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Memoir of the early History of Auxiliary Tables for the Computation of Life Contingencies. By FREDERICK HENDRIKS, Esq., Actuary of the Globe Insurance Company.

As nearly all those readers whom the subject-matter of the following pages is likely to interest, are already fully acquaihted with the nature and acknowledged utility of the *Commutation Method,* or arrangement of the preparatory Tables for Annuity and Assurance calculations into certain forms called the D,N,S,M, and R columns; it will, therefore, be unnecessary to enlarge on those points; and the facts to be presently adduced will have reference simply to the early history of that system of computation, or, more strictly, to the date when, and the person by whom, it was first *really* promulgated. In bringing such facts forward, it is requisite to disclaim the remotest intention of urging that full merit is not likewise due to those English writers who have either foreshadowed or extended the method in this country, and who, although *not* its *first* discoverers, were undoubtedly its improvers.

The method was arrived at *separately* by English and by German authors, but the priority in the respective dates will be seen fairly to belong to the *latter.*

The circumstances to be brought forward in support of this view appear to have hitherto escaped any notice in England. There can scarcely be a doubt that when the late Mr. Francis Baily* drew up his well-known Memoir on Barrett's Formulæ,which was read to the Royal Society in June, 1812, his practical mind, and honourable endeavours to prevent, as he says, "Mr. Barrett's astonishing labours sinking into oblivion," offered a guarantee of his being perfectly unaware that formulæ, applicable to the same purposes, and in a more complete form, had been published (though *not* in England) some 27 years *previously;* otherwise it may be justly presumed that he would not have characterised Barrett's invention as an entirely new one, or its principles as "opening a *new* and a wide field to the analyst," without adding parenthetically—"at least new in England." (Note A, page 15.)

Mr. Barrett's unsuccessful attempt to publish his Life Annuity Tables, dates from 1811, and Mr. Baily's letter to him of the 17th June of that year is printed in the proposals for publication. Baily mentions that 25 years was the extent of time Barrett had devoted to the preparation of those Tables. This would bring their first conception to about the date of 1786.

The method was made public in England by Mr. Baily's "Appendix to the Doctrine of Life Annuities and Assurances," printed in 1813; whilst in Germany it had been published in a far more advanced form so far back as the year 1785, the date of the following work:— "Einleitung zur Berechnung der Leibrenten und Anwartschaften die

* Since President of the Royal Astronomical Society.

vom Leben und Tode einer oder mehrerer Personen abhangen mit Tabellen zum practischen Gebrauch, von Johann Nicolaus Tetens, Professor der Philosophie und Mathematik zu Kiel." Leipzig: bei Weidmanns Erben und Reich, 1785.

The above title is as follows in English:—Introduction to the computation of Life Annuities and Reversions which depend on the life and death of one or more persons, with Tables for practical use, by John Nicholas Tetens, Professor of Philosophy and Mathematics at Kiel." Leipsic: Weidmann's Heirs, and Reich, 1785.

It will be preferable to append in a note, instead of inserting here, some particulars as to who and what the learned Professor was; (see Note 1;) and before giving translations of such portions of his work as bear on the subject under consideration, it will be interesting to remark that he commences his somewhat lengthy Preface by observing, that on account of the changes that had occurred in the Calemberg Widows' Fund, it fell to his duty to make himself acquainted with the branch of arithmetic developed in his "Introduction," when noticing that the various questions which arose in that occupation concerning the applicability of the general Tables of Mortality, and respecting the greatness of the risk of the Fund, and other enquiries, were of much importance; and finding these nowhere cleared up, in pursuing his investigations many other results of consequence were arrived at, and they suggested the idea of an original introduction in which he could add to what was known on the subject, such particulars as he thought "most concise and to the purpose."

The changes in the Calemberg Widows' Fund, which Tetens refers to as the cause of his work, are thus mentioned in the communication from Mr. Oeder, of Oldenburgh, which Dr. Price inserted in his work on Life Annuities (see 1st vol. page 132, 6th Edition), where, after noticing the case of the Bremen Institution, which promised Annuities to widows for a payment on admission of a sum equal to one yearly payment of the Annuity purchased, and an annual contribution during marriage of 15 per cent., or a little less than one-seventh of the Annuity, he goes on to say:—

"The States of the Duchy of Calemberg, of which Hanover is the capital, established in 1767 a like scheme, but on terms still more deficient; for, though it differed from the two former schemes in paying a regard to the ages of married persons; yet, notwithstanding several augmentations, the contributions required by it did not, two years ago, come up to half the value of the Annuities. Great numbers, influenced probably by the lowness of the terms and the authority of the States, have been induced to encourage this institution. In 1779 it had Annuities to pay to 600 widows, and consisted of no less than 3800 members or subscribers, whose widows would be entitled to Annuities. In consequence of a rapid increase, its insufficiency was not then become palpable enough to force either a dissolution or a timely and effectual reformation. It was, therefore, likely to lay the foundation of great confusion and distress."

From the preceding extracts it will be seen that these "foreign bubbles" (as Dr. Price terms them) gave rise to the work of Professor Tetens, and the coincidence with the motives assigned by Mr. Baily, in introducing Barrett's Formulæ is somewhat remarkable. The following are Baily's prefatory remarks:—

"Nothing, perhaps, tended so much to destroy the numerous *Bubble Societies* which sprang up about forty years ago; nothing, probably, opened so effectually the eyes of the public with respect to their delusive schemes, as the publication of more correct and comprehensive Tables of the Values of Life Annuities, whereby the true value which ought to be given in such cases, was more accurately determined. And, as one improvement in science generally leads on to another, this naturally opened the way to a more complete and comprehensive investigation of the subject; so that, at the present day, a new, a distinct, and an interesting branch of analysis has arisen, which was unknown to mathematicians of a former period."

Tetens gives in his Preface a condensed view of the researches in the Theory of Life Annuities made by Dr. Halley, De Moivre, and others. (Note No. 2.) The works of Simpson he speaks of with much praise, and then introduces his own contributions to the subject in the following terms:—

"Of the more recent British authors on this matter only Morgan and Price have come to my knowledge, as having produced anything excellent or original. In the general theory both agree with Simpson. Morgan has indicated a double method of computing Annuities, which has, in all respects, much value. Dr. Price, who, with justice, much recommends it, seems, however, to ascribe to it somewhat too great pre-eminence. *I have not found it so easy or short as not to have had reasons for devising yet another, and for preferring this latter method.* One can compare them and judge.

"Dr. Price has made many original remarks of consequence in practice, especially in the fourth edition of his 'Observations on Reversionary Payments,' in two volumes, 1783.

"In the year 1752, when the above-mentioned '*Select Exercises*' of Simpson appeared, it seemed that in Germany one was not yet so much as acquainted with the *elements* of these calculations. We had Widow's, Orphans', and Dead People's Funds; but of what description? All projected at random, or upon an approximative, but in these cases generally deceptive calculation, without just principles, or upon such as have also had the result, which the like ill-calculated projects generally give rise to. Still, we do not need to blush on this account, for such practical follies have likewise occurred among our neighbours on the other side of the sea. They have Provident Associations, which are nearly as faulty as the most faulty of ours, and which are of more recent date than the writings from which they could instruct themselves of a better administration. Reason in books is also with them; not reason in practice. But what might surprise is this, that we in Germany remain behind hand in the science itself; we who are, moreover, so very active in transplanting among us new foreign discovered knowledge. In 1706 *Euler* first propounded his method of computing Widows' Annuities. It was considered of importance as a new discovery, and certainly the illustrious man had found it out of his own accord, but it had no quality less than that of novelty: it was the same method of computing according to the Mortality Tables, as had long been known to the English. (Note No. 3.) From that time forward we have made it a study. Messrs. Seyberth, Kritter, Lambert, Fuss, Oeder, Lous, Bugge, and Florencourt, have laboured therein with intelligence and advantage. We may say that we are equal to the Britons in those branches of it which immediately appertain to our requirement, and, if one looks at it strictly, we excel them in some one or other. But if we wish to be just, we must admit that Simpson had gleaned in no German book, and also that much might be learnt from *Price*, after one has likewise mastered all that has been written on the subject by our countrymen.

"This position of the science, and of so useful a science, was the circumstance which chiefly incited me to pursue it further than I at first intended, and further, perhaps, than the business which led me to it would have required for itself alone.

"The Britons have long been our teachers; or, properly, they could have been so. May not the German, then, embrace the resolution, so to bring it about, that in their turn they can learn something from us, if they will. I would fain incite and give opportunity to that purpose." (Note B, see page 19.)

This praiseworthy sentiment of Professor Tetens was expressed, however, at a period when a long course of political troubles was about to prevent any extended communication of ideas between Great Britain and Germany reciprocally, and to postpone till more peaceful days, the establishment on the continent of the various Provident Annuity and Insurance Institutions, which have since been founded on correct principles, and to the appreciation of which, and as offering valuable information, the work of Professor Tetens had long been available.

The following *translated extracts*, with occasional remarks and notes, have now to be submitted, and the reader is requested, after their perusal, to ask himself the question, Whether, reviewing the circumstances adduced, must it not be concluded that Tetens was the first discoverer of the modern system of arranging Life Tables? If the present memoir be the means of satisfying its readers of that fact, then its object will have been fully carried out, and the care bestowed on its compilation amply recompensed.

Translated Extracts, with remarks and notes:

"*Practical method of computing the mean Duration of Life and Life Annuities according to any given Table of Mortality.*

"The methods of computation hitherto applicable to these subjects either only lead to *nearly correct values*, being *Methods of Approximation*, or else they give such values quite correctly. Recourse was had to the former because the latter appeared diffuse and laborious. But they are not so, when one is but provided with the auxiliary Tables which appertain to the calculation of interest.

"By means of a new Auxiliary Table which can be made in accordance with the Table of Mortality by which it is to be reckoned, and at the rate of interest proposed for its foundation, the whole labour, as well for Life Annuities as for the mean duration of life, may be reduced to one division. The preparation of that table requires nothing more than an easy addition when regard is had to the *Duration of Life* only, but demands somewhat more trouble if it be extended to the *calculation of Life Annuities*,—it would not therefore be desirable to make it for one single Annuity of the kind. But then it gives simultaneously all values of Life Annuities as well as all durations for every age at once. I will here annex it as applied to Sussmilch's Table of Mortality.—(Note 4, p. 5). It is the model for others of the like description."—(Note 5, p. 15.)

TABLE T. (1).

A. Age.	B. Living according to Sussmilch.	C. The Numbers in B discounted for the years of the age, r being $= 1·04$.	D. Sum of the Numbers in B added upward from beneath.	E. Sum of the Numbers in C added upward from beneath.	A. Age.	B. Living according to Sussmilch.	C. The Numbers in B. discounted for the years of the age, r being $= 1·04$.	D. Sum of the Numbers in B added upward from beneath.	E. Sum of the Numbers in C. added upward from beneath.
0	1000	1000	28988	12431·48					
1	750	721·15	27988	11431·48	51	291	39 37	4934	455·74
2	661	611·13	27238	10710·33	52	282	36·69	4643	416·37
3	618	549·40	26577	10099·20	53	273	34·15	4361	379·68
4	593	560·90	25959	9549·80	54	264	31·75	4088	345 53
5	579	475·90	25366	9042·90	55	255	29 49	3824	313·78
6	567	448·11	24787	8567·00	56	246	27 36	3569	284·29
7	556	422·51	24220	8118·89	57	237	25 34	3323	256·93
8	547	399 69	23664	7696·38	58	228	23·44	3086	231·59
9	539	378·69	23117	7296 69	59	219	21·65	2858	208·15
10	532	359·40	22578	6918·00	60	210	19 96	2639	186·60
11	527	342·33	22046	6558·60	61	201	18·37	2429	166·54
12	523	326·66	21519	6216·27	62	192	16·87	2228	148 17
13	519	311·70	20996	5889 61	63	182	15·38	2036	131·30
14	515	297·40	20477	5577·91	64	172	13·98	1854	115·92
15	511	283·74	19962	5280·51	65	162	12·66	1682	101 94
16	507	270·69	19451	4996·77	66	152	11·42	1520	89 28
17	503	258·23	18944	4726·08	67	142	10·26	1368	77·86
18	499	246·32	18441	4467 85	68	132	9·17	1226	67·60
19	495	234·95	17942	4221·53	69	122	8·15	1094	58·43
20	491	224·09	17447	3986·58	70	112	7·19	972	50 28
21	486	213·27	16956	3762·49	71	103	6 36	860	43·09
22	481	202·96	16470	3549 22	72	94	5·58	757	36·73
23	476	193·13	15989	3346 26	73	85	4·85	663	31·15
24	471	183·75	15513	3153 13	74	77	4·23	578	26·30
25	466	174·80	15042	2969 38	75	69	3·64	501	22·07
26	461	166·28	14576	2794·58	76	62	3·15	432	18·43
27	456	158·15	14115	2628 30	77	55	2 68	370	15·28
28	451	150·40	13659	2470·15	78	49	2·30	315	12·60
29	445	142·69	13208	2319·75	79	43	1 94	266	10·30
30	439	135·35	12763	2177·06	80	37	1·61	223	8·36
31	433	128·37	12324	2041·71	81	32	1·33	186	6 75
32	427	121·72	11891	1913·34	82	28	1·12	154	5·42
33	421	115·39	11464	1791·62	83	24	0 93	126	4 30
34	415	109·37	11043	1676·23	84	20	0·74	102	3·37
35	409	103 65	10628	1566·86	85	17	0·61	82	2·63
36	402	97·95	10219	1463·21	86	14	0·48	65	2·02
37	395	92·55	9817	1365·26	87	12	0·40	51	1·54
38	388	87·41	9422	1272·71	88	10	0·32	39	1 14
39	381	82·53	9034	1185·30	89	8	0·24	29	0·82
40	374	77·90	8653	1102·77	90	6	0·18	21	0·58
41	367	73·50	8279	1024·87	91	5	0·14	15	0·40
42	360	69·33	7912	951·37	92	4	0·11	10	0·26
43	353	65·36	7552	882·04	93	3	0·08	6	0·15
44	346	61·60	7199	816·68	94	2	0 05	3	0·07
45	339	58·04	6853	755·08	95	1	0·02	1	0·02
46	332	54·65	6514	697·04	96				
47	324	51·28	6182	642 39					
48	316	48·09	5858	591·11					
49	308	45·07	5542	543 02					
50	300	42·21	5234	497·95					

"This Table constructed, nothing more is requisite for the computation of the mean duration of life, than that when the age of a person is a, the number in the column D against the next following year $a + 1$, be divided by the number, in column B against the given age, and $\frac{1}{2}$ added thereto.

"The number in the column D against the year $a + 1$, inc uding the sum of the numbers of living in B, from the year $a + 1$ upward, the latter included, to the end of the Table. Let A be the number at which we set out, or the number living at the given age, which stands against the year of this age in column B, then the number in column D against the next following year is—

$$= \overset{1}{-}A + \overset{2}{-}A + + \overset{x}{-}A$$

This being then divided by A and $\frac{1}{2}$ added, gives—

$$\frac{\overset{1}{-}A + \overset{2}{-}A + + \overset{x}{-}A}{A} + \tfrac{1}{2} = E\,\overline{a}$$

"Example.—Let the given age be 15 years, then $A = 511$. The number standing under D, against the 16th year, is 19451. Now,

$$\frac{19451}{511} + \frac{1}{2} \text{ is } = 38{\cdot}06 + 0{\cdot}5 = 38{\cdot}56$$

This is the mean duration of life.

"To find the value of a *Life Annuity* of 1, that is for the *whole year*, in which the intermediate periods are not reckoned, and the rent itself is payable always at the end of the year, one divides the number from column E, which stands against the age with 1 year added, by the number in column C against the given age itself.

"EXAMPLE.—Let the given age be 20 years, then the number 3762·49 stands in the column E against 21 years, and 224·09 in the column C against 20 years. Now, $\frac{3762{\cdot}49}{224{\cdot}09} = 16{\cdot}79$. This is the net present value of a Life Rent of 1 for a person aged 20 years, when the interest of money is reckoned at 4 per cent.

"The values found are correct only to two decimal ciphers when the numbers in column B are only taken to so many places. Should a table be prepared with nicety, then one must extend it further, up to 5 figures. In the preceding example one finds by the numbers employed, $\lambda\ \overline{20} = 16{\cdot}7901$; but the more exact required value is 16·7905.—(Note 6, see page 15).

"Moreover it becomes evident from the arrangement of the Auxiliary Table itself, that the method of computation is accurate. The number against the year 20 in column C, viz.: 224·09 is the number of living in that year in column B (491), discounted back for 20 years, expressed thus $\frac{491}{r^{20}}$

The next numbers of the same column C are the following numbers of the living discounted.

$$\frac{\overset{1}{-}A}{r^{20+1}} \qquad \frac{\overset{2}{-}A}{r^{20+2}} \qquad \frac{\overset{3}{-}A}{r^{20+3}}$$

and so farther, where A again represents the number of the living at commencement of the age of 20 years, $\overset{1}{-}A$ that for the first following year, $\overset{2}{-}A$ for the second, and so on. But now the number in column E, at 21, gives the sum

$$\frac{\overset{1}{-}A}{r^{20+1}} + \frac{\overset{2}{-}A}{r^{20+2}} + + \frac{\overset{x}{-}A}{r^{20+x}}$$

therefore this number divided by the number of column C, against 20 years, is equivalent to

$$\frac{1}{A : r^{20}} \cdot \left(\frac{\overset{1}{-}A}{r^{20+1}} + \frac{\overset{2}{-}A}{r^{20+2}} + + \frac{\overset{x}{-}A}{r^{20+x}} \right)$$

and when instead of 20, a is substituted generally for any given age, it is equal to

$$\frac{1}{A : r^{a}} \cdot \left(\frac{\overset{1}{-}A}{r^{a+1}} + \frac{\overset{2}{-}A}{r^{a+2}} + + \frac{\overset{x}{-}A}{r^{a+x}} \right)$$

$$= \frac{1}{A} \cdot \left(\frac{\overset{1}{-}A}{r} + \frac{\overset{2}{-}A}{r^{2}} + + \frac{\overset{x}{-}A}{r^{x}} \right)$$

$=$ the present worth of the Life Annuity at the age a, $= \lambda\,\overline{a}$.

"I consider this method the most excellent of all, especially because one may also use the so prepared auxiliary tables with much advantage in the valuation of Joint Life Annuities, as I will show hereafter. Besides, as column C gives some trouble, one can deviate from it in regard to single life annuities by an easier arrangement.

"1st—Instead of the discounted numbers of the living, one takes the similarly *discounted yearly decrements* and places these in column C, and their sums from beneath upward in column E. The Decrements are much smaller numbers, therefore the discounting succeeds more easily and rapidly.

"2nd—If the given age be a, then one divides the number against a in column E, by the original number A.

"3rd—This quotient being again multiplied by r^a, one thus arrives at *the present value of the sum of* 1 *payable at the decease of a person aged a.* The 21st Table contains these values according to Sussmilch's Table, and at the rate of interest $r = 1{\cdot}04$.

"4th—This product deducted from 1 and the remainder multipied by the *present value of the perpetuity* gives the value of the Life Annuity of 1, whence its worth is easily ascertained for any other amount of yearly rent.

"I will not therefore insert such a Table here myself. Its construction is easily intelligible. One sees that by this plan the labour of the arrangement is shortened, but then again more computation is wanted than by the before described process. However, such methods are very serviceable when we only seek for one or more Life Annuities by a certain Table of Mortality, without requiring to have them all.

"If the preceding plan be used, then the sum at the age *a*, which one takes from the table (calling it *S*)

$$= \frac{\Delta \mathrm{A}}{r^{a}} + \frac{\overset{1}{\Delta^{-}}\mathrm{A}}{r^{a+1}} + + \frac{\overset{x}{\Delta^{-}}\mathrm{A}}{r^{a+x}} \cdot$$

$\Delta \mathrm{A}$, $\overset{1}{\Delta^{-}}\mathrm{A}$, $\overset{x}{\Delta^{-}}\mathrm{A}$, being the decrements following each other of the initial number A at age *a*, each discounted back to the commencement of the life. Thence is

$$\frac{r^{a-1}S}{\mathrm{A}} = \frac{1}{\mathrm{A}} \cdot \left(\frac{\Delta \mathrm{A}}{r} + \frac{\overset{1}{\Delta^{-}}\mathrm{A}}{r^{2}} + + \frac{\overset{x}{\Delta^{-}}\mathrm{A}}{r^{x}}\right)$$

This last quantity being again multiplied by r, is *the present value of a sum of* 1 *payable at the death of a person of the given age* a (*the worth of an inheritance of* 1 *which is to be received at the death of the person.*) This value will be used in many problems, therefore a special table for it was necessary. (See Note 7, page 16.)

Calling this value T then $\lambda\ \overline{a}$ is $= p\ (1 - \mathrm{T})$

Let the given age be 40 years; then by the 21st Table the value $\mathrm{T} = 0{\cdot}4737$, and, because by the same rate of interest $r = 1.04$, p is $= 25$,—then we have $(1 - 0{\cdot}4737) \times 25 = 13.1575$ for the Life Annuity or $\lambda\ \overline{40}$.

When the values of T have not already been computed from the values of the Life Annuities, in that case one could obtain the latter from the former. The English call the Life Annuity *the value of the life (Der Werth des Lebens).* The said values T might according to this analogy be termed *values of the death.* But they have already other names, and these are willingly retained in order to avoid the overstrained affected expressions of 'finding the value of the life from the value of the death' and the like.

" *Third Method of calculating Life Annuities.*

"The arrangement necessary in the following method is still easier:—

"1. Instead of the *yearly decrements* their differences are taken, and each of these discounted back to the commencement of the age as before. Rejecting the lives under 10 years of age, as their Life Annuities seldom come into notice, then by Sussmilch's Table, the first *difference of the decrements* $5 - 4 = 1 = \Delta \mathrm{A} - \overset{1}{\Delta^{-}}\mathrm{A}$ and the following ones up to the 20th year are all $= 0$, and all the subsequent ones either 1 or 0. Consequently the column of discounted numbers, which hitherto has required the most computation, can be transcribed merely from the Table of discounted sums. (Table No. 2 at end of the work.)

"2. One sums these *discounted differences* from beneath upward. But here it is to be noticed, that where the following *decrement* is greater than the one preceding, when $\overset{n+1}{\Delta^{-}}\mathrm{A} > \overset{n}{\Delta^{-}}\mathrm{A}$, then will the *difference* $\overset{n}{\Delta^{-}}\mathrm{A} - \overset{n+1}{\Delta^{-}}\mathrm{A}$, and also the discounted difference be *negative*, and must therefore be *deducted* from the described *summation* from beneath upward.

"3. To use these sums for any other age *e. g.* 20 years, the sum belonging to 20 years must again be multiplied by r^{20}. Let the so found sum of the differences for a commencing number be represented by N;

"4. One subtracts the number N from the fiist decrement of the commencing number A, that is from Δ A, and one multiplies the remainder by $\frac{rp}{A}$. This is also again the value of the reversion. (Note 8, page 17.) Therefore,

"5. That value itself being deducted from unity, and multiplied by the present value of the *perpetuity*, gives the value of the *Life Annuity* of 1. This is expressed in Symbols

$$\lambda \; \overline{a} = \left(1 - rp \, \frac{(\Delta \, A - N)}{A}\right) p.$$

SECTION 61.

"We can still further take the *differences of the differences* instead of the differences themselves, discount them back in the same manner, as before, and sum the discounted numbers backward from the end. With Sussmilch's Table of Mortality nothing is gained thereby, nor have we smaller numbers, or fewer to discount, than when the differences themselves are used. With other Tables it may be the reverse. The computation then to be made is, it is true, somewhat more contracted, but that is unimportant. In the Appendix I have treated of this more at length.

"There will also be found there another method of obtaining the value of Life Annuities by means of the *decrements* and values of *temporary Annuities certain*." (Zeitrenten.)

Extract from Appendix above referred to (page 157):—

"Another easier method of obtaining the value of Life Annuities is the following, but it does not present so convenient a general arrangement at one age as at another. One multiplies the present values of the temporary Annuities following each other in the 3rd Table, $\int \frac{1}{r}$, $\int \frac{1}{r^2}$, $\int \frac{1}{r^3}$, and so on, up to $\int \frac{1}{r^{x-1}}$, when x is the *complement of the life*, (Altersergänzung) each into the decrements belonging thereto, which are but small numbers, for example, $\int \frac{1}{r}$ is multiplied into $\Delta \overset{1}{-} A$, that is, into the decrement of the *second year*. Further $\int \frac{1}{r^2}$ into $\Delta \overset{2}{-} A$, and so forth.

$$\therefore \Delta \overset{1}{-} A \int \frac{1}{r} + \Delta \overset{2}{-} A \int \frac{1}{r^2} + \Delta \overset{3}{-} A \int \frac{1}{r^3} + +$$

$$\Delta \overset{x-1}{-} A \int \frac{1}{r^{x-1}} = A \lambda \overline{a}$$

(See note 9, page 18.)

Section 62.

In reverting to his own system (Table T (1) of this paper), Tetens gives a detailed account of "Morgan's Method of Computing Life Annuities by a *double reckoning*, of which one is at the same time the proof of the other." Our author then adds that the reason of his quoting such method is that Dr. Price prefers it to every other; but "wherefore" "he sees no just ground." He subsequently, however, recommends Morgan's method when a check is desired upon the figures in his new Table. He compares the two systems thus (page 102):—"This method (Morgan's) has decidedly its value, but I imagine, however, that in conciseness (Kürze) it is not to be compared with the one which I have before indicated." Tetens goes on to say that the preparation of his own Auxiliary Table (No. T (1) just quoted) "requires absolutely no division, but only multiplication and addition, when one has already the discounted numbers."

The Table of discounted numbers here means a Table of the present value of one receivable at the end of any number of years at a given rate of interest. (Table 2 of Smart's work.) Tetens notices that Morgan also publishes those discounted values. Morgan's Table is, however, for the rate of 4 per cent. only, whilst that of Tetens is also for 3, 5, and 6 per cent., being a Table computed for his work by Herr von Drateln, and the values are worked out to 10 places of decimals, being four places more than in Morgan's, and two more than in Smart's Table.

Section 85.

"By means of such an Auxiliary Table as the preceding one (§ 56) for the computation of Life Annuities, one finds the *Deferred Life Annuities* (aufgeschobene Leibrenten) in a similar manner by one division. The sum of the discounted numbers of living in column E, against the year $a + n + 1$ has to be divided by the discounted number of living, standing in column C against the year a, and the quotient is what is sought. . . . . When $a = 20$ and $n = 16$, then there stands against the year $a + n + 1 = 37$, in the column E, 1365·26 and against 20 years in the column C, 224·09. Then $\frac{1365.26}{224.09} = 6.093$ is the result."

Section 117.

"Method of computing the average duration of the joint lives of two persons.

"I will here insert the Auxiliary Table which I have already mentioned (chap. 1, § 89). It serves to compute with facility the value of a Life Annuity, increasing according to the natural order of numbers, also the average duration of *joint lives*, and, in the following section, the *values of Annuities on joint lives.*"

TABLE T (2)

A. Year	B. Decrements.	C. Differences of the Decrements.	D. Sums of the Sums of the living.	E. Sums of the Sums of the discounted Numbers of living.	A. Year	B. Decrements.	C. Differences of the Decrements.	D. Sums of the Sums of the living.	E. Sums of the Sums of the discounted Numbers of living.
0	250	+161	857202	210074·50					
1	89	+ 46	827219	197643·02	51	9	0	59589	4339 76
2	43	+ 18	799231	186211 54	52	9	0	54655	3884·02
3	25	+ 11	771993	175501·21	53	9	0	50012	3467·65
4	14	+ 2	745416	165402·01	54	9	0	45651	3087·97
5	12	+ 1	719457	155852·21	55	9	0	41563	2742·44
6	11	+ 2	694091	146809·31	56	9	0	37739	2428·66
7	9	+ 1	669304	138242·31	57	9	0	34170	2144·37
8	8	+ 1	645084	130123·42	58	9	0	30847	1887·44
9	7	+ 2	621420	122427·04	59	9	0	27761	1655 85
10	5	+ 1	598303	115130·35	60	9	0	24903	1447 70
11	4	0	575725	108212·35	61	9	— 1	22264	1261·10
12	4	0	553679	101653·78	62	10	0	19835	1094·56
13	4	0	532160	95437·48	63	10	0	17607	946 39
14	4	0	511164	89547·87	64	10	0	15571	815·69
15	4	0	490687	83969 96	65	10	0	13717	699·17
16	4	0	470725	78689·45	66	10	0	12035	597·23
17	4	0	451274	73692·68	67	10	0	10515	507·95
18	4	0	432330	68966·60	68	10	0	9147	430·09
19	4	— 1	412889	64498·75	69	10	0	7921	362·49
20	5	0	395947	60277·22	70	9	+ 1	6827	304·06
21	5	0	378500	56290·64	71	9	0	5855	253·78
22	5	0	361544	52228·15	72	9	+ 1	4995	210·69
23	5	0	345074	48978·93	73	8	0	4238	173 96
24	5	0	329085	45632.67	74	8	+ 1	3575	142·81
25	5	0	313572	42479·54	75	7	0	2997	116·51
26	5	0	298530	39510·16	76	7	+ 1	2496	94·44
27	5	— 1	283954	36715·58	77	6	0	2064	76·01
28	6	0	269839	34087 28	78	6	0	1694	60·73
29	6	0	256180	31617·13	79	6	+ 1	1379	48·13
30	6	0	242972	29297·38	80	5	+ 1	1113	37·83
31	6	0	230209	27120·32	81	4	0	890	29·47
32	6	0	218885	25078·61	82	4	0	704	22·72
33	6	0	206994	23165·27	83	4	+ 1	550	17·30
34	6	— 1	195530	21373·65	84	3	0	424	13·00
35	7	0	184487	19697·42	85	3	+ 1	322	9·63
36	7	0	173859	18130.56	86	2	0	240	7·00
37	7	0	163640	16667·35	87	2	0	175	4 98
38	7	0	153823	15302·09	88	2	0	124	3·44
39	7	0	144401	14029·38	89	2	+ 1	85	2·30
40	7	0	135367	12844·08	90	1	0	56	1·48
41	7	0	126714	11741·31	91	1	0	35	0·90
42	7	0	118435	10716·44	92	1	0	20	0·50
43	7	0	110523	9765·07	93	1	0	10	0·24
44	7	0	102971	8883·03	94	1	0	4	0·09
45	7	— 1	95772	8066·35	95	1	+ 1	1	0·02
46	8	0	88919	7311·27	96	0	0	0	
47	8	0	82405	6614·23					
48	8	0	76228	5971·84					
49	8	— 1	70365	5380·73					
50	9	0	64828	4837·71					

SECTION 118.

"This Table is only, however, made according to Sussmilch's Table of Mortality, but can be easily constructed according to any other order of dying. It requires the before given Table for single lives (§ 56). From the latter it can be prepared in less than an hour's time. It consists entirely in an easy addition of numbers. The whole work which can be called in any way laborious, is, that one must compute in the first Table for single lives the *discounted numbers of living* (being the numbers for column E). All the rest is then as good as completed.

"The Table before us contains in column C, the *difference of the* decrements in B (what A and B. are one perceives at a glance). The difference of one decrement from the next following is placed at the side of the former, and this is *positive* when the following decrement is smaller, and *negative* in the contrary case.

"The column D of this Table contains the sum of the numbers in column D of the former Table. The sums of the living for every year are summed anew from beneath upward, so that *the number, which stands against any year, is the sum of all the numbers which stand in the former Table against the same year, and which follow from it to the end of the Table.* There stands in the present Table against the 15th year the number 490687. This is the sum of 19962 against the 15th year in the former Table, and all following numbers together.

The *sums of the living* in the former table for a certain age a are $\overset{1}{-}A + \overset{2}{-}A + \overset{3}{-}A + + \overset{x}{-}A$. If this series be symbolized by $\int \overset{1}{-}A$ then the number in the present Table, column D, for the same year, can be distinguished by $S. \int \frac{1}{A}$. For it contains

$$\left(\overset{1}{-}A + \overset{2}{-}A + \overset{3}{-}A + +\right) + \left(\overset{2}{-}A + \overset{3}{-}A + +\right) + \left(\overset{3}{-}A + +\right) + +$$

$$= \int \overset{1}{-}A + \int \overset{2}{-}A + \int \overset{3}{-}A + + = S. \int \overset{1}{-}A$$

"Thus far the Table extends to the computation of the *duration of joint lives.* The last column E appertains to the calculation of Annuities which are payable so long as the joint lives endure. This column E is in like manner constructed by summation of the column E of the former Table, as the numbers of such latter Table from the discounted numbers of living. When the series of *the discounted numbers of living* for a given age

$$= \frac{1}{r^{a}} \left(\frac{\overset{1}{-}A}{r} + \frac{\overset{2}{-}A}{r^{2}} + \frac{\overset{3}{-}A}{r^{3}} + + \frac{\overset{x}{-}A}{r^{x}} \right)$$

is represented by $\int \frac{\overset{1}{-}A}{r^{a}+1}$, then that is the number which stands in the former Table against the year a. The one, however, which stands in the present Table against a in the column E, is

$$= \frac{1}{r^{a}} \left(\frac{\overset{1}{-}A}{r^{1}} + \frac{\overset{2}{-}A}{r^{2}} + \frac{\overset{3}{-}A}{r^{3}} + + \frac{\overset{x}{-}A}{r^{x}} \right.$$

$$+ \frac{\overset{2}{-}A}{r^{2}} + \frac{\overline{\ }A}{r^{3}} + + \frac{\overset{x}{=}A}{r^{x}}$$

$$+ \frac{\overset{3}{-}A}{r^{3}} + + \frac{\overset{x}{-}A}{r^{x}} + + \Big)$$

$$= S. \int \frac{\overset{1}{-}A}{r^{a} + 1}$$

"In using the last Table, one must have the columns B and C of the first at same time before one —I feared that the Table might become too wide for the space of the page, otherwise I would again have here introduced those columns."

SECTION 119.

"From this Table may be *found by one division the values of Life Annuities increasing according to the natural order of numbers* L $\overline{a}$ (§ 76.)

"The number *at the age a* + 1 in *column E* must be divided by the *discounted number of living* in column C of the *first* Table. The quotient is the required value of the *increasing Life Annuity.* (§ 89.)

"For the quotient is—

$$= \frac{1}{r^{a}} \cdot \frac{1}{A : r^{a}} \cdot \left(\frac{\overset{1}{-}A}{r^{1}} + \frac{\overset{2}{-}A}{r^{2}} + + \frac{\overset{x}{-}A}{r^{x}} \right.$$

$$\left. + \frac{\overset{2}{-}A}{r^{2}} + + \frac{\overset{x}{-}A}{r^{x}} \right)$$

"For example—Let the given age be 20 years = a. The number standing against 21 years is 56290·64. The discounted number of living in the first Table is 224.09. Thence $\frac{56290.64}{224.09} = 251.64$. (Note 10, p. 19.) According to the hypothesis of equal decrements (gleichförmigen Absterben) the value was 248·805." (Chap. 1, § 79.)

Professor Tetens then gives some approximative and other rules for determining the average duration of joint lives, and the values of Annuities thereon, generally by addition and substraction of differences of which the Table last given is the element of foundation. It is to be remarked that the practical importance of preparing Joint Life Tables for every combination of ages was not of absolute necessity at the time when he wrote; moreover, his plan of obtaining values for joint lives was suggested by its applicability to the Sussmilch Table of Mortality, in which the differences from the ages of 10 to 96 were either 1 or 0.

The subject of this paper has now taken up more space than was at first intended, and, whilst apprehending that it may prove comparatively uninteresting to the general reader, yet to the actuary, or to any person to whom the progress of the theory of Life Contingencies carries either information or instruction, it is ventured to observe that the consideration of the early history of the theory of Life Contingencies, without reference to Professor Tetens' researches, which are here first introduced to his notice, would be, not only incomplete, but inconsistent with a correct view of the system in its whole history and practice.

June, 1850. F. H.

Note No. 1. (Page 2.)

Our available authorities respecting Johann Nicolaus Tetens are the "Encyclopädie der Deutschen Nationalliteratur" by Dr. O. L. B. Wolff, 7th volume. Leipzig, 1842, and the "Biographie Universelle," 45th volume. Paris, 1826.

Tetens was born September, 1737, at Tetensbüll, Eiderstädt, in the Duchy of Schleswig; he studied from 1755 to 1758 at Rostock and Copenhagen—was appointed in 1763 professor of Physics at Butzow, and occupied from 1765 to 1770 the post of Director of the Educational Establishment which he had newly organized there. In the year 1776 he was invited to Kiel and named to the Professorship of Philosophy and Mathematics in the University of that town. In 1789, he removed to Copenhagen, where he died 19th August, 1807, after having for nearly 20 years occupied some very honourable posts in the administration, as Assessor in the Financial College and Director of Revenue—next in 1791 as Danish State Counsellor and Deputy in the Financial College, and lastly as Privy Counsellor. From 1787, he was a Fellow of the Royal Danish Society.

His German writings are—

No. 1.—Von den vorzüglichsten Beweisen für das Dasein Gottes. Butzow and Wismar, 1761. "On the most prominent proofs of the existence of God."

No. 2.—Ueber den Ursprung der Sprache und Schrift. 8vo. Butzow and Wismar, 1772. "On the origin of Speech and of Writing."

No. 3.—Ueber die allgemeine speculative Philosophie. Butzow, 1775. "Upon general speculative philosophy."

No. 4.—Philosophische Versuche über die menschliche Natur und ihre Entwickelung. 2 vols. 8vo. Leipzig, 1777. "Philosophical Essay on Human Nature and its Development."

No. 5.—Gedanken über einige Ursachen, warum in der Metaphysik nur wenige ausgemachte Wahrheiten sind. Butzow, 1780. "Thoughts respecting some reasons why in Metaphysics there are but few determined truths."

No. 6.—Einleitung zur Berechnung der Leibrenten und Anwartschaften. 8vo. Leipzig, 1785. "Introduction to the Calculation of Life Annuities and Reversions." The second part, forming another volume of the same work, was published at Leipzig, 1786.

No. 7.—(written in French). Voyage sur les côtes de la Mer du Nord, pour y observer la construction des Digues. 8vo. Leipzig, 1788.

No. 8.—(written in French). Considerations sur les Droits réciproques des puissances belligérantes et des puissances neutres sur Mer. 8vo. Copenhagen, 1805.

No. 9.—Tetens also edited the following work in Latin. "Jens Kraftü prælectiones mechanicæ cum additamentis, latinè redditæ." 4to. Butzow, 1773.

Of the above, only Nos. 1 to 5 inclusive are mentioned by Wolff, and only Nos. 2, 3, 6, 7, 8, and 9 in the Biographie Universelle.

In the general catalogue of the British Museum Library we have only found one of the nine works, viz., No. 8 in the above list—"Considerations sur les Droits, &c."

The preceding account of Professor Tetens will be most appropriately concluded with the following remarks from the "Encyclopädie der Deutschen Nationalliteratur."—

"Tetens acquired for himself a celebrity especially by reason that in opposition to those of his time, he treated practical philosophy intelligibly to all, and knew how to bring it forward for that purpose in a clear and luminous manner."

Note No. 2. (Page 3.)

The following passage on the above subject is too remarkable to be omitted:

"These Annuity calculations have long been known in England. Halley, whom we have to thank for so many useful discoveries, came upon them already at the end of the preceding century. Huygens, before him, had taught how to compute probabilities. His principles Halley applied to the Registers of Death, when they were brought into order, and formed the method upon which to compute Life and Widows' Annuities. Some still call such method 'Halley's,' which is quite as reasonable as when one calls the well-known Theorem in Geometry, '*the* Pythagorean'; but some who thus named it, did so in lack of their own conviction of its accuracy, and wished thereby to set it up as such a method as might rest on Halley's reputation. But the reflection upon this is not unserviceable, that it is the identical method which exists in the nature of things, which follows necessarily from it, and therefore is one which every person, if he must not necessarily come upon it, will, however, arrive at it at first, and also in truth every one has come to, who has sought a method *upon right principles.*"

Tetens has *not* put the last three words in italics, but it is better to do so as they contain the pith of the whole paragraph.

Note 3. (Page 3.)

Vide "Milne's Treatise on the Valuation of Annuities and Assurances." Vol. 1, p. 16.

Note A. (Page 1.)

See also Note at page xxv, of "Baily's Doctrine of Life Annuities," particularly the concluding passage, viz.:—"Those useful and interesting parts of the science which relate to the subject of Reversions, Survivorships, and Assurances, together with their several applications to the various purposes of life do not enter into any of the foreign treatises which I have had an opportunity of seeing."

Note 4. (Page 5.)

This Table of Mortality was arranged by Baumann upon the average results of various observations (chiefly made in Brandenburg), and was first inserted in the "Gôttliche Ordnung" of Sussmilch. 4th Edition, 1775.

Note 5. (Page 5.)

It will lead to more clearness of perception of the extracts and notes, if the two tables of Tetens be symbolised thus:—the following by T (1), and the other, a few pages forward, by T (2). A distinction indeed is necessary, as there are columns in each table with like headings.

Note 6. (See Page 6.)

From the preceding example the arrangement will be easily perceived, where the numbers at ages of $a+1$, $a+2$, $a+3$, &c., of columns E, of Tables T (1), and T (2), correspond with those at ages of a, $a+1$, $a+2$, &c. of modern notation; in which structural relation of the ordinary D and N columns, Tetens' arrangement is similar to that of Barrett, which is still preserved by Mr. Farr.

Tetens' remark on the calculation of his Table T (1) to a greater number of decimals, has induced us to check it in that manner between the ages of 95 and 90, which affords an opportunity of annexing the usual columns computed to the extended number of decimal places. Each of those columns will be presently referred to, and their enunciation in Professor Tetens' work briefly noticed. They are as follows:—

1st.—Column D of modern notation, $= l_a \cdot v^a$, being column C of Tetens' first Table T (1)—

95	—	·02408978
94	—	·05010674
93	—	·07816651
92	—	·10839089
91	—	·14090815
90	—	·17585337

2nd.—Column N of modern notation $= D_a + D_{a+1} + \cdots$ corresponding with N_{a-1} of the usual form of arrangement before described, and being column E of Tetens' first Table T (1)—

95	—	·02408978
94	—	·07419652
93	—	·15236303
92	—	·26075392
91	—	·40166207
90	—	·57751544

3rd.—Column S of modern notation $= N_a + N_{a+1} + \cdots$ corresponding with S_{a-1} of the usual form, and being column E of Tetens' Second Table T (2)

95	—	·02408978
94	—	·09828630
93	—	·25064933
92	—	·51140325
91	—	·91306532
90	—	1·49058076

4th.—Column C of modern notation $= d_a \cdot v^a$ and being also the *suggested* Table C of Tetens (see page 7), the reversion being considered as receivable *at decease* of the "*cestuy qui vie.*"

95	—	·02408978
94	—	·02505337
93	—	·02605550
92	—	·02709772
91	—	·02818163
90	—	·02930890

5th.—Column M of modern notation $= C_a + C_{a+1} + \cdots$ corresponding exactly with the usual form, and being the suggested Table S of Tetens (see page 8)

95	—	·02408978
94	—	·04914315
93	—	·07519865
92	—	·10229637
91	—	·13047800
90	—	·15978690

6th.—Column R, of modern notation $= M_a + M_{a+1} + \cdots$

95	—	·02408978
94	—	·07323293
93	—	·14843158
92	—	·25072795
91	—	·38120595
90	—	·54099285

Note No. 7. (See page 8).

Table XXI. is the one Tetens here refers to. In expressing a discounted value he always gives the elementary symbol $\frac{s}{r}$ equivalent in present notation to $\frac{s}{(1+i)^n}$ where i is the rate of interest and n, the number of years. By r, alone, in Tetens' work, is to be understood the amount of 1 at the end of 1 year at the given rate of interest. The C column which Tetens has just described is precisely the C column of modern notation for sums receivable *at decease*, and the column which Tetens calls S (id. est., the *summation* of C) is column M of the modern system.

As Tetens has not given an illustration in numbers, let his rule be applied to the following example :—

Required the value of 1, payable *at decease* of a person aged 90, taking the rate of Mortality from Sussmilch's Table and Interest at 4 per cent.

From the 5th column of the preceding note, Tetens' column S, at 90 would be ·1597869. The initial number A, at age a, is the number of living at age of 90 $=$ 6 (see Table T (1)). Then, by the Rule,

$$\frac{6)\ {\cdot}1597869}{{\cdot}02663115} = \frac{1}{A}\left(\frac{\Delta A}{r} + \frac{\Delta \overset{1}{-} A}{r^2} + + \frac{\Delta \overset{x}{-} A}{r^x}\right)$$

$$\begin{array}{rl} & \cdot02663115 \\ r^a \text{ inverted} = & 33911\cdot43 = \text{amount of 1 in 90 years at 4 per cent.} \\ \hline & 7989345 \\ & 1065246 \\ & 26631 \\ & 2663 \\ & 2397 \\ & 80 \\ & 8 \\ \hline & \cdot9086370 \text{ Answer.} \end{array}$$

On referring to Table XXI of Tetens' work the value at age of 90 is given at ·9086, the decimals not being carried further.

It is evident, therefore, that his formula for the youngest or any other life in the Table, viz.—

$$T = \frac{1}{A} \cdot \left(\frac{\Delta A}{r} + \frac{\overset{1}{\Delta} A}{r^2} + + \frac{\overset{x}{\Delta} A}{r^x} \right) . r^a$$

is equivalent in modern notation to

$$\frac{d_a \cdot v^a + d_{a+1} \cdot v^{a+1} + + d_{a+x} \cdot v^{a+x}}{l_a \cdot v^a}$$

or

$$\frac{C_a + C_{a+1} + + C_{a+x}}{l_a \cdot v^a} = \frac{M}{D}$$

which will present a second proof of its accuracy, for $\frac{M_{90}}{D_{90}}$ is by the extended columns Nos. 5 and 1 in preceding note $= \frac{\cdot15978690}{\cdot17585337} = \cdot908637$, as before.

Note 8, page 9.

Tetens has not given an example of this method, in numbers, but as it may be said of numerical, as well as of pictorial illustrations, that

" Segniùs irritant animos demissa per aurem
Quam quæ sunt oculis subjecta fidelibus,"

it will be of use to apply it to the same question as in the preceding note, viz.—to find the value of a Reversion to 1 receivable at the *decease* of a person aged 90, in which case the discounted differences will be as follows :—

Δ 95 — Δ 96 = 1 which, multiplied by the present value of 1, receivable at end of 96 years, becomes ·02316325

Δ 94 — Δ 95	= 0	....
Δ 93 — Δ 94	= 0	....
Δ 92 — Δ 93	= 0	....
Δ 91 — Δ 92	= 0	.... ..
Δ 90 — Δ 91	= 0	....

Sum of discounted differences = ·02316325

Multiply the above by the amount of 1 in 90 years = 34 . 11933 . . . } 33911·43

$$\begin{array}{r} 6948975 \\ 926530 \\ 23163 \\ 2316 \\ 2084 \\ 69 \\ 7 \\ \hline \cdot7903144 \end{array}$$

Let the last product be represented as in the rule by N; it has then to be subtracted from Δ A or the first *decrement* at the age of A; $\therefore$ in the present case

$$\begin{aligned} \Delta A &= 1\cdot \\ N &= 0\cdot 7903144 \\ \hline \text{remainder} &= \cdot 2096856 \end{aligned}$$

which has to be multiplied by $\frac{r\,p}{A}$ (or in modern notation,

$$\left.\frac{(1+i)\times p}{l_m}\right) = \frac{1.04 \times 25\cdot}{6} = \frac{26\cdot}{6} = 4\cdot 3333$$

$$\begin{array}{r} \cdot 2096856 \\ 33333\cdot 4 \\ \hline 838742 \\ 62906 \\ 6291 \\ 629 \\ 63 \\ 6 \\ \hline \end{array}$$

T $=$ ·908637 which is the value of the reversion, as before; and to obtain the *value of the annuity* at age of 90, using the formula $p\,(1 - T) = \lambda\,\overline{a}$, we have 25 (1 — ·908637) = 25 × ·091363 = 2·284075 —(See Tetens' 16th Table, page 571, the value to 4 places being there 2.2841).

Note 9. (See page 9.)

Let the last example of the preceding note, viz., the computation of the value of an Annuity of 1, at age of 90, per Sussmilch's Mortality and Interest at 4 per cent., be solved by Tetens' fifth method.

$$\begin{aligned}
&\overset{1}{\Delta}A \times \int \frac{1}{r} &&= 1 \times 0\cdot 9615385 = 0\cdot 9615385 \\
&\overset{2}{\Delta}A \times \int \frac{1}{r^2} &&= 1 \times 1\cdot 8860947 = 1\cdot 8860947 \\
&\overset{3}{\Delta}A \times \int \frac{1}{r^3} &&= 1 \times 2\cdot 7750910 = 2\cdot 7750910 \\
&\overset{4}{\Delta}A \times \int \frac{1}{r^4} &&= 1 \times 3\cdot 6298952 = 3\cdot 6298952 \\
&\left(\begin{aligned} &\overset{5}{\Delta}A \times \int \frac{1}{r^5} \\ = &\overset{x-1}{\Delta}A \times \int \frac{1}{r^{x-1}} \end{aligned}\right) &&= 1 \times 4\cdot 4518223 = 4\cdot 4518223 \\
& && \text{Sum} \quad = \quad \overline{13\cdot 7044417}
\end{aligned}$$

This last sum is equal to A $\lambda\,\overline{a}$ of Tetens formula, A being the number of living, which at the age of 90 is $=$ 6, we have therefore for $\lambda\,\overline{a}$ or the value of the Annuity $\frac{13\cdot 70444}{6} = 2\cdot 28407$. Were it not that the subject has already taken so much space, it could be shown how the interesting formula just exemplified would clearly suggest a continuous arrangement similar to that of Mr. Thompson, described in the Post Magazine of 12th May, 1849, the only difference being, that in Tetens' formula the *decrements* and *not* the numbers living, form the element of operation.

Note 10. (See page 13.)

This formula of Tetens', viz., $\frac{E_{a+1} \text{ (of Table T (2))}}{C \text{ (of Table T (1))}}$ is precisely $\frac{S_a}{D_a}$ of the modern system of notation, the E_{a+1} of Table T (2) being equivalent to S_a of the usual form.

Note B. (See Page 4).

Some notice of the authors cited by Tetens may be useful, as this passage will doubtless come under the eye of some readers who have not paid much attention to the works of foreign writers on the subject; and Tetens' mention of them is apt to mislead, as implying at first sight, that they were all *German*; that such is *not* the case will presently be seen, and our impression is, that in alluding to those writers, Tetens' leading idea was to distinguish them as *continental* writers, for it would not be fair to suppose that he considered the fact of their having written in the transactions of German learned academies as constituting a nationality.

The celebrated Leonard *Euler*, was born and educated in Switzerland, and resided the greater part of his life in Russia. Christian *Huygens* was a Dutchman. (Notice of their treatises on the subject of Life Annuities and Probabilities will be found in the Essay on Probability, by Sir William Lubbock, and J. D. Bethune). John Henry *Lambert* was a Swiss, but resided some time in Berlin, his treatise on Annuities appeared in the Beytrage, 1st part, and in the Leipsic Magazine for 1780. He was justly considered one of the most learned men of the 18th century. The following extract from one of his biographers will not be out of place :—

"La formule par laquelle il remplaca la table de Mortalité construite sur les registres de Londres, &c., sont des exemples mémorables de cette aptitude qu'il avait acquise, comme il le dit lui même, en pratiquant fréquemment la construction géométrique des résultats que lui donnait l'analyse; en sorte que l'inspection des formes géométriques lui rappelait sans effort les formules correspondantes, et que, par conséquent, les tables ou séries de faits une fois construites, il avait tout prête une combinaison de calculs propres ã les représenter tres approximativement."

John Peter *Sussmilch* was born at Berlin in 1708, studied theology at the Universities of Halle and Jena, was an army chaplain during Frederick William's Silesian campaign, and afterwards obtained church patronage at Berlin. The work upon which his fame rests was the "Göttliche Ordnung," being a treatise "On the Divine decree in the variations of the human race, with regard to births, deaths, &c." This work is frequently quoted by Malthus, Price, Milne, Tetens, and others; it was the result of much study and sound judgment, and seems to have been well patronized. The first edition appeared at Berlin, 1742; the second in 2 vols. 8vo., with additions, 1761; a third edition in 1765; and a fourth edition in 1775, after his death. This last edition contains the Table quoted by Tetens, and comprises a third volume, the whole edited by Baumann.

Lous was Professor of Mathematics and Navigation at the Academy of Sea Cadets at Copenhagen, and *Bugge* was Professor of Astronomy in the University of the same place. Both were Fellows of the Royal Danish Academy of Sciences. Some notice of their labours may be seen in Dr. Price's 2nd volume (page 439, 6th edition).

Carl Chassot *de Florencourt* was author of the "Abhandlungen aus der juristischen und politischen Rechenkunst." Altenberg, 1781, in 4to. Price notices him as the ingenious author of a mathematical treatise on political arithmetic, published in Germany in 1781."

Oeder, of Oldenburg, was apparently Dr. Price's mentor in all that concerns that part of the latter's work, which refers to German authors or Provident Institutions. Some mention of Oeder's own writings will also be found there.

With respect to *Kritter*, we have before us the titles of three works by him on the subject of Widows' Funds, printed at Gottingen and Hamburg, in 1767, and 1768.

Nicholas von *Fuss* (an author also quoted by Milne in his first volume) was born at *Basle* in 1755, and resided for the greater part of his life at Petersburg, as Professor of Mathematics in the Marine corps. He was named a Counsellor of State in 1800, and died in 1826.

Tetens himself, although from writing in the German language he claims to belong to German literature, may still be more correctly considered as a *Dano-German*, having been born in Schleswig, and having resided either in that duchy or in Copenhagen, during the whole of his life. As six years elapsed between the publication of his work, and the date of Dr. Price's death, it is very likely that Tetens (regard being had to his express desire to instruct the British public in some measure) may, through Mr. Oeder or otherwise, have taken means to communicate his researches to Dr. Price. From Morgan's Biography of the latter it will be perceived that he had not only a superfluity of occupation in those years, but that he laboured under increasing ill health, which interrupted the publication of the fourth edition of his work, a matter he had much at heart, and for which he was constantly collecting materials. The consequence of his illness was that the second volume of the work (which would have contained any *additional particulars* on foreign subjects) was entrusted to the care of Mr. William Morgan. (See the latter's "Memoirs of Dr. Price," pp. 173 and 174).

Are Dr. Price's MS. collections yet extant? if so, they would perhaps clear up this point.