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# A METHOD OF DETERMINING THE RATE OF INTEREST INVOLVED IN A GIVEN TRANSACTION

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In Todhunter's *Compound Interest* (4th edition, chapters 8 and 9) several methods are given for the approximate determination of the rate of interest involved in a given transaction. In these methods there is no adequate treatment of the errors contained in the results. No simple means are given whereby the computer can tell to how many significant figures the results can be trusted.

The purpose of this paper is to explain a simple method applicable to many practical problems by which (*a*) we approach more and more closely the true rate of interest sought, and (*b*) at each stage we obtain limits between which we are sure that the true rate lies. The method depends on (i) ordinary inverse interpolation, and (ii) a knowledge of the signs of the successive derivatives of the function involved. Incidentally, the method has applications to problems other than those of compound interest.

Let $f(x)$ be monotonic within the range $x_0 < x < x_1$. That is, $f'(x)$ cannot change sign within this range. Either $f'(x) \geqslant 0$ and $f(x)$ is not decreasing, or $f'(x) \leqslant 0$ and $f(x)$ is not increasing. If $f'(x) = 0$ at any point, then we have a point of inflexion there and not a turning point. We exclude the case where $f'(x) = 0$ at all points within the range; i.e. where $f(x)$ is a constant.

Then if $c$ lies between $f(x_0)$ and $f(x_1)$ in value, the equation $f(x) = c$ has one root, $\rho$ say, within the given range. There will be no loss of generality if we suppose the scale of $x$ chosen so that $x_0 = 0$ and $x_1 = 1$. We then have that $f'(x)$ does not change sign within the interval $0 < x < 1$ and that the required root $\rho$ lies within this interval.

In what follows we shall refer to the unique polynomial of degree $n-1$ whose graph passes through the $n$ points $(x_1, f(x_1)) \ldots (x_n, f(x_n))$ as "the polynomial through $f(x_1) \ldots f(x_n)$".

Now $f(x)$ can be expanded by difference formulae in various ways. For example

$$f(x) = f(0) + x\Delta f(0) + \frac{x^{(2)}}{2!} f''(\xi_1), \qquad \ldots\ldots(1)$$

where $\xi_1$ lies between the greatest and least of the numbers 0, 1 and $x$ [Ref. (1), p. 210]. That is, $f(x)$ is expressed as the linear function through $f(0)$ and $f(1)$ plus a remainder term.

Also $$f(x)=f(1)+(x-1)\,\Delta f(1)+\frac{(x-1)^{(2)}}{2!}f''(\xi_2), \quad \ldots\ldots(2)$$

where $\xi_2$ lies between the greatest and least of the numbers 1, 2 and $x$. Here, $f(x)$ is expressed as the linear function through $f(1)$ and $f(2)$ plus a remainder term. Ignoring the remainder terms we get two linear equations

$$f(0)+x\Delta f(0) \qquad =c, \quad \ldots\ldots(3)$$

$$f(1)+(x-1)\,\Delta f(1)=c, \quad \ldots\ldots(4)$$

whose solutions $x_1$ and $x_2$ are approximations to the required root $\rho$.

Now let $f''(x)$ have a constant sign over the interval $0<x<2$; for example suppose it to be positive. Then in (1) the remainder term $\frac{x(x-1)}{2!}f''(\xi_1)$ is negative over the interval $0<x<1$. Hence, the line $y=f(0)+x\Delta f(0)$ passing through $f(0)$ and $f(1)$ lies above the curve $y=f(x)$ within this interval. On the other hand, in (2) the remainder term $\frac{(x-1)(x-2)}{2!}f''(\xi_2)$ is positive over the interval $0<x<1$. Hence, the line $y=f(1)+(x-1)\,\Delta f(1)$ passing through $f(1)$ and $f(2)$ lies below the curve $y=f(x)$ within this interval.

Next let $\Delta f(0)$ and $\Delta f(1)$ have the same sign; for example, suppose that they are positive. Then, as is clear from Fig. 1, we have $x_1<\rho<x_2$. The reader can easily verify that there are three other useful cases, viz.

$f''(x)$ negative, $\Delta f(0)$ and $\Delta f(1)$ positive, $x_2<\rho<x_1$,

$f''(x)$ positive, $\Delta f(0)$ and $\Delta f(1)$ negative, $x_2<\rho<x_1$,

$f''(x)$ negative, $\Delta f(0)$ and $\Delta f(1)$ negative, $x_1<\rho<x_2$.

In practice it may be found that the limits of uncertainty obtained by the above method are inconveniently large. If so, the method can be immediately extended.

For example,

$$f(x) = f(0) + x\Delta f(0) + \frac{x^{(2)}}{2!}\Delta^2 f(0) + \frac{x^{(3)}}{3!} f'''(\xi_3), \quad \ldots\ldots(5)$$

where $\xi_3$ lies between the greatest and the least of the numbers 0, 2 and $x$. Here $f(x)$ is expressed as the quadratic function through $f(0)$, $f(1)$ and $f(2)$ plus a remainder term.

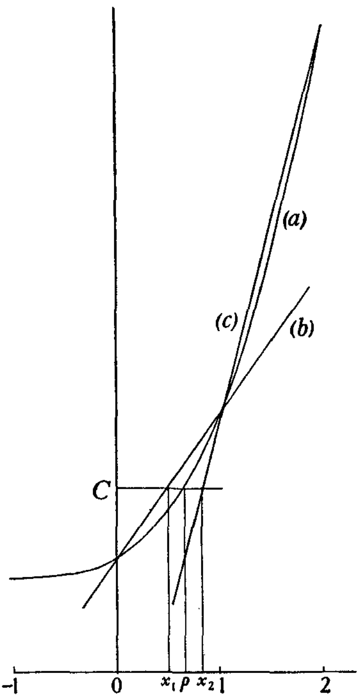


Fig. 1. (*a*) Curve $y = f(x)$. (*b*) Straight line through $f(0)$ and $f(1)$. (*c*) Straight line through $f(1)$ and $f(2)$.

Also

$$f(x) = f(-1) + (x+1)\Delta f(-1) + \frac{(x+1)^{(2)}}{2!}\Delta^2 f(-1) + \frac{(x+1)^{(3)}}{3!} f'''(\xi_4), \quad \ldots\ldots(6)$$

where $\xi_4$ lies between the greatest and the least of the numbers

$-1$, $1$ and $x$. That is, $f(x)$ is expressed as the quadratic function through $f(-1)$, $f(0)$ and $f(1)$ plus a remainder term. Ignoring the remainder terms we get two quadratic equations

$$f(0)+x\Delta f(0)+\frac{x^{(2)}}{2!}\Delta^2 f(0)=c, \qquad \text{......(7)}$$

$$f(-1)+(x+1)\Delta f(-1)+\frac{(x+1)^{(2)}}{2!}\Delta^2 f(-1)=c. \qquad \text{......(8)}$$

Now, the function

$$f(0)+x\Delta f(0)+\frac{x^{(2)}}{2!}\Delta^2 f(0)-c$$

has the value $f(0)-c$ at $x=0$ and the value $f(1)-c$ at $x=1$. Since these quantities are of opposite sign, equation (7) has one root, $x_3$ say, in the interval $0<x<1$. Similarly equation (8) has one root, $x_4$ say, in this interval. These two quantities are approximations to the required root $\rho$. It will be shown in the examples given later that they can be conveniently obtained by iteration (2).

Next, suppose that $f'''(x)$ has a constant sign over the interval $-1<x<2$. For example suppose that it is positive. Then in (5) the remainder term $\frac{x^{(3)}}{3!}f'''(\xi_3)$ is positive within the interval $0<x<1$. Hence, the parabola

$$y=f(0)+x\Delta f(0)+\frac{x^{(2)}}{2!}\Delta^2 f(0),$$

passing through $f(0)$, $f(1)$ and $f(2)$, lies below the curve $y=f(x)$ within this interval. Similarly in (6) the remainder term

$$\frac{(x+1)^{(3)}}{3!}f'''(\xi_4)$$

is negative within the interval $0<x<1$. Hence, the parabola

$$y=f(-1)+(x+1)\Delta f(-1)+\frac{(x+1)^{(2)}}{2!}\Delta^2 f(-1),$$

passing through $f(-1)$, $f(0)$ and $f(1)$, lies above the curve $y=f(x)$ within this interval. Fig. 2 shows this case for $\Delta f(0)$ positive. It is clear that $x_4 < \rho < x_3$. The reader will find that the remaining three cases are

$$f'''(x) \text{ negative}, \quad \Delta f(0) \text{ positive}, \quad x_3 < \rho < x_4,$$
$$f'''(x) \text{ positive}, \quad \Delta f(0) \text{ negative}, \quad x_3 < \rho < x_4,$$
$$f'''(x) \text{ negative}, \quad \Delta f(0) \text{ negative}, \quad x_4 < \rho < x_3.$$

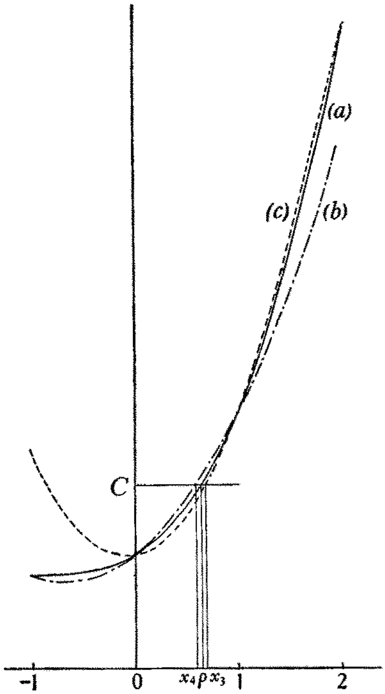


Fig. 2. (*a*) Curve $y = f(x)$. (*b*) Parabola through $f(-1)$, $f(0)$ and $f(1)$. (*c*) Parabola through $f(0)$, $f(1)$ and $f(2)$.

It will be found in many cases that the uncertainty in the value of $\rho$ has been considerably reduced, due to the quadratic interpolation polynomials lying closer to the curve $y = f(x)$ than did the linear interpolation polynomials used in the first attempt.

In general, suppose that $f(x)$ is tabulated for the $n+1$ values $x_1 x_2 \ldots x_{n+1}$, where these values are written in ascending order, but need not be equidistant, and $n > 2$. Suppose also that $c$ lies between $f(x_k)$ and $f(x_{k+1})$ in value where $2 < k < n$ and that

$f'(x)$ does not change sign within the interval $x_k < x < x_{k+1}$, so that the equation $f(x) = c$ has one root within this interval.

$f(x)$ can be expanded in the form (1)

$$f(x) = P_1(x) + \frac{(x-x_1)\dots(x-x_n)}{n!} f^{(n)}(\xi_1), \qquad \dots\dots(9)$$

where $P_1(x)$ is the polynomial of degree $n-1$ passing through $f(x_1) f(x_2) \dots f(x_n)$ and $\xi_1$ lies between the greatest and least of the numbers $x_1$, $x_n$ and $x$.

It can also be expanded in the form

$$f(x) = P_2(x) + \frac{(x-x_2)\dots(x-x_{n+1})}{n!} f^{(n)}(\xi_2), \quad \dots\dots(10)$$

where $P_2(x)$ is the polynomial of degree $n-1$ passing through $f(x_2) f(x_3) \dots f(x_{n+1})$ and $\xi_2$ lies between the greatest and least of the numbers $x_2$, $x_{n+1}$ and $x$.

Ignoring the remainder terms we get two equations

$$P_1(x) = c, \qquad \dots\dots(11)$$

and

$$P_2(x) = c. \qquad \dots\dots(12)$$

Since $P_1(x_k) = f(x_k)$ and $P_1(x_{k+1}) = f(x_{k+1})$, equation (11) has at least one root in the interval $x_k < x < x_{k+1}$. Similarly equation (12) has at least one root in this interval. In many practical cases it will be found that each equation can have but one root within the interval. We shall consider this case only and call the two roots $x'$ and $x''$ respectively.

Next suppose that $f^{(n)}(x)$ does not change sign over the interval $x_1 < x < x_{n+1}$. Then it is easily seen that the remainder terms in (9) and (10) have opposite signs within the interval $x_k < x < x_{k+1}$. Thus one interpolation polynomial will lie above $y = f(x)$ within this interval; the other polynomial will lie below it and $\rho$ will lie between $x'$ and $x''$ in value (cf. Fig. 2). We can expect in most cases that as $n$ is increased the interpolation polynomials will approach the curve $y = f(x)$ more and more closely, and $x'$ will approach $x''$ in value. Admittedly, cases can be found where this will not be true (3). When dealing with the usual case of equidistant values of $x$, it will be advisable to arrange matters so that the interval $x_k$ to $x_{k+1}$ lies as near the middle of the range $x_1$ to $x_{n+1}$ as possible, since the effect of this will usually be to minimize the differences between the true curve and the interpolation polynomials (4).

This establishes a method of solving equations, designed to be of use when (*a*) values of the function can readily be obtained from tables for certain values of the variable, but are difficult to calculate for intermediate values, and (*b*) the first and the $n$th derivatives of the function are known to be of constant sign over suitable ranges.

It must be emphasized that a knowledge of the differences alone is not sufficient. Thus though $\Delta f(0)$ may be positive, this by itself does not preclude $f'(x)$ from changing sign many times between $x=0$ and $x=1$.

In compound interest problems, we often have to solve equations of the form $f(i)=c$, where $f(i)$ is one of the two following types:

$$\text{(I)}\quad f(i)=a_0+a_1v+a_2v^2+\ldots+a_nv^n,$$

where $a_r \geqslant 0$ for all values of $r$, and $i>0$.

That is, $f(i)$ is the present value of certain future payments. Then

$$f'(i)=-(a_1v^2+2a_2v^3+\ldots+na_nv^{n+1}),$$

$$f''(i)=+(2a_1v^3+2.3a_2v^4+\ldots+n(n+1)a_nv^{n+2}),$$

etc.

So we obtain an infinite number of derivatives and all those of odd order are negative, while all those of even order are positive.

$$\text{(II)}\quad f(i)=a_0(1+i)^n+a_1(1+i)^{n-1}+\ldots+a_n,$$

where $a_r \geqslant 0$ for all values of $r$, and $i>0$.

That is, $f(i)$ is the amount of certain past payments. Then

$$f'(i)=na_0(1+i)^{n-1}+(n-1)a_1(1+i)^{n-2}+\ldots+a_{n-1},$$

$$f''(i)=n(n-1)a_0(1+i)^{n-2}+\ldots+2a_{n-2},$$

etc.

So the first $n$ derivatives are positive and subsequent ones are all zero.

The technique described above is applicable to these functions and we proceed to discuss some examples.

*Example* 1 (see Todhunter, pp. 174–227). The equation to be solved is

$$110v^{50}+2{\cdot}75a_{\overline{50}|}=105.$$

We shall employ the following difference table of the function involved:

| $i$ | $x$ | $f(x)$ | $\Delta f(x)$ | $\Delta^2 f(x)$ | $\Delta^3 f(x)$ |
|---|---|---|---|---|---|
| ·0200 | −2 | 127·2830 | | | |
| | | | −9·0785 | | |
| ·0225 | −1 | 118·2045 | | +·8740 | |
| | | | −8·2045 | | −·0937 |
| ·0250 | 0 | 110·0000 | | +·7803 | |
| | | | −7·4242 | | −·0833 |
| ·0275 | 1 | 102·5758 | | +·6970 | |
| | | | −6·7272 | | |
| ·0300 | 2 | 95·8486 | | | |

This table is much more extensive than will be required in most practical cases.

The cubic through $f(-2)$, $f(-1)$, $f(0)$ and $f(1)$ can be written (5)

$$110{\cdot}0000-8{\cdot}2045x+{\cdot}39015\,(x+1)\,x-{\cdot}01562\,(x+1)\,x\,(x-1)$$
$$=P_5(x), \text{ say.}$$

Omitting the last term gives the quadratic through $f(-1)$, $f(0)$ and $f(1)$; call it $P_3(x)$. Omitting the last two terms gives the linear function through $f(-1)$ and $f(0)$; call it $P_1(x)$. Similarly the cubic through $f(-1)$, $f(0)$, $f(1)$ and $f(2)$ can be written

$$110{\cdot}0000-7{\cdot}4242x+{\cdot}34850x\,(x-1)-{\cdot}01389x\,(x-1)\,(x-2)$$
$$=P_6(x), \text{ say.}$$

Omitting the last term gives the quadratic through $f(0)$, $f(1)$ and $f(2)$; call it $P_4(x)$. Omitting the last two terms gives the linear function through $f(0)$ and $f(1)$; call it $P_2(x)$.

The six equations $P_r(x)=105$ where $r=1$ to 6 all have one root within the interval $0<x<1$, and the required root of $f(x)=105$ will lie between the roots of each pair of equations of the same degree.

The solutions of $P_1(x)=105$ and $P_2(x)=105$ are $x={\cdot}60942$ and $x={\cdot}67347$ respectively, or in terms of $i$, $i={\cdot}0265236$ and $i={\cdot}0266837$. So if $\rho$ be the true rate of interest, the first attempt gives

$${\cdot}02652<\rho<{\cdot}02669,$$

and this result could be obtained by the above method using the three tabulated values $f(-1)$, $f(0)$ and $f(1)$ to five significant figures.

Next, the equation $P_3(x) = 105$ can be written in the form

$$x = \cdot 60942 + \cdot 04755\,(x+1)\,x,$$

which is a convenient one for solution by iteration. Choosing the initial trial value of the root as $x_0 = \cdot 66$, the solution runs

$$\begin{aligned} x_0 &= \cdot 66, \\ x_1 &= \cdot 60942 + \cdot 05210 = \cdot 66152, \\ x_2 &= \cdot 60942 + \cdot 05226 = \cdot 66168, \\ x_3 &= \cdot 60942 + \cdot 05228 = \cdot 66170 = x_4. \end{aligned}$$

That is, the value of the root which lies between 0 and 1 is $x = \cdot 66170$, to five places. The corresponding value of $i$ is $\cdot 02665425$.

Similarly, the equation $P_4(x) = 105$ can be written in the form

$$x = \cdot 67347 + \cdot 04694x\,(x-1).$$

By iteration we can readily show that the root that lies between 0 and 1 has a value $x = \cdot 66298$. The corresponding value of $i$ is $\cdot 02665723$.

So the second attempt gives

$$\cdot 026654 < \rho < \cdot 026658.$$

This result could be obtained by the above methods using the four tabulated values $f(-1)$, $f(0)$, $f(1)$ and $f(2)$ to six significant figures, as given in Todhunter, p. 226. In all, thirteen estimates of $\rho$ are given in Todhunter. Already we have obtained limits within which only four of these thirteen values lie (see p. 181 formula ($7a$), p. 183 formula ($8a$), p. 190 formula (14) and p. 227 formula (18)). These four are perhaps the most difficult to compute.

Continuing, the equation $P_5(x) = 105$ can be written

$$x = \cdot 60942 + \cdot 04755\,(x+1)\,x - \cdot 00190\,(x+1)\,x\,(x-1).$$

Choosing the initial trial value of the required root as $x_0 = \cdot 6623$ the solution runs

$$\begin{aligned} x_0 &= \cdot 6623, \\ x_1 &= \cdot 60942 + \cdot 05235 + \cdot 00077 = \cdot 66254, \\ x_2 &= \cdot 60942 + \cdot 05238 + \cdot 00071 = \cdot 66251, \\ x_3 &= \cdot 60942 + \cdot 05237 + \cdot 00071 = \cdot 66250 = x_4. \end{aligned}$$

So the value of the root that lies between 0 and 1 is $x = \cdot 66250$. The corresponding value of $i$ is $\cdot 026656250$.

Similarly $P_6(x) = 105$ can be written

$$x = \cdot 67347 + \cdot 04694x(x-1) - \cdot 00187x(x-1)(x-2),$$

for which the required root is $x = \cdot 66241$, when $i = \cdot 026656025$. Thus $\cdot 0266560 < \rho < \cdot 0266563$. This fixes $\rho$ with an uncertainty of 3 in the seventh decimal place and confirms the "true" value of $\rho$ quoted in Todhunter for purposes of comparison, which has been obtained by some more complicated process than any illustrated in the text.

Seven figures were employed in the tabulation of $f(x)$ to ensure that no complications arose due to "rounding off". Some readers may consider that six figures would have been sufficient. The arithmetic is readily done with the aid of an arithmometer. Clearly, the method can be extended if the value of the root is required more accurately.

*Example* 2 (see Todhunter, p. 199). The equation to be solved is

$$102v^{14} + 3a_{\overline{14}|} = 108.$$

The following difference table was employed:

| $i$ | $x$ | $f(x)$ | $\Delta f(x)$ | $\Delta^2 f(x)$ |
|---|---|---|---|---|
| ·0200 | −1 | 113·622 | | |
| | | | −3·235 | |
| ·0225 | 0 | 110·387 | | +·109 |
| | | | −3·126 | |
| ·0250 | 1 | 107·261 | | +·106 |
| | | | −3·020 | |
| ·0275 | 2 | 104·241 | | |

The use of the linear functions through $f(-1)$ and $f(0)$ and through $f(0)$ and $f(1)$ gave $\cdot 02434 < \rho < \cdot 02441$. The use of the quadratic functions through $f(-1)$, $f(0)$ and $f(1)$ and through $f(0)$, $f(1)$ and $f(2)$ gave

$$\cdot 0244007 < \rho < \cdot 0244013.$$

Clearly, $\rho = \cdot 02440$ to four significant figures. Todhunter gives $\rho = \cdot 02441$. The slip (which has no effect on the final answer) was due to (i) employing the linear function through $f(0)$ and $f(1)$, which gives too high an estimate, and (ii) slightly increasing this error by rounding off to five decimal places.

*Example* 3 (see Todhunter, p. 199). The equation to be solved is

$$73\,(1+i)^{17}+46 s_{\overline{17}|}=1200.$$

The function was tabulated for $i=\cdot 030$, $\cdot 035$, $\cdot 040$ and $\cdot 045$. The use of two linear functions gave $\cdot 03716<\rho<\cdot 03729$ and the use of two quadratic functions gave $\cdot 037194<\rho<\cdot 037196$.

Further accuracy could be obtained in both these examples by continuing the method.

So far, when solving $f(i)=c$, $f(i)$ has been of type (I) or type (II). In this section one or two functions of other types will be discussed.

For example, instead of solving the equation $a_{\overline{n}|}=c$ we might take the equation $1/a_{\overline{n}|}=1/c$ (see Todhunter, pp. 178, 186 and 225). Or for a single debenture, instead of solving the equation $1+k=g a_{\overline{n}|}+v^n$ we might write it in the form $g=i+k/a_{\overline{n}|}$ (see Todhunter, p. 204).

In these the original equation $f(i)=c$ has been transformed into a new equation, say $g(i)=d$. In many cases we shall obtain more accuracy for about the same amount of work, when applying the methods of this paper to the transformed equation, if the ratios $\Delta^2 g/\Delta g$, $\Delta^3 g/\Delta g$, ... are smaller than the corresponding ratios $\Delta^2 f/\Delta f$, $\Delta^3 f/\Delta f$, .... But to assess the error in our solution we require that the derivatives of $g(i)$ be of constant sign over suitable ranges. Difficulties often arise in connexion with this point. For example consider the successive derivatives of such simple functions as $\dfrac{1}{a_{\overline{n}|}}$ and $\dfrac{1}{s_{\overline{n}|}}$.

We first note that since

$$\frac{1}{a_{\overline{n}|}}=i+\frac{1}{s_{\overline{n}|}},$$

$$D\frac{1}{a_{\overline{n}|}}=1+D\frac{1}{s_{\overline{n}|}}$$

and

$$D^p\frac{1}{a_{\overline{n}|}}=D^p\frac{1}{s_{\overline{n}|}},$$

where $D\equiv\dfrac{d}{di}$ and $p>1$.

Next
$$D\frac{1}{s_{\overline{n}|}} = -\frac{Ds_{\overline{n}|}}{(s_{\overline{n}|})^2}$$

and so is negative for $i>0$ and $D\frac{1}{a_{\overline{n}|}}$ is positive for $i>0$.

Then by quite elementary but very tedious means it can be proved that $D^2\frac{1}{s_{\overline{n}|}}$ is positive and $D^3\frac{1}{s_{\overline{n}|}}$ is negative for $i>0$. Hence, $D^2\frac{1}{a_{\overline{n}|}}$ is positive and $D^3\frac{1}{a_{\overline{n}|}}$ is negative. But the writer has discovered no means of dealing with the fourth and higher derivatives, beyond noting by an examination of the first four differences of $\frac{1}{s_{\overline{n}|}}$ that $D^4\frac{1}{s_{\overline{n}|}}$ in general changes sign somewhere between $i=0$ and $i=\cdot 08$ and that the point of change varies for different values of $n$.

So in the following examples it will be inadvisable to apply the methods of this paper beyond the first two stages.

*Example* 4. The equation given in Example 1 can be transformed into

$$110i - 5/a_{\overline{20}|} = 2\cdot 75.$$

Here, the first three derivatives of the function do not change sign for $i>0$. The following difference table was employed:

| $i$ | $x$ | $g(x)$ | $\Delta g(x)$ | $\Delta^2 g(x)$ |
|---|---|---|---|---|
| ·0225 | −1 | 2·3074080 | | |
| | | | +·2663015 | |
| ·0250 | 0 | 2·5737095 | | −·0002155 |
| | | | +·2660860 | |
| ·0275 | 1 | 2·8397955 | | −·0002090 |
| | | | +·2658770 | |
| ·0300 | 2 | 3·1056725 | | |

The use of the two linear functions through $g(-1)$ and $g(0)$ and through $g(0)$ and $g(1)$ gave $\cdot 0266549 < \rho < \cdot 0266564$. Note that only four of Todhunter's thirteen estimates of $\rho$ lie within these limits, which have been obtained by solving two linear equations. Only three values of the function, tabulated to six significant figures, are required.

The use of the two quadratic functions through $g(-1)$, $g(0)$ and $g(1)$ and through $g(0)$, $g(1)$ and $g(2)$ gave

$$\cdot 026656103 < \rho < \cdot 026656111,$$

thus determining $\rho$ with an uncertainty of one in the eighth decimal place.

The method cannot be carried further due to lack of knowledge of the fourth derivative of the function.

*Example* 5. The equation given in Example 3 can be transformed into

$$1127/s_{\overline{n}|} - 73i = 46.$$

Again, the first three derivatives of the function do not change sign for $i > 0$. The function was tabulated for $i = \cdot 030$, $\cdot 035$, $\cdot 040$ and $\cdot 045$. The use of two linear functions gave $\cdot 03714 < \rho < \cdot 03722$ and the use of two quadratic functions gave $\cdot 0371947 < \rho < \cdot 0371957$. No particular advantage arises in this case by transforming the original equation, and the method cannot be carried further for lack of knowledge of the fourth derivative.

After the reader's attention has thus been drawn to cases where practical difficulties arise, it is hoped that he will feel that a very large field remains, in which the methods of this paper can be easily and usefully applied.

## REFERENCES


(1) STEFFENSEN. *Interpolation*, pp. 22 *et seq.*

(2) WHITTAKER and ROBINSON. *Calculus of Observations*, p. 79.

(3) STEFFENSEN. *Interpolation*, p. 39.

(4) FREEMAN. *Actuarial Mathematics*, p. 76 and the references on p. 77; or *Mathematics for Actuarial Students*, Part II, pp. 71 *et seq.*

(5) FREEMAN. *Actuarial Mathematics*, pp. 123 *et seq.*, or *Mathematics for Actuarial Students*, Part II, pp. 142 *et seq.* See also Kerrich, "Systems of Osculating Arcs", *J.I.A.* Vol. LXVI, p. 111, for a simple mnemonic for writing down difference formulae.