in both average ages causes the rates of natural increase to rise more steeply than the corresponding reproduction rates. With these two corrections the rate of natural increase is a more efficient measure of reproductivity than the corresponding reproduction rate.

(III) *Negative values.* The negative values of ρ_f, or the rates of natural decrease, obtained throughout the thirties by considering the female sex were the cause of many gloomy prophecies. The male rate, which has equal claims as a measure of the population reproductivity, is only negative for the short period of 2 years. This would provide an inadequate basis for gloomy prophecies. Further, it may well be that the male reproduction rate for these 2 years is too low rather than too high and if full data were available we might have been able to show that the population was more than replacing itself throughout the 1930's. This possibility is justified by Table 9 which shows the proportion of the female population in a given age-group marrying in a given year. We could deduce from this table that the proportion of females married at a given age fell from just after the mid-1920's until the early 1930's. The results of section 3 tell us that in these circumstances the net reproduction rate (and corresponding true rate of natural increase) would for some years after the early 1930's underestimate the reproductivity. It is likely that the negative values obtained during the 1930's for ρ_f do not indicate a failure of the population to replace itself, but are the direct result of fluctuations in marriages in a community always more than replacing itself.

Table 9. Proportion of females in a given age-group marrying in a given year—Australia

Age-group	Year										
	1927	1929	1931	1932	1934	1936	1938	1940	1942	1943	1944
–19	·0283	·0270	·0242	·0251	·0256	·0275	·0276	·0337	·0407	·0359	·0381
20–24	·0842	·0778	·0618	·0674	·0773	·0859	·0909	·1186	·1345	·1031	·1035
25–29	·0465	·0427	·0331	·0396	·0475	·0536	·0543	·0643	·0588	·0427	·0410
30–34	·0178	·0163	·0120	·0139	·0177	·0205	·0212	·0249	·0247	·0187	·0181
35–39	·0097	·0088	·0063	·0067	·0083	·0094	·0103	·0123	·0132	·0105	·0107
40–44	·0058	·0052	·0039	·0042	·0044	·0051	·0056	·0066	·0080	·0069	·0068

(IV) *Upward fertility trend.* After the mid-1930's the steady upward trend in the values of Table 9 seems to indicate a steady increase in the proportion of females marrying at a given age up to the year 1942 after which the proportions appear to be constant. The results of section 3 would suggest that, while the reproductivity has increased, it is considerably overestimated by these measures and, assuming it remains constant, these measures may be expected to fall gradually to lower values.

(V) *Possible future movements.* It is possible that the 'proportions of females married at a given age' may return to pre-war values. If this occurs and if fertility remains constant, section 3 proves that R_0 will fall to an unduly low value, and, even if the population is more than replacing itself, negative values of ρ (or values of $R_0 < 1$) may be obtained. It should then be remembered that these values simply indicate a faulty index, and the underestimate at this time should be balanced against the overestimate of some years earlier.

6·4. *Replacement index* J_2

The values of J_2 in Table 10 give an indication of the extent to which this simple index can be used to measure reproductivity. Over the 12 years, during which time great variations have taken place in reproductivity, J_2 moves parallel to R_0^f within a range of error of less than 2%. In the early years J_2 exceeds R_0^f by just over 4% and in the early 1940's by about $4\frac{1}{4}$%. The variation from parallel of the two curves follows closely the change in the average age of net reproductivity of women (Fig. 5). The figures thus confirm the relation between these indices given in paragraph 1·3.

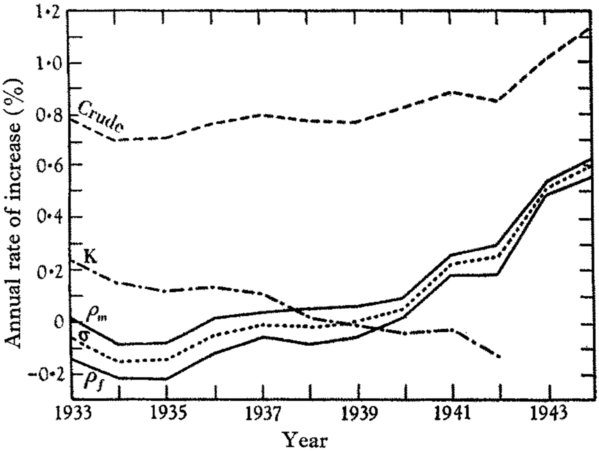

Fig. 5. Measures of reproductivity—Australia 1933–44.

6·5. *Karmel formula*

It is rather difficult to explain the course of the Karmel rates in practice. In section 3 we showed that a temporary change in the 'proportions married' has little immediate effect on the Karmel rates, but causes discrepancies over the next 20 years. The upward trend in the 'proportions married' during the 1930's will therefore exert little influence on the Karmel rates in Table 9. The high initial values are probably a reflexion of the high post-war marriage rates and the steady decline a result of the decreasing proportions married up to 1931.

6·6. *Clark-Dyne formula*

No application of this formula to Australian data has been made because the form in which birth data are published scarcely justifies it. In the *Demography Bulletin*, Australian legitimate confinements for a given year are published according to curtate duration of marriage and mother's age at confinement in quinquennial age-groups. The Clark-Dyne formula requires fertility to be expressed in the form of Table 1; that is, we must relate the births of a given year to the relative marriages. As the latter are given per calendar year the births would need to be given according to marriage duration expressed in the form of calendar years. The mother's age should be given in age-groups at marriage rather than at confinement.

To determine the fertility in the form of Table 1 from the present data would involve averaging greatly varying annual marriages and the broad age-groups

Table 10. Values of various measures of reproductivity for Australia 1933–44

Year	Crude natural increase (%)	Gross reproduction rate		R_0^m	R_0^f	J_2	K_0	S_0	ρ_m (%)	ρ_f (%)	σ (%)	κ (%)
		Male	Female									
1933	·786	1·142	1·052	1·003	·959	·981	1·075	·963	·009	—·143	—·061	·224
1934	·707	1·102	1·030	·969	·939	·962	1·050	·910	—·093	—·215	—·151	·151
1935	·709	1·105	1·030	·973	·939	·965	1·037	·916	—·082	—·216	—·141	·112
1936	·770	1·138	1·060	1·003	·967	·995	1·041	·970	·009	—·114	—·048	·125
1937	·799	1·149	1·075	1·014	·981	1·014	1·035	·996	·042	—·065	—·007	·106
1938	·783	1·151	1·069	1·016	·976	1·011	1·006	·992	·048	—·084	—·013	·018
1939	·772	1·156	1·080	1·020	·986	1·025	·998	1·005	·061	—·050	·008	—·005
1940	·825	1·170	1·102	1·033	1·007	1·051	·988	1·044	·099	·023	·070	—·038
1941	·892	1·232	1·154	1·089	1·054	1·100	·994	1·149	·261	·185	·226	—·019
1942	·857	1·247	1·156	1·102	1·056	1·101	·961	1·164	·297	·191	·251	—·123
1943	1·035	1·350	1·257	1·193	1·148	1·194	—	1·369	·538	·484	·514	—
1944	1·146	1·388	1·289	1·225	1·176	1·216	—	1·442	·617	·563	·592	—

would introduce errors in the age transformation. Little satisfactory information is available for Australia regarding typical proportions of females married at a given age. This has to be used in conjunction with the above fertility.

So great are the approximations that the results could not be used with any confidence. Calculations were not therefore carried out.

6·7. *Remarks*

After all this, the first question one would naturally ask is: 'Well, what course *did* the reproductivity take over the last 15 years?' The honest answer to this question is 'We don't exactly know'. The general direction of the change as suggested by the net reproduction rates is probably correct but the actual figures are more misleading than is generally supposed. If we must use this method it would be better to base our prophecies on the joint rate of natural increase, considered in conjunction with previous variations in the 'proportions married'.

The additional information about population trends given by the Clark-Dyne formula must surely justify the small change in presentation of the annual births; that is, their tabulation according to mother's age at marriage and calendar year of marriage. This information can be obtained from the present birth registration form. From these data we could determine reproduction movements with confidence.

The values obtained by applying the Clark-Dyne formula to female data would give us accurate *relative* annual figures. The male sex must be considered if an *absolute* measure of reproductivity is desired. For this purpose the determination of a joint rate of increase by the Clark-Dyne method is suggested. This would involve the tabulation of female births according to father's age and male births according to mother's age. Given births each year in this form and figures for the proportions married either every 5 or every 10 years then we could determine an absolute measure which would correctly reveal inherent population tendencies.

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ABSTRACT OF THE DISCUSSION

The President (Mr A. H. Rowell) said that all the members would wish to join with him in extending to Mr Pollard a particularly warm welcome as the winner of the Rhodes Prize Essay competition, who had flown all the way from Sydney to attend the sessional meeting of the Institute.

Mr A. H. Pollard, in introducing his paper, thanked the President for his words of welcome and the Council for their consideration in arranging, for his benefit, that the present meeting should be held so close to the date of the Centenary Assembly.

The paper was, he realized, one the subject of which was not included in the present syllabus, nor were papers in that field often discussed before the Institute. It might be of interest to mention, therefore, how it was that he became interested in this subject. Some years previously, he had been present at a meeting of a small number of economists who were discussing the economics of a declining population. When they came to the statistical aspects of population movements they turned to him and asked him a number of awkward questions, thinking that as an actuary he would be an expert in that field. Not wanting to let himself or the profession down, he did not confess to them that he had not heard before of the net reproduction rate about which they talked so much. He had a very awkward evening, but he took great care before the next gathering to spend many hours in the Sydney Public Library, reading the works of A. J. Lotka and others. Some years later two papers appeared in an Australian journal, the first by Karmel and the second by Clark and Dyne. It was largely to the inspiration provided by those two papers that the present paper was due.

Mr P. R. Cox, in opening the discussion, said that the paper dealt with a very interesting topic and a suitable subject for actuaries to examine, though unfortunately it was one not often raised at Institute meetings. A reconsideration of the means of measurement of reproductivity was especially opportune at the present time, because the study of fertility had been making big strides recently and ideas about it were undergoing some changes. The author's contribution was therefore particularly welcome. It covered a wide field and raised some interesting technical questions. Section 5 consisted of an ingenious development of Lotka's mathematical formulae of the stable population in an attempt to eliminate inconsistencies between the sexes. There were some elegant formulae at this point, especially on p. 307, which could not fail to appeal to the mathematically minded.

The subject of fertility was suitable for actuaries to discuss because their training gave them the power to get down to fundamentals in demographic matters and it was very important at the beginning of the discussion to consider what was really fundamental in fertility. The author had chosen a general title for his paper, though he had really dealt only with certain specific points. He had based his analysis upon ideas as to the nature of fertility which were undefined and which, when brought to light, might possibly arouse some criticism. Again, mathematical developments in a very complex subject of this kind involved the danger that in their more remote processes there might be some departure from reality. That danger was enhanced by the fact that in most countries the data required for a full analysis of the position were lacking. It might almost be said that it was necessary to devote attention as much to collecting fresh information and fresh data as to developing what might amount to untested hypotheses.

In order to emphasize the essentials in fertility, certain significant facts were worth noting. First, there had been a big decline in fertility since the last century, due to the use of contraception. Secondly, there had been a rise, probably followed by a fall, in differential fertility, i.e. the extent to which various social classes of the population differed from each other in their fertility experience. That indicated, apparently, an increasing use of contraception, starting with the upper classes and proceeding downwards throughout the whole of the population. It was known that illegitimate births were small relative to the total, and that infertility was probably not a very large factor.

He deduced, therefore, that a very large part of fertility consisted of a process of family building according to the plans which married couples made as to the families which they wanted. Fertility and of course reproduction were in fact dependent on human volition. They occupied, therefore, a special position in statistical analysis, and their treatment should not be the same as that of mortality or sickness. If a family-building plan was interfered with by war or economic depression it might afterwards tend to readjust itself; this was a phenomenon not found in a mortality or sickness experience. Psychological factors and social currents might be more important than economic causes, because they were more permanent. That might explain the anomaly that families had fallen steadily while economic circumstances were improving in many countries.

There was also what might be termed a 'generation' effect, by which the additions to families were correlated more with the time when the parents married than with the time when the child was born. Many parents tended to adhere to the size of family which was popular in their youth.

In consequence, one should seek to study fertility principally through the five following essential factors: family size; time at which the marriage of the parents took place; ages of parents at marriage; time at which the birth of the child took place; and possibly also the social class of the parents.

Examining the paper with those points in mind, it would be observed that age attained played a very large part in the analysis, especially in section 5; but age attained was not one of the fundamentals. Consequently, an analysis using it as the principal point of reference was dangerous, and had to be watched with care. In particular, to total f_x, the fertility rate, was open to objection because of possible generation and other effects. There was a difference here between fertility and reproduction. Σf_x might be regarded, perhaps, as a tolerable index of fertility, but not as a measure of reproduction.

The author had referred to duration of marriage. This was not, perhaps, fundamental in fertility, though it went hand in hand with family size to some extent. An example of the difference between the two was given recently by Hajnal in a paper* in which he showed the effects of a postponement of family building due to a war or economic depression. As the duration of marriage increased the chance of having a child generally fell; hence, if the births of children were deferred through economic or other causes and then the family was made up later, there would be a small decline in the fertility rates at the early durations but a large increase at the later durations. There was a risk that such results might be misleading. That would not arise in an analysis of fertility according to family size. However, there were no suitable data as to family size in most countries, and in the circumstances the analysis by age and duration of marriage, associated by the author with the names of Clark and Dyne, must be regarded as the best of the indices which were discussed in the paper.

Unfortunately, the durational effects were not brought into the analysis in section 5. They would, if brought in there, impose some very grave additional mathematical problems.

In section 3 of the paper there was some material which might be open to criticism. The proportion married was assumed to change from p_x to P_x according to a cosine blending function, and consequently in Table 5 many of the extra marriages in the first three years occurred at the older ages; later the position was reversed. He wondered whether that was necessarily likely to occur in practice. Again, while the proportion married was changing it had been assumed that fertility rates by age and duration of marriage remained constant; but an increase in the proportion married might very well bring in marriages of a different type from those which were there before. People who perhaps would not have married in ordinary circumstances now had a chance to do so, and their fertility might well be quite different from that of those marriages already in being. Moreover, owing to stresses occasioned by war or by economic circumstances the age at marriage might change, so that, for instance, people who would normally have married at age 20 now married at (say) 24; but they would tend to bring into their

* *Population Studies*, Vol. 1, No. 2.

marriage at 24 the family-building ideas appropriate to age 20. That upset the analysis by age and duration to quite a considerable extent.

The effect of the changes in the proportions married on the Karmel index which were illustrated on pp. 302–304 depended on the figures shown in Table 5, and he thought they might be a little misleading. The effect of the increase in marriages on the reproduction rate R_0 could, however, be appreciated from general reasoning without the detailed analysis.

With regard to section 5, the anomaly that male data were liable not to give the same results as female arose frequently in demographic analysis, and a crucial test of the satisfactory nature of the work done was that the results should be the same whether obtained by one means or the other. Judging by recent researches, however, and in particular those of Karmel, it seemed doubtful whether it was possible to reconcile male and female fertility functions based on attained age alone.

The masculinity rate, i.e. the proportion of male to female births, was almost constant at all times. If k were the ratio of female to male births at any time t it would then be possible to substitute $k \times \mathrm{M}$ functions for the F functions in equations (13) and (14), so that on the left-hand side of (13) one would have $k\mathrm{M}(t)$ and on the right-hand side of (14) $k\mathrm{M}(t-y)$. It would also be possible to substitute x for y in (14) without any loss of generality. Equating $k \times$ the right-hand side of (14) to the right-hand side of (13) after these substitutions (each left-hand side being $k\mathrm{M}(t)$) we had:

$$k^2 \int \mathrm{M}(t-x)\, \xi(x)\, dx = \int \mathrm{M}(t-x)\, \phi(x)\, dx.$$

He suggested that there were only two possible interpretations of that equation. Either the relationship between the fertility functions $\xi(x)$ and $\phi(x)$ was dependent on the values of $\mathrm{M}(t-x)$, i.e. it was influenced by the course of births in the past; or else $k^2\xi(x) = \phi(x)$ at every age, which meant that the chance of a man having a male child was proportionate to the chance of a woman having a female child at every age. The second interpretation was out of accord with reality, so that the relationship between $\xi(x)$ and $\phi(x)$ must depend on the course of births in the past, and, as t increased, that course would not be constant, and so the relationship between $\phi(x)$ and $\xi(x)$ would change.

That, as Karmel had suggested, struck at the root of the Lotka mathematical system, because it meant that the population could not tend towards a stable state with constant fertility rates for both sexes. This threw considerable doubt on the worth of the reproduction rate, and the recent researches which had been made in this subject had brought out the big difference between male and female rates and the great difficulty of reconciling them. It did not seem that that difficulty was really circumvented by combining male and female rates in a double integral. Population mathematics would have to be founded upon the other ideas about the nature of fertility which he had mentioned earlier, and would have to become a good deal more complex before this very difficult problem could be solved.

There were one or two small points of detail to which he would like to refer. In section 3·2 of the paper the author suggested that expression (9) could be made to yield an analytic expression for the births at time t by assuming π_x to change in accordance with an inverse tangent function, since $\int \tan^{-1} kt\, dt$, $\int t \tan^{-1} kt\, dt$, etc., were integrable. He was glad that the author did not pursue that idea, because there was a danger in using a mathematical expression simply because it was a function which could be employed in order to solve the equation and not because it was likely to represent the actual experience.

In section 1·3 three forms of replacement index were mentioned. J_2 had often been associated with the name of the statistician Dr Burgdörfer, and it would be of interest if the author could say how he came into the history of the development of that index.

At the top of the next page it was said that the Replacement Index 'is the only useful index when age-specific fertility is not available'. Possibly, for the benefit of students, some mention might have been made of substitute reproduction rates, in which the experience of another place, class or time could be used without serious inaccuracy where it was known not to be dissimilar.

Mr W. G. Bailey said that at first sight he agreed with the opener with regard to section 5 of the paper, that equations (13) and (14) were subject to the strictures placed upon them by Karmel's analysis. On second thoughts, however, he was of opinion that once the author had arrived at equations (15) and (16) he could claim to have jettisoned (13) and (14), and that the assumption in his model was that the product of $\phi(x)$ and $\xi(y)$ was constant for variations in time. If that was so, then it would appear that the model was self-consistent, and escaped the criticism which the analysis in Karmel's paper would place upon any assumption of the constancy of $\phi(x)$ and the corresponding function for y. The argument against it would be the author's analysis in section 5·10, but there he thought that the author was really saying that if one wished to calculate what might be described as the Lotka reproduction rates for males and females separately it would be found that his value lay between the two of them. He was not going to say what he thought of the model which demanded that that product be constant; he had not investigated it. It was quite probable that the model required would be a very peculiar one, but it was a model, and the author was entitled to claim that if use could be made of it he was justified in putting it forward.

He had been disappointed to find that the author had been content to follow the lead of the population mathematicians and confine his attention to that side of the subject. On reflexion, however, he decided that he had no right to be disappointed, because, after all, the author wrote the paper about three years ago to satisfy the conditions of the Rhodes competition. Moreover, he realized that the author had written an essay and not either a text-book or a long treatise on the subject.

The trouble about demography was that so far it had been under the domination of the population mathematicians, whereas normally one would expect the mathematician to be a servant of the investigator. The cause of that, as the opener had said, was lack of data, and particularly lack of exposed to risk; and that, he thought, explained to a large extent why it was such a long time since the Institute had had a paper by an actuary on this subject. The result had been, in his view, the production of models the main purpose of which appeared to be to give the maximum scope for the peculiar technique of the population mathematician, without any particular regard to the purpose of the models or even to their utility. In fact, he did not think that he had ever seen it stated what purpose had been behind the model or the index derived.

To take the net reproduction rate, there had been two orthodox interpretations of that index. The first was that it showed the rate at which women now in child-bearing ages were replacing themselves in the next generation. The alternative was that it showed the rate at which female children now born would replace themselves in the next generation. The first merely speculated in regard to future mortality—not too dangerous a thing to do, though dangerous enough. The second interpretation, if used, speculated on mortality and fertility. If the net reproduction rate was adjusted to fit in with the Karmel or Clark-Dyne formulae, he would submit that even the first interpretation became objectionable as a reproduction index, because it introduced the factor of marriage.

He made the same distinction between reproduction and fertility as the opener, and he agreed that a full-dress investigation was desirable, taking into account the factors which had been mentioned; but he would like to see, if models were being constructed, some attempt to construct a model which was not one which held good only at infinity, but one which would provide some information regarding the period when the population was passing through a stage resulting from a lack of balance amongst the sexes. He would like also to see provision made in such a model for the hypothetical effect of a change in marriage rates resulting from a change in that balance, and of a change in fertility rates as affected by change in the marriage rates. It seemed to him that the dynamic conception of the change in population was considerably more important than what the condition of the population might be expected to be if it ever reached a stable condition.

One of the reasons for the veneration of the net reproduction rate was that it appeared to give a criterion against which it was possible to measure the results of current fertility. Actually it was a very poor criterion. The publicists took the view that if the net reproduction rate fell below 1, then the population was bound to fall, whereas it was known that that was not true; it was possible to put up for a considerable period with a net reproduction rate below 1 without in the end suffering a fall in population.

So far he had been talking about reproduction. With regard to fertility, he thought that there it was not necessary to be quite so hard on the reproduction rate, whether that devised by the two Australians mentioned in the paper or by the author himself. The net reproduction rate could be further improved by the introduction of birth parities, but that was open to the objection that it neglected generation effects. If the data were available it would of course be possible to have a generation index, but he thought that an index of current fertility was needed for studying correlation. After all, not very much would be known about population until a study was made of the correlation between fertility rates and economic or other factors.

Hajnal suggested that a study should be made of family size, and said that he put forward that suggestion solely on the ground that it conduced to more stable indices. Personally, he was not sure that, for the study of the effect of sudden economic changes, a stable index was desirable; there was the danger of ironing out those very fluctuations which it was desired to correlate to the factors concerned. Again, with regard to occupational fertility, he doubted whether the amount of data required to do the job with absolute thoroughness would ever be available, and also whether they were concerned with the past at all when studying occupational fertility. He thought that it was a moot point, and in that respect, and for the purpose of fertility analyses alone, he thought that the author's joint reproduction index was excellent; and if it could be applied, as the author said, to the Clark-Dyne formula, and also to other formulae taking account of the number of children already born, he thought that the author would have added a very valuable piece of research to this body of knowledge.

Mr H. O. Worger said that he wanted to cast some doubts on the very basic indices which were discussed in the paper, the Karmel and Clark-Dyne formulae. They were defined as the integral with respect to y of $l_y\, m_y$ multiplied, in the Karmel case, by a constant, and, in the Clark-Dyne index, by another variable. In that integral, m_y was the force of marriage, the instantaneous ratio of women marrying to all women. Now, that could lead, as was pointed out, to the integral $\int_0^\infty l_y\, m_y\, dy$ being greater than l_0, and that, he thought, was because all women were not exposed to the risk of marriage. Only single women were exposed to the risk of marriage, and therefore he thought that in this particular formula l_y should represent the number of single women at age y out of l_0 females born and m_y should be the force of marriage to single women.

It might be thought, in that connexion, that it would be necessary to have a double-column table, with an l-married column and a force of becoming unmarried, transferring the married back to the single; but that was not so, as the effect of becoming unmarried and remarried, repeated as often as desired, could be allowed for in the function ${}_yb_r$. If the rate of marriage of single women altered quite considerably, then, if there was an increase at the younger ages but no alteration at the older ages, the ratio m_y which was used in expressions (5) and (6) at the older ages must fall; but if the force of marriage to single women was used, allowance would automatically be made for the alteration in the force of marriage.

It was possible to allow for the ${}_yb_r$ function by taking only first marriages. In other words, the duration was taken from the first time that the woman was married, ignoring any subsequent 'unmarriages' and remarriages. In that way there would be included in ${}_yb_r$ the effects of becoming 'unmarried' and remarrying and having further children.

He thought that that index was most useful at the present time, modified as he suggested, because the Government were taking considerable interest in the reproduction and the fertility rates; and in some countries, and perhaps in Great Britain in future, Governmental action might be taken to increase the number of births. The most probable way of doing that, and the way which had been adopted in other countries in the past, was to encourage earlier marriages. If the reproduction rate was based on the marriages of single women it would be possible to make an approximate estimate of the effect of this encouragement of earlier marriage as soon as statistics were available giving the new modified rates of marriage, if the assumption was made that a woman who married for the first time at age *y* would have the same future fertility from marriage as a woman who married in the past at age *y*, and that the effect on reproduction was merely that of having more marriages at the younger age.

Of course, there was the point that many people merely planned to have a fixed number of children, and if they got married at a younger age they would perhaps still have the same number, which would upset this assumption; but at any rate the index would be more useful than those discussed in paragraphs 2·5 and 2·6 of the paper.

There was one other point to which he would like to draw attention. A very broad grouping was used in the paper. With a function whose second derivatives were small compared with their first, the group rate might be, and usually was, a fairly good approximation to the central rate for the group; but with marriages, where the forces of marriage and the forces of issue to married women changed so very rapidly with age, he thought that even quinquennial age grouping was liable to introduce quite a serious error into the calculations. It was true that published statistics seldom gave data by individual ages, but many of the indices in question were being investigated by the Government, who would have access to the individual ages, in view of the fact that the statistics were collected in individual ages and afterwards grouped for publication.

Mr P. H. Karmel (a visitor) expressed his thanks to the President and the Council for having invited him to attend the meeting, particularly as the paper under discussion was on work in which he was very interested, and also because he happened to be a fellow-countryman of the author of the paper. (The speaker then summarized briefly some notes that he had prepared on the male and female conflict which have been included as a written contribution at the end of the report of the discussion.)

Mr W. A. B. Hopkin (a visitor) began by expressing the thanks of the Royal Commission on Population, of which he was a member, for the honour which the Institute had done them by inviting him to take part in the discussion, and also his personal thanks. He thought that this was a suitable occasion for a recognition of the great part which Australia was playing in building up demography. Both in relation to the size of the country and to the achievements of other countries, Australia stood very high in the demographic field, and it was a fortunate circumstance that there were present that evening two Australians who had made considerable contributions to the subject.

The paper was a very valuable contribution, perhaps particularly in demonstrating the kind of effect which changes in age at marriage could have on crude indices like reproduction rates and annual volumes of births and so on. He thought that the author's graphs should be engraved on the minds of all demographers; they represented something which, once one had fully absorbed it, must alter one's attitude to the interpretation of historical periods like that of the last fifteen years.

So far as concerned the solutions which the author proposed to the problems which were discussed, he was conscious of one or two points of doubt. There was first of all a point which the opener had raised, and which he would like to carry a little further. The opener had pointed out that it was not safe to assume that the age at marriage was changing while at the same time the fertility by age at marriage was left unchanged. Personally, he wanted to emphasize that the result which the author obtained in the paper—i.e. the result that the Clark-Dyne formula gave in fact a better measure of the fundamental trend of reproductivity than what the author called the Karmel formula—was derived simply from the author's original assumption that fertility defined in relation to women who married at each given age was constant. In fact, it might very well be the case that when people married younger the balance of fertility between different ages at marriage would change. It was possible to look at the matter the other way round and to ask 'If the age at marriage changes, why should the average size of family change at all?' He did not want to say dogmatically that it would not change—he thought that probably it would—but he did not believe that it was at all likely to change in precisely the same way as if the fertility of marriages taking place at each age remained unaltered. In practice, therefore, it might well be that a formula like the Karmel formula gave a better measure of what was happening to reproductivity when the age of marriage changed than one like the Clark-Dyne; and he dissented from the author's conclusion, or from what seemed to him to be the implication of what the author said, namely that it was possible to be fairly confident that the Clark-Dyne formula would give the best answer.

With regard to the sex conflict, he thought that it might be of some interest if he gave one or two figures which showed concretely the importance of the point which Mr Karmel made, which seemed to him to be fatal to the joint rate suggested by the author. It involved assuming an indefinite flexibility in the sex ratio, and that was the last thing which one had the right to expect to be flexible. His former colleague, Mr Hajnal, who was unfortunately no longer in this country, had had an opportunity of seeing the paper before he left and had written a little note about it. He did not intend to read the whole of that note, because the ground had been largely traversed by Mr Karmel; but it worked out for a few periods for England and Wales the sex ratio which would be implied by a joint reproduction rate on the author's lines:

Period	Ratio of male to female births
1910–12	·99
1920–22	·97
1931	·99
1938	1·0

There were two things which stood out there. The first was that the sex ratio changed quite appreciably from one period to another, and the second was that in all four cases it was quite different from any sex ratio which had ever in fact been recorded in this country, since there had never been a sex ratio in this country, he thought, below 1·04 or thereabouts. That seemed to him to be a fatal objection to that particular method of reconciling the male and female reproduction rates.

He would like to add one thing to what Mr Karmel said on the question of reconciliation. He felt sure that the mathematical demographers, if they kept on trying, would succeed in the end in finding a formula which reconciled male and female natality and made it possible, therefore, to work out a reproduction rate which could be said in some sense to be a unique rate, expressing the tendencies of the particular period the effect of which on reproductivity one was trying to measure.

He did not think, however, that that would answer the question in more than a formal mathematical sense. The point was that one was here faced with a real question, not a mathematical one, if he might make what was perhaps a dubious distinction. If one looked at the marriage rates of some recent period in this country, or indeed in almost any country, it would be found that the male marriage rates were inconsistent with the female marriage rates. It would be impossible to expect both sets of rates to continue in the future, because that would imply the continuance of an abnormal sex ratio in the population. The sex ratio in the population was going to change; that was a fact about which it was possible to be fairly confident. When it changed, the relation between male and female marriage rates would change. It was not known exactly how, of course, though he was not convinced that it was impossible to collect a certain amount of information which would be relevant to the question; but no process which consisted merely in mathematically reconciling the rates for an earlier period would indicate what in fact was likely to be the change. It depended on real forces, such as, broadly speaking, which sex was dominant in marriage. He did not believe that what was likely to happen was something which came fairly well in the middle between the two; he thought it was much more likely that one sex was more important than the other in determining how many people got married and at what ages; and he had his own ideas as to which sex it was. Perhaps it would be dangerous for him to lay himself open to attack on a question where it would be very hard to find solid evidence in justification. His main point, however, was that it was a real question, and that to answer it realistically it would be necessary to discover which was in fact the dominant sex in matters of marriage.

In conclusion, he would like to say that those who were professionally concerned with the study of population must be now, and would remain, extremely grateful to the author for having put some of these issues before them in such a very clear way, and for having advanced the study of the subject in an important degree.

The Hon. Secretary read the following contribution to the discussion from **Prof. R. A. Fisher, F.R.S.**:

I should like to congratulate Mr Pollard on his exceedingly ingenious and satisfying treatment of the joint rate of increase developed in section 5. The fact is that we do not know, and have no ready means of finding out, the extent to which an increase in the number of males in a given age distribution affects the fertility of females at different ages. Consequently no formulation is possible based on factual experience to supply an expectation of births from given numbers with known age distributions of the two sexes. Mr Pollard's novel approach, arriving at a unique and practically useful compromise between the observable rates of reproduction for the two sexes separately, is, therefore, a contribution of substantial value to the advancement of the subject.

Mr H. W. Haycocks said that after listening to the discussion it seemed to him that there was much confusion over what population mathematicians were trying to do. The confusion arose largely because of failure to distinguish carefully between mere index numbers of fertility and the more elaborate process of constructing population models. Why did demographers construct such models? Model construction was a methodological device common to all the sciences, including the social sciences. The main difference between the natural and social sciences was that in the latter a model was thought to be most appropriate only when its basic postulates were very closely associated with the human activities and decisions in which one was interested. The investigator usually abstracted from human behaviour in general and set up, so-to-speak, a very simple 'world' governed by this particular behaviour. In other words, one worked out the logical implications of certain human decisions.

He thought that population theory was a social science, and therefore any worthwhile theory could be understood only in terms of some basic human decisions and activities. Such activities should form the basis of the postulates underlying the theory. In this connexion he thought that marriage and fertility rates according to age were poor concepts. Demographers, perhaps for want of better data, had been tempted to construct elaborate models on the basis of unsuitable concepts. Those concepts had served their purpose in the construction of mere index numbers, but something better was required when the problem of reproductivity was to be considered.

The fundamental concepts for a mathematical theory were: first, mortality; secondly, the age probabilities of marriage in respect of both spinsters and bachelors; thirdly, the sex-ratio at birth; and fourthly, the probabilities according to ages at marriage of a married couple having 0, 1, 2, etc., children respectively. The data on which these probabilities would be based should, of course, be classified in generations. If it was possible to obtain a consistent mathematical theory involving these concepts, one would obtain much better models than those now met with.

Of the above four concepts, the two important ones were the marriage and family probabilities. The factors in social life influencing these probabilities could be divided broadly into those affecting reproductivity and those which were irrelevant to this problem. For instance, in the case of marriage probabilities the relevant factors were those which brought about permanent shifts in the average age at marriage and those which so changed the set of probabilities that the cumulative effect was to reduce the number of married couples relative to the possible number. Irrelevant factors were those, usually economic, which merely caused temporary changes in the shape of the probability distribution. Such factors, however, would influence the ordinary marriage rates and it was for this reason that these rates were poor concepts.

The family probabilities would take no account of variations in the spacing of births over time. These variations were also generally due to variations in economic conditions and brought about fluctuations in the ordinary age-fertility rates. They were, however, irrelevant to the problem of reproductivity.

For those reasons, he thought that serious objections could be made to all the index numbers mentioned in the paper if they purported to be more than mere index numbers of fertility. In the case of the more elaborate Clark-Dyne formula it should be pointed out that the b_r's used were not additive. A particular b_r would depend on the social and economic conditions during the previous r years, but it was associated with a b_{r-t} derived from a group of lives in respect of which the future b_r would depend on the social and economic conditions during the next t years. He was reminded of a Part III question which was set many years ago, namely, under what circumstances is it legitimate to assert that $p_x . p_{x+1} = {}_2p_x$? If now the p_x's were multiplied by measures of fertility it became much more difficult to give a reasonable answer.

He had listened to Mr Karmel with much interest, as he himself had applied Mr Karmel's technique to the author's treatment of the joint reproduction rate, and had arrived at the same conclusion that the sex-ratio equation had been overlooked. In this connexion it was well to remember that a particular set of values of the ξ and ϕ functions might imply in the resulting model an abnormal sex ratio. The trouble was that the values of the functions might have arisen out of abnormal social conditions such as a war or an economic depression so that it was absurd to incorporate these values in a model which completely ignored the very factors which had brought about the abnormality. In the case, for instance, where a war had reduced the male population at the younger ages and had brought about the abnormality, a realistic model would postulate a population subject to perpetual war which would bring about the abnormal mortality conditions required. If this were not done then the surplus females in the model would be secured by an unrealistic sex-ratio at birth. It followed that if the author used two schedules of fertility, male and female, to construct a hypothetical population subject to normal mortality, he could do so provided only that those two schedules themselves had been derived from a population which had been fairly stable for a number of years. But would a model then serve any useful purpose? His remarks did not apply with anything like the same force if only index numbers of fertility were being considered.

Mr W. Perks, in closing the discussion, said that the trouble about the subject under discussion was that the mathematicians were trying to do too much. The subject of the paper was 'The Measurement of Reproductivity', i.e. the extent to which the population was reproducing itself. That idea was essentially a generation idea, and all the troubles which had been referred to in the discussion arose out of attempts to measure a generation concept by means of data taken over a short period, and particularly the fertility data over a short period. To his mind, that attempt was doomed to failure, and it was his hope that the very gallant attempt which had been discussed that evening would be the last attempt of that kind.

There had been mentioned that evening models which were internally inconsistent and models which, while consistent, were unrealistic. He had tried to understand the particular model which the author had devised, the joint rate, and his conclusion was that it was internally consistent but did lead, in certain special cases, to unrealistic results. He had in mind particularly the case where the data were obtained from a population which had been subject to abnormalities, such as male war deaths or the migration of one sex in different proportions from the other.

He believed that what had gone wrong in the subject was the taking over of the life-table technique and the technique of standardized death-rates into the fertility field. Everyone realized that fertility was a function of other things besides age; duration of marriage came in, and birth order, i.e. the number of children that a woman had already had before she was exposed to the risk of further fertility. If one gave careful consideration to a three-variable fertility function depending on age, duration of marriage and number of previous children at a particular point of time, following a period in which fertility had changed and marriage rates had changed, one would, he thought, come to the conclusion that those basic three-variable rates were dependent on the particular proportions of women exposed in the various cells in the three-variable set-up. Since fertility in marriage had changed in the recent past, those proportions must change in the near future, and the net effect was that any attempt to build up a life-table model or a standardized rate involved the use of a probability process which was quite unsound. It meant multiplying together rates which were not independent or rates which were not properly conditionally dependent, and he thought that that was at the root of a good deal of the trouble which the mathematicians had found in trying to set up these models.

That was on the technical side, but what about the demographers who tried to express what it was that they were doing? They said 'We will assume that present mortality and present fertility habits remain constant in the future', but what happened was that they did not assume fertility habits remaining constant; they assumed certain fertility rates remaining constant, and that was not the same thing. Indeed, their starting-point was, in his view, meaningless, and what they did in keeping certain rates constant was unsuitable.

He was convinced that the only satisfactory way of studying reproductivity was in generations, and, if it was desirable to obtain early information about changing reproductivity, it was necessary to obtain samples of the population year by year for, say, quinquennial ages of adult males and females. That was the technique which was wanted to see how reproductivity was changing. It was not necessary to have samples of the size that were taken recently by the Royal Commission. In his view, a sample involving hundreds of thousands of people was not sampling at all in the sense in which statisticians had come to think about it. It was possible to learn a good deal about reproductivity by taking quite small samples—a matter of a few thousands each year.

He believed that what required to be done in this field was more and more analysis of data. There appeared to have been a change of approach in recent times by some of the demographers, and in saying that he had in mind particularly some work in 'Population Studies' by Mr Hopkin and by Mr Hajnal. Actuaries could do nothing but applaud that type of work, and many would hope that young actuaries would join in that work in analysing fertility data, because he was sure that actuaries had a great deal to contribute in that field.

He would, however, like to utter a word of warning to any young actuaries who might be considering entering this field. For a long time it was thought that human affairs were not susceptible to sound statistical treatment because of human freewill or volition. Because success had been obtained with statistical methods when dealing with human mortality there was no valid reason for expecting the same success with fertility; they must not expect the same stability or the same rational progression in marriage and fertility statistics. He was certain that actuaries fully appreciated this, but he was not so sure that it was fully appreciated by other workers in this field.

He thought that it was very instructive to go back to the discussion which took place in the Institute about ten years ago on a paper on the population question by Mr Honey, and to re-read some of the things which were said by actuaries and some of the things which were said by the non-actuarial demographers. He had in mind particularly what Mr Rowell and Sir William Elderton said on that occasion. The discussion was full of the then current pessimism on the fertility and population question, but Mr Rowell and Sir William Elderton took a balanced view. They pointed out the part played by fashion and by current sets of values, and the possibility of a swing of the pendulum. How right they were and how wrong were the demographers.

He thought that they should not attempt to forecast fertility. There had been a spate of forecasting and an infuriating chorus of propaganda on the subject of the population problem. In recent years statements had been made again and again about the 'falling birth-rate' and the 'declining population', but the truth was that the birth-rate touched bottom in the early 1930's, when there was a shortage of husbands due to the 1914–18 war and a postponement of marriages due to the depression and the population had never declined and did not show any signs of declining for a good many years to come. Actuaries long ago knew that the real population problem was not a decline in total numbers but the ageing of the population, the fact that the number of old people was rapidly growing, much more rapidly than the total population. It was interesting to see that demographers were now giving more emphasis to that aspect. A recent publication of *P.E.P.* no longer talked about the imminent decline of the population, but about a possible point at which it would become stationary, and it put the emphasis on the growing numbers of old people.

He believed, as he had said before at meetings of the Institute, that one of the most important matters to actuaries in their professional work and in the population problem was the future of old-age mortality. They would all have seen the announcement by the Registrar-General about the mortality in the first quarter of 1948. He was sure, from what he had seen of the mortality figures so far, that 1948 would represent a record low mortality at the old ages.

What was going to happen to fertility in the next few years? In his view, nobody knew. The demographers would find reasons for thinking that the present situation was not as bright as it seemed. Their general line was to be pessimistic, to find reasons why any favourable trend was not going to last. His own view was that the births would keep up reasonably well, and at any rate there would be enough births to maintain the total population. There was a set-back in the autumn of 1947, and that fitted in very well with the onset of the coal crisis. There had been some recovery from that set-back. He wondered whether there would be an impact from the dollar crisis of July 1947. Even with that set-back, however, the number of births in this country was in his view quite satisfactory.

The President, in proposing a vote of thanks to the author, said that the members present could congratulate themselves and the author upon a discussion which had been very much to the point and conducted at a high level. At one time it had been hoped to have present Mr Rhodes, the donor of the essay prize, but he had not felt able to make the trip from the United States. Some time ago, however, Mr Rhodes expressed a wish to see a copy of the paper in its essay form, and in a letter afterwards said that he was very gratified that his main aim of encouraging the younger men had been accomplished, and that even greater good would be realized if, in the case of the subject of the paper, young and older men alike were tempted to develop matters further.

Whatever might be the lines of future research, members must be very grateful to the author for preparing his essay, which, with the wider publicity now given to it, would, he hoped, do much to confirm actuaries in their already obvious intention to devote to demographic study the increased attention which it undoubtedly deserved. As had already been mentioned, the revised examination syllabus would give it much greater recognition. With those facts in mind, there was no doubt that the paper was a timely and valuable contribution towards the development of a very important and highly topical subject.

Mr A. H. Pollard, in reply, said that he had not yet had access to Mr Karmel's latest paper, nor had he had an opportunity of reading the contribution which Mr Karmel made that evening. Since writing the paper he had done a fair amount of work on the subject, and had considered at some length some of the points that had been brought up in the discussion, and he would not find it possible to deal with them briefly. He would, however, endeavour to do so in a written reply, when he had had the opportunity of reading and studying the contributions to the discussion.

Mr P. H. Karmel writes:

The fundamental equation from which Lotka derives his theory of the stable population and the true annual rate of natural increase is of the form

$$B(t)=\int_0^\infty B(t-x)\,l(x)\,b(x)\,dx, \tag{1}$$

where $B(t)$ is births at time t; $l(x)$ is the probability of a child born surviving to age x; $b(x)$ is the specific fertility rate at age x. It is not difficult to prove that, irrespective of the initial age distribution, for large t, B (t) becomes an exponential function with a relative rate of increase r, where r is the real root of the equation

$$1=\int_0^\infty e^{-rx}\, l(x)\, b(x)\, dx. \qquad (2)$$

These equations are generally made to refer to the female sex, but clearly they can be made to refer to the male sex also, so that we have

$$B_F(t)=\int_0^\infty B_F(t-x)\, l_F(x)\, b_F(x)\, dx, \qquad (3)$$

$$B_M(t)=\int_0^\infty B_M(t-y)\, l_M(y)\, b_M(y)\, dy, \qquad (4)$$

where the subscripts F and M refer to females and males respectively. Equations for the derivation of r_F and r_M analogous to (2) can also be written down. In addition to (3) and (4), there is a third equation relating $B_F(t)$ and $B_M(t)$, namely

$$B_M(t)=mB_F(t), \qquad (5)$$

where m is the masculinity at birth. The functions $l_F(x)$, $b_F(x)$, $l_M(y)$, $b_M(y)$ and the constant m are the given conditions defining the system. It is immediately seen that there are three equations ((3), (4), (5)) and two unknowns ($B_M(t)$ and $B_F(t)$) so that the system is overdeterminate. In short, if r_F and r_M are derived from (3) and (4), they will generally be different, so that m cannot be taken as given. This may be termed the male-female conflict.

Faced with this conflict we can do one of two things. Either one of the three equations can be dropped; or the basic conditions of the system can be redefined. Generally the first line has been adopted by demographers. The usual practice is to omit (4) which is the same as assuming that $b_M(y)$ is left free to vary as the population moves from the actual to the stable age-sex structure, which is also the same as assuming that the fertility of females is independent of the relative male-female supply. Such an assumption might be very nearly correct for a completely promiscuous population, but certainly it is not correct for a population where monogamy is practised and illegitimacy unimportant, which are the very characteristics of most modern populations. Similarly, the omission of (3) implies that the fertility of the males is independent of the relative male-female supply, an equally untenable assumption. The omission of (5) is clearly impossible; such an omission implies that males produce males without the intervention of females and vice versa—a completely unrealistic assumption. Further, this would imply that the masculinity at birth would, for large t, approach 0 or ∞, i.e. either the male or female population would outstrip the other.

It is thus seen that the dropping of any one of the three equations leads to untenable assumptions, untenable because they are unrealistic. Hence it is necessary to consider reformulating the conditions upon which the system is based, i.e. we must consider the reformulation of the basic mortality and fertility functions. As far as I am aware no such explicit reformulation has been published before Mr Pollard's paper, but a suggestion of R. R. Kuczynski in *Fertility and Reproduction* (1932) should be considered. In a chapter in that book in which the male-female conflict is briefly discussed Kuczynski makes a suggestion which he does not follow up in any of his later works, namely, that the two sexes be pooled and reproduction rates for the total of the two sexes worked out. This implies the following system of equations:

$$B_T(t)=\int_0^\infty B_T(t-x)\, l_T(x)\, b_T(x)\, dx, \qquad (6)$$

$$B_M(t)=\frac{m}{1+m}B_T(t), \qquad (7)$$

$$B_F(t)=\frac{1}{1+m}B_T(t), \qquad (8)$$

where the subscript T refers to total of males and females combined and $b_T(x)$ is defined as total births to persons aged x divided by twice the total population aged x, since every birth will be counted twice.

However, although the system of equations (6), (7) and (8) is determinate, this solution of the conflict is unsatisfactory, because $b_T(x)$ must depend on the sex distribution of the population and this will vary as the actual population approaches the stable one.

I now come to Mr Pollard's solution. He assumes, in effect, that a line of descent alternates between males and females, i.e. that males have female children and females have male children. As a result of this, transcribing his equations (15) and (16) into my notation, he obtains

$$B_F(t) = \int_0^\infty \int_0^\infty B_F(t-x-y)\,\phi(x)\,\xi(y)\,dx\,dy, \tag{9}$$

$$B_M(t) = \int_0^\infty \int_0^\infty B_M(t-x-y)\,\phi(x)\,\xi(y)\,dx\,dy. \tag{10}$$

In addition, a third equation, not explicitly quoted by Mr Pollard, must be written down:

$$B_M(t) = m(t)\,B_F(t), \tag{11}$$

where $m(t)$ is the masculinity at birth expressed as a function of time. There are then three unknowns ($B_M(t)$, $B_F(t)$, $m(t)$) and three equations. Furthermore, for large t (9) and (10) become exponential functions with the same value of r, so that $m(t)$ approaches a constant value.

There are two powerful criticisms which can be levelled against this system. First, the functions $\phi(x)$ and $\xi(y)$ do not correspond to reality. The males do not produce female children exclusively and the females male children any more than males produce male and females female. The fact that Mr Pollard's system does not lead to ridiculous results, as does the latter assumption, conceals the unreality of his basic conditions but does not remove this fundamental criticism. Mr Pollard's generations alternate between male and female descendants, so that by taking a joint generation of males followed by females he gets the same ultimate rate of growth whether he starts with males or females. Secondly, Mr Pollard's system implies peculiar behaviour on the part of the masculinity at birth. The asymptotic value of $m(t)$ is determined (see section 5·9) by the initial conditions which are the values of $B_M(t)$ and $B_F(t)$ for t less than the upper age of reproduction. But these initial values of $B_M(t)$ and $B_F(t)$ will depend on the particular sex-age distribution of the actual population from which we start. This in itself is unrealistic, for surely the masculinity at birth is the one factor which we can suppose to remain constant and fixed by factors external to the system of equations. Nor can we suppose that $m(t)$ has its asymptotic value from the start, for that would imply that the initial age distribution should conform to certain conditions, but that this distribution should be completely unrestricted in its character is an essential part of the whole theory of a stable population. It must be concluded that, although ingenious, Mr Pollard's solution is no real solution because the basic conditions which enter into it do not conform to the real facts of population growth.

I should now like to offer a solution of my own. This solution conforms closely enough to reality, but it should be made clear at the start that its practical utility is likely to be very limited. It is of interest, however, as it demonstrates the sort of approach necessary to get a satisfactory solution. In the first place it must be obvious that no solution of the problem can be acceptable which does not take into account that each child has both a male and female parent *at the same time*. Some function of mating must therefore be introduced. Fortunately, this is not an insuperable difficulty, as it can be introduced by means of nuptiality functions. To simplify the discussion, it is assumed that persons only marry once and there is no divorce or illegitimacy. Given the conditions of male nuptiality and premarital mortality, a function $u_M(x, y)$ can be estimated, which gives the probability at birth of a male marrying at age y a female aged x. Similarly, a function $u_F(x, y)$ can be developed based on the conditions of female nuptiality and premarital mortality, which gives the probability at birth of a female marrying at age x a male aged y. Given a knowledge of post-marital mortality in so far as it affects the dissolution of marriages, two more functions, $M_M(x, y)$ and $M_F(x, y)$, can be developed representing respectively the expectation at birth that a male will be living in the married state at age y with a female aged x based on male conditions of nuptiality and the expectation at birth that a female will be living in the married state at age x with a male aged y based on female conditions of nuptiality. Finally, we require $b(x, y)$, being the specific fertility rate of female births to couples aged x and y, and m, the masculinity at birth. Then we have the following system of equations

$$B_F(t) = \int_0^\infty \int_0^\infty B_F(t-x)\, M_F(x, y)\, b(x, y)\, dx\, dy \tag{12}$$

$$B_M(t) = \int_0^\infty \int_0^\infty B_M(t-y)\, M_M(x, y)\, mb(x, y)\, dx\, dy, \tag{13}$$

$$B_M(t) = mB_F(t). \tag{14}$$

This system of equations is also overdeterminate, but the inconsistency between (12) and (13) is seen to lie not in the fertility rates but in the conditions of nuptiality. We could assume male nuptiality is dominant, i.e. female nuptiality adjusts itself as the population moves from the actual to stable positions, and drop (12); or we could make the reverse assumption and drop (13). These equations at least show where the real conflict lies.

But cannot we reformulate what we mean by nuptiality conditions and avoid the conflict? Suppose that the males have a basic pattern of nuptiality which would obtain if the relative supply of males and females did not enter into the matter. Let us designate this 'the potential male nuptiality' and express it by the notation $u^*_M(x, y)$, an analogous function of $u_M(x, y)$. Similarly 'the potential female nuptiality' will be $u^*_F(x, y)$. Now in any year t, if the males were perfectly dominant, the number of marriages of males aged y to females aged x would be

$$B_M(t-y)\, u^*_M(x, y) \tag{15}$$

and if the females were perfectly dominant it would be

$$B_F(t-x)\, u^*_F(x, y). \tag{16}$$

It seems reasonable to assume that in practice a compromise would be effected. Let us take a mean value between the two:

$$[B_M(t-y)\, u^*_M(x, y)]^\lambda \; [B_F(t-x)\, u^*_F(x, y)]^{1-\lambda}, \tag{17}$$

where λ is a positive fraction. This can be written

$$[B_F(t-y)]^\lambda\, [B_F(t-x)]^{1-\lambda}\, \bar{u}(x, y), \tag{18}$$

where
$$\bar{u}(x, y) = m^\lambda\, u^*_M(x, y)^\lambda\, u^*_F(x, y)^{1-\lambda}. \tag{19}$$

Clearly $\bar{u}(x, y)$ represents the distribution of marriages by age of bride and bridegroom which would eventuate in a stationary population with radix of m males and 1 female. Corresponding to $\bar{u}(x, y)$ we can write $\overline{M}(x, y)$ for the distribution of married couples in such a population. It follows that the number of married couples aged x and y in year t will be

$$[B_F(t-y)]^\lambda\, [B_F(t-x)]^{1-\lambda}\, \overline{M}(x, y) \tag{20}$$

and the number of female births will be

$$B_F(t) = \int_0^\infty \int_0^\infty [(B_F(t-y)]^\lambda\, [B_F(t-x)]^{1-\lambda}\, \overline{M}(x, y)\, b(x, y)\, dx\, dy. \tag{21}$$

The number of male births is given by

$$B_M(t) = \int_0^\infty \int_0^\infty [B_F(t-y)]^\lambda\, [B_F(t-x)]^{1-\lambda}\, \overline{M}(x, y)\, m\, b(x, y)\, dx\, dy, \tag{22}$$

and
$$B_M(t) = m\, B_F(t). \tag{23}$$

But (21) and (22) are identical so that we have two unknowns and only two equations. If we now substitute $B_F(t) = Qe^{rt}$ in (21) we find that it is a solution if r is a solution of the characteristic equation

$$1 = \int_0^\infty \int_0^\infty e^{-ry\lambda}\, e^{-rx\lambda\,(1-\lambda)}\, \overline{M}(x, y)\, b(x, y)\, dx\, dy. \tag{24}$$

It can easily be shown in the usual way that there is only one real root of (24) and that it is greater in magnitude than the real parts of the complex roots. If this real root is r_0, then we have for large t

$$B_F(t) = Q_0 e^{r_0 t}, \tag{25}$$

and the other customary propositions about the stable population immediately follow.

The solution of (24) to any degree of accuracy required is quite straightforward. If we write

$$H_0 = \int_0^\infty \int_0^\infty \overline{M}(x, y)\, b(x, y)\, dx\, dy, \tag{26}$$

then from (24)

$$M_{x,y}(-r\,\overline{1-\lambda},\, -r\lambda) = \frac{1}{H_0}, \tag{27}$$

where M is a bivariate moment generating function, that is

$$K_{x,y}(-r\,\overline{1-\lambda},\, -r\lambda) + \log_e H_0 = 0, \tag{28}$$

where K is a bivariate cumulant generating function. Hence r can be solved from

$$\sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \left\{ k_{ij} \frac{(-r\,\overline{1-\lambda})^i}{i!} \frac{(-r\lambda)^j}{j!} \right\} + \log_e H_0 = 0, \tag{29}$$

where k_{ij} is the cumulant of ith order in x and jth order in y in the $\overline{M}(x, y)\, b(x, y)$ distribution and $k_{00} = 0$. To the second order we have

$$0 = -r(\lambda k_{01} + \overline{1-\lambda}\, k_{10}) + \frac{r^2}{2}(\lambda^2 k_{02} + 2\lambda\, \overline{1-\lambda}\, k_{11} + \overline{1-\lambda}^2 k_{20}) + \log_e H_0, \tag{30}$$

and to the first order

$$r = \frac{\log_e H_0}{\lambda k_{01} + (1-\lambda)\, k_{10}}. \tag{31}$$

It should be noted that the denominator is a weighted average of the mean ages of fathers and mothers at the birth of their children in the $\overline{M}(x, y)$ distribution.

The following points should be noted in reference to the above.

(1) The system breaks down under certain extreme conditions. For example, if the males marry females considerably younger than themselves and the population is very rapidly increasing, then there may be insufficient males to provide the compromise number of bridegrooms. Research on this question indicates that for values of r of less than ·03 in magnitude this would not occur, unless the mean difference of age between brides and bridegrooms were very much greater than is usually found. Values of r as great as ·03 in magnitude are never found in practice.

(2) If the males are perfectly marriage-dominant, i.e. $\lambda = 1$, (21) reduces to an equation similar to (13) and (12) can be omitted. On the other hand, if the females are perfectly marriage dominant, i.e. $\lambda = 0$, (21) reduces to an equation similar to (12) and (13) can be omitted. The usual method of calculating only female rates is thus seen to be a special case of this system, i.e. where females are perfectly marriage-dominant.

(3) The net reproduction rate as ordinarily calculated can have two meanings, namely, the ratio of female (male) births to deaths per annum in a population with a stationary age distribution but with the given fertility rates, and the progeny of a single generation of females (males). Now in the above system the first meaning can be attached to H_0, but not the second meaning. The progeny of a female generation will be given by

$$H'_{0F} = \int_0^\infty \int_0^\infty e^{-r(y-x)\lambda}\, \overline{M}(x, y)\, b(x, y)\, dx\, dy, \tag{32}$$

and of a male generation by

$$H'_{0M}=\int_0^{\infty}\int_0^{\infty} e^{-r\,(y-x)\,(1-\lambda)}\overline{M}\,(x, y)\, b\,(x, y)\, dx\, dy. \tag{33}$$

Thus the two interpretations put on a single net reproduction rate as ordinarily calculated are seen to have two different values in this new system. In a stationary population they would, of course, all equal unity.

(4) In the stable population the number of marriages of couples aged x and y will be proportional to

$$e^{-ry\lambda}\, e^{-rx\,(1-\lambda)}\, \bar{u}\,(x, y) \tag{34}$$

and the number of married couples aged x and y will be proportional to

$$e^{-ry\lambda}\, e^{-rx(1-\lambda)}\, \overline{M}\,(x, y). \tag{35}$$

(5) There seems to be no reason why λ should not be regarded as a function of x and y if it is thought desirable. This would not affect the fundamental theory, although the solution of r would become a little more complicated.

(6) In this system the definition of the nuptiality conditions of a population is a modification of that usually adopted. The usual practice is to define the nuptiality conditions in terms of separate male and female nuptiality tables. Here the nuptiality conditions are defined in terms of a single distribution $\bar{u}\,(x, y)$ which is an average of the two fundamental distributions $u_M^*\,(x, y)$ and $u_F^*\,(x, y)$, the potential male and female nuptiality tables. Hence $\bar{u}\,(x, y)$ can be taken as independent of the sex-age distribution of the population, whereas the customary male and female nuptiality tables cannot be so taken.

(7) In order to estimate r, we need to know $\overline{M}\,(x, y)$, λ and $b\,(x, y)$. This latter presents little difficulty, but it does not take much consideration to realize that the determination of the former two functions would be, from a statistical point of view, almost an impossibility. They might be ascertained in a population which had not had its sex distribution affected by external disturbances like wars and migrations and in which nuptiality had not exhibited any great temporary fluctuations, but one can think of no important modern population fulfilling those requirements.

The above is a strictly formal solution of the male-female conflict. As yet it does not seem to be of practical utility, but the implications of the conflict are clearly seen in it. From the practical point of view we are still left with conflicting male and female true annual rates of natural increase, and it still remains largely a matter of judgment where precisely the unique true rate, if there be such a rate, lies.

Mr Pollard has subsequently written as follows:

Before discussing the various points raised by speakers I would like to make quite clear the real task I set myself in this paper. The aim was to consider the relative merits of the various single-figure indices which could be obtained for most countries from the data which they publish each year. I was therefore concerned only with indices that we could actually determine, or at least approximate to, from the data available, and not with indices that we would like to be able to calculate if the data were available. In short, I was concerned with practical possibilities, rather than with theoretical desirabilities. Bearing in mind this difference of outlook, I heartilv agree with many of the points made by the speakers.

I agree with Mr Worger that we should relate marriages to the numbers exposed to the risk of marriage, but the fact is we do not know the numbers so exposed each year. I also am one of those who wish to introduce 'number of previous births' as a factor, but here too, even for census years, the number 'exposed' is not generally analysed as we would desire and the likelihood of having this information available each year is at present remote. (I had to go back to the 1921 Australian census for data for my parity studies and even this was not entirely satisfactory.) I realize, too, the danger, pointed out so clearly by several speakers, of adding values of $f\,(x)$ or of ${}_y b_r$ obtained for a single year and the advantage of a general approach, but the problem I have set myself is not really one of forecasting but of obtaining a reasonable index giving the inherent performance of a particular year, however unusual we may know that year to be. Every single-year index must *necessarily* be open to this criticism in some form or other. Nevertheless, like an influenza mortality index for 1919, although it is dangerous for forecasting, it is a reasonable measure of that year's performance.

While explaining my concentration on indices that are practically possible rather than on those that are theoretically desirable, I would like to emphasize, as I have not done before, the great advance made by Mr Karmel in his 1944 paper when he put forward the idea of relating the births in a given year, specified according to year of marriage, to the relative marriages given in the demographic statistics of previous years. Mr Bailey referred to the fact that our chief difficulty was that the 'exposed to risk' were not available. Mr Karmel's suggestion is an ingenious one which helps us out of the difficulty if we take marriage duration as a factor.

In discussing section 3, several speakers took exception to the fact that, although the proportion married was assumed to change, the fertility was assumed to be constant as in Table 1 and they consequently challenged the implication that the Clark-Dyne formula gave the best answer. The explanation is that Clark and Dyne obtained the data for Table 1 for Queensland for the years 1938 and 1944 and showed that, notwithstanding the change in the proportions married, the marriage fertility in this form was, for all practical purposes, the same. The graphs of section 3 are not, therefore, based on false assumptions, but do represent approximately the position from 1938–44 in Queensland where C_0 measures satisfactorily the replacement of the female sex. When the marriage fertility varies, it would of course have to be allowed for. Mr Cox pointed out that the cosine blending function, and hence Table 5, may not correctly represent the actual transition. That is so but, however the change takes place, the difference between either R_0 or K_0 and C_0 is likely to be great. The cosine blending function represents a smooth change, and the less smooth the change the greater the divergence between the graphs is likely to be. Mr Cox also pointed out that the shape of the R_0 curves could have been anticipated from general reasoning. Nevertheless, I think it is useful to know the *actual* difference, that false negative values may occur, and how R_0 and K_0 differ.

Before discussing the male *versus* female rate problem, I would like to refer to the criticism that the treatment of the joint rate of increase is unsatisfactory because it is based on age only. That is not so, as the reference to a 'joint rate of increase by the Clarke-Dyne method' in the last paragraph would suggest. If we are given marriage rates and, for the opposite sex, marriage fertility according to marriage duration, or some other forms, we can deduce from them the corresponding age-specific rates in the ultimate population. By assuming these age-specific rates to remain constant we can obtain a reproductivity index which depends on the factors which determine fertility and also makes some allowance for the sex anomaly.

The aim of section 5 is, like that of other sections, a very practical one—namely, to obtain a *simple*, *single-figure* index which is better than that obtained by considering one sex only. It is, I think, agreed that the rate allowing for the sex anomaly should lie between the two rates obtained by considering each sex separately. Should we then take the arithmetic mean, or the geometric mean, or some other arbitrary mean between the two rates? In section 5, I am simply suggesting an index which lies between the two indices obtained from each sex separately but which, instead of being some arbitrary mean value, is an intermediate value which has some useful theoretical properties. The aim of section 5, I repeat, was not to give an exhaustive treatment of a problem in mathematical demography, but only to suggest a simple index of reproductivity, the mathematics being introduced there only to bring out the properties of the index suggested. It is probably better to use an index with some useful properties than to take an arbitrary mean. Mr Karmel proposes a solution to the sex anomaly, but not to the problem I was attempting to solve, namely to find a simple index. Mr Karmel's solution is far too complicated for this purpose. In any case, the data required for it are not available and, furthermore, after converting the conflict to a marriage conflict, Mr Karmel proceeds to take an arbitrary average, which therefore does not get us much further.

Incidentally, the marriage function should not be $u(x, y)$—a function of x and y only—but should be a function of the relative supply of males and females at these and at other alternative ages. For as the population moves towards the ultimate age-sex distribution, this marriage function will alter as the relative supply of sexes at various ages alters. I did pursue this line at one stage and obtained an approximate marriage formula allowing for all these factors but, naturally, it was too complicated to be of use in this connexion.

The portions of section 5 that I would like to have deleted before the paper was read are paragraphs 5·8 and 5·9. These paragraphs are related to the chief objection raised to the joint rate of increase, namely that it leads to conflicting values of the sex ratio at birth. The following treatment will make the position clearer.

If we have a stable male-female population increasing at rate σ and if the number of male births at a particular time is l_0^m, then the number of males aged x is $l_x^m e^{-\sigma x}$.

The number of females l_0^f born at that particular time is therefore

$$\int_0^\infty l_x^m e^{-\sigma x} m(x)\, dx = l_0^m \int_0^\infty e^{-\sigma x} \phi(x)\, dx.$$

(Incidentally the number of males born at this particular time similarly

$$= \int_0^\infty l_y^f e^{-\sigma y} f(y)\, dy \qquad = l_0^f \int_0^\infty e^{-\sigma y} \xi(y)\, dy$$

$$= l_0^m \int_0^\infty \int_0^\infty e^{-\sigma(x+y)} \phi(x) \xi(y)\, dx\, dy \quad = l_0^m$$

as it should be.)

The sex ratio at birth therefore

$$= \frac{l_0^m}{l^f} = \frac{1}{\int_0^\infty e^{-\sigma x} \phi(x)\, dx} = \int_0^\infty e^{-\sigma y} \xi(y)\, dy.$$

If we happened to be dealing with a population in which the sex ratio had been X, irrespective of parents' age or sex, then the value of ρ_f obtained would be given by

$$X = \int_0^\infty e^{-\rho_f y} \xi(y)\, dy.$$

Now
$$\int_0^\infty e^{-\sigma y} \xi(y)\, dy \gtreqless \int_0^\infty e^{-\rho_f y} \xi(y)\, dy,$$

according as
$$\sigma \lesseqgtr \rho_f.$$

Therefore the ultimate sex ratio at birth $\gtreqless$ the constant past ratio according as

$$\rho_f \gtreqless \sigma.$$

This, in a sense, may be an anomaly. It simply means that as the age-sex distribution adjusts itself towards the ultimate distribution, if X remains constant, $\phi(x)$ and $\xi(y)$ will alter slightly, as indeed we would expect. This is a similar phenomenon to the marriage rate adjustment (pointed out above in the criticism of Mr Karmel's solution) which occurs as the supply of marriageable males and females varies as the present age-sex distribution tends to the ultimate one.

The fact remains that, *for the year under consideration*, the joint rate of increase gives the inherent unique rate at which the population is increasing. As time goes on, with constant real nuptiality and fertility, $\phi(x)$, $\xi(y)$, the nuptiality functions, etc., will all vary slightly as the supply of the sexes adjusts itself. The most satisfactory ultimate population would be that given by taking σ and X to be constant.

While admitting there are theoretically more satisfying indices I would say, in conclusion, that I favour as a practical index the annual determination of the female

Clark-Dyne rate of increase adjusted in accordance with the correction found necessary at the previous census year, when a joint Clark-Dyne rate of increase will have been determined.

[If B (t) is presumed to be known for $-\infty \leqslant t < 0$, equation (17) may be solved formally in terms of Laplace transforms in a way similar to that described by Vajda, *J.S.S.* Vol. VI, pp. 80–82. If H (t) is defined to be that part of

$$\int_0^\infty \int_0^\infty \mathrm{B}\,(t-x-y)\,\phi\,(x)\,\xi\,(y)\,dx\,dy$$

which depends on values of B $(t-x-y)$ for $-\infty \leqslant t-x-y < 0$ and is therefore a known function,

$$\mathrm{H}\,(t) = \int_0^\infty \int_0^\infty \mathrm{B}\,(t-x-y)\,\phi\,(x)\,\xi\,(y)\,dx\,dy - \int_0^t \int_0^{t-y} \mathrm{B}\,(t-x-y)\,\phi\,(x)\,\xi\,(y)\,dx\,dy,$$

and the solution of equation (17) can be shown to be

$$\mathrm{B}\,(t) = \mathrm{L}^{-1}\left\{\frac{\mathrm{L}\{\mathrm{H}\,(t)\}}{1-\mathrm{L}\{\phi\,(t)\}\,\mathrm{L}\{\xi\,(t)\}}\right\}.$$

Eds. *J.I.A.*]