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MINIMUM SOLVENCY MARGIN OF A GENERAL INSURNCE COMPANY: PROPOSALS AND CURIOSITIES

ROBERTO DARIS
GIANNI BOSI

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Minimum solvency margin of a general insurance company: proposals and curiosities

Roberto DARIS - Gianni BOSI

Dipartimento di Matematica Applicata
"Bruno de Finetti", Università di Trieste, Italy

Summary

An analytical model is presented for the determination of the minimum solvency margin of a general insurance company. The technical risk proportional to the standard deviation of the aggregate claim amount and the financial risk represented by a multiplying factor are both considered. Further, the ruin probability criterion and the zero expected utility approach starting from a simple solvency condition are compared.

1 Model description

Denote by

U the required solvency margin,

P the risk premium income net of reinsurance,

λ the aggregate safety loading coefficient,

X the aggregate claim amount net of reinsurance,

j the rate of return on investment.

The solvency condition relative to a certain accounting period $[0,1]$ is represented by the following inequality:

$$U(1+j) + P(1+\lambda)(1+j) - X > 0.$$

In this way, it is assumed that the premiums P are collected at time 0 and invested at the random rate j, together with the solvency margin U, in order to match the random aggregate claim amount X to be settled at time 1 or to be put into reserve for outstanding claims.

According to the ruin probability criterion, we choose as the minimum solvency margin U_{MIN} the minimum U satisfying the identity

$$prob\{U(1+j) + P(1+\lambda)(1+j) - X > 0\} = 1 - \epsilon,$$

which is equivalent to

$$prob\{U + P(1+\lambda) \leq X - (U + P(1+\lambda))j\} = \epsilon.$$

Once the normal approximation for the independent random variables X and j (and then for the difference $X - (U + P(1+\lambda))j$) has been assumed, we get

$$\begin{aligned} prob\{U + P(1+\lambda) &\leq E(X) - (U + P(1+\lambda))E(j) \\ &+ \sigma[X - (U + P(1+\lambda))j]\, Z\} = \epsilon, \end{aligned} \tag{1}$$

where Z is a normally distributed random variable with mean zero and standard deviation one.

If we denote by i the deterministic rate of inflation, and we let $E(X) = P(1+i)$, the identity (1) is equivalent to

$$\frac{(1+E(j))(U+P(1+\lambda))-P(1+i)}{\sigma(X-(U+P(1+\lambda))j)} = z_\epsilon, \tag{2}$$

with z_ϵ percentile of Z corresponding to the ϵ ruin probability.

In order to determine U, it is convenient to consider the property

$$\sigma(X-(U+P(1+\lambda))j) = \alpha\left[\sigma(X)+(U+P(1+\lambda))\sigma(j)\right], \tag{3}$$

with $\sqrt{0.5} \le \alpha \le 1$ (see appendix 1).

Putting (3) into (2), we finally obtain

$$U = \frac{1}{1+E(j)-\alpha z_\epsilon \sigma(j)}\left[\alpha z_\epsilon \sigma(X)+P(1+i)\right]-P(1+\lambda), \tag{4}$$

and we choose

$$U_{MIN} = \frac{1}{1+E(j)-\sqrt{0.5}z_\epsilon \sigma(j)}\left[\sqrt{0.5}z_\epsilon \sigma(X)+P(1+i)\right]-P(1+\lambda) \tag{5}$$

as the minimum[1] solvency margin (i.e., the minimum safety reserve).

In particular, given $\epsilon = 0.2\%$, we have

$$U_{MIN} = \frac{1}{1+E(j)-2\sigma(j)}\left[2\sigma(X)+P(1+i)\right]-P(1+\lambda). \tag{6}$$

In the sequel, $c(j)$ will stand for the risk coefficient $\frac{1}{1+E(j)-2\sigma(j)}$.

We note that it is reasonable to assume $c(j) > 0$. In fact, only a very risky investment can lead to $2\sigma(j) - E(j) > 1$.

From (6), we can observe that

$$\begin{aligned} a)\ if\ \ E(j) &> 2\sigma(j)\ \ (riskless\ investment\ \diamond\ c(j) < 1) \\ U_{MIN} &< [2\sigma(X)+P(i-\lambda)] \end{aligned}$$

$$\begin{aligned} b)\ if\ \ E(j) &< 2\sigma(j)\ \ (risky\ investment\ \diamond\ c(j) > 1) \\ U_{MIN} &> [2\sigma(X)+P(i-\lambda)] \end{aligned}$$

[1] It is easy to prove that U is an increasing function of α.

$$\begin{aligned} c)\ if\ \ E(j) &= 2\sigma(j)\ \ (neutral\ investment\ \diamond\ c(j)=1) \\ U_{MIN} &= [2\sigma(X) + P(i-\lambda)]. \end{aligned}$$

In case b), for example, U_{MIN} should cover

1. the technical risk $2\sigma(X)$,
2. the amount $P(i-\lambda)$ (if $i > \lambda$),
3. the financial risk (measured by the multiplying factor $c(j)$).

In order to have some practical applications[2] of this model, let us consider figure 1.

Figure 1

$P = 84.42$
$\lambda = 3\%$
$i = 3\%$
$\gamma = 15\%$
$P_N = 100$
$\sigma(X) = 9$

investment	j	$E(j)\%$	$\sigma(j)\%$	$c(j)$	U_{MIN}
REAL ESTATE ASSETS	j_1	6	3	1	18
BONDS	j_2	5	10	1.17	35.85
EQUITIES	j_3	20	25	1.43	63.15

In the last column you can find the minimum solvency margin, expressed as percentage of P_N (premium income net of reinsurance), corresponding to three different kinds of investment.

[2] γ is the expenses loading coefficient, and the assumption $\sigma(X) = 9$ allows us to compare U_{MIN} with the minimum solvency margin required by EC regulation ('73). For a practical estimation of $\sigma(j)$ and $\sigma(X)$, see Daris [3], Daykin, Pentikainen and Pesonen [4], and Rantala [5].

Figure 2 considers the more realistic case of mixed investments[3].

Figure 2

REA α_1 %	BON α_2 %	EQ α_3 %	$E(j)_{\%}$	$\sigma(j)_{\%}$	$c(j)$	U_{MIN}
10	80	10	6.6	8.38	1.11	29.55
10	65	25	8.85	8.92	1.09	27.45
20	40	40	11.2	10.78	1.11	29.55
0	80	20	8	9.43	1.12	30.60
0	70	30	9.5	10.25	1.12	30.60
0	60	40	11	11.66	1.14	32.70
0	50	50	12.5	13.46	1.16	34.80

We conclude our considerations about the previous model observing that (6) can be rewritten as follows:

$$U + \lambda P = 2c(j)\sigma(X) + \left[c(j) - \frac{1}{1+i}\right] E(X). \tag{7}$$

Observe that the aggregate safety amount $U + \lambda P$, which is necessary to guarantee a solvency situation with probability 0.2%, is a linear combination of $\sigma(X)$ and $E(X)$ with coefficients[4] $2c(j)$ and $c(j) - \frac{1}{1+i}$.

[3] Once the independence of j_1, j_2 e j_3 has been assumed, $\sigma(j) = \sqrt{\alpha_1^2\sigma^2(j_1) + \alpha_2^2\sigma^2(j_2) + \alpha_3^2\sigma^2(j_3)}$ holds. Since j_2 e j_3 are positively correlated in practice, it should be noted that U_{MIN} is underevaluated in the latter four cases.

[4] Identity (7) generalizes $U + \lambda P = z_\epsilon\sigma(X)$ in the case when the solvency condition $U + P(1+\lambda) - X > 0$ is adopted (see Beard, Pentikainen and Pesonen [1]).

2 Expected utility approach

It may be interesting to compare the ruin probability criterion and the zero expected utility approach when the solvency condition is simply

$$U + P - X > 0.$$

In the first case, it is well known that, if a normal approximation for X is adopted, and $P = E(X)$, then the condition $prob\{U + P - X > 0\} = 1 - \epsilon$ leads to

$$U_{MIN} = z_\epsilon \sigma(X). \tag{8}$$

On the other hand, if we consider, for example, the exponential utility function $u(x) = B\left(1 - e^{-\frac{x}{B}}\right)$, the solvency margin U can be determined as the amount satisfying the following zero expected utility condition:

$$E\left(B\left(1 - e^{-\frac{U+P-X}{B}}\right)\right) = 0.$$

Under the previous assumptions, we can easily find

$$U = B \ln E\left(e^{\frac{\sigma(X)}{B} Z}\right).$$

Once a second degree approximation for the cumulant generating function of $\frac{Z}{B}$ has been used, we may choose as the minimum solvency margin

$$U_{MIN} = \frac{1}{B}\frac{\sigma^2(X)}{2}, \tag{9}$$

where $\frac{1}{B}$ (equal to $-\frac{u''(x)}{u'(x)}$) is the well known risk aversion coefficient[5].

The comparison of (8) and (9) yields to the following relation between $\frac{1}{B}$ and z_ϵ:

$$\frac{1}{B} = \frac{2z_\epsilon}{\sigma(X)}. \tag{10}$$

Therefore, if we assume a ruin probability equal to 0.3% and a standard deviation $\sigma(X)$ equal to 6.5% of the premium income (net of reinsurance)

[5]Even if we use a quadratic utility function, the same expression of U_{MIN} is obtained (see appendix 2).

P_N, or equivalently (from (8)), $U_{MIN} = 0.18P_N$, just like in EC regulation (see Campagne [2]), it is somewhat surprising that

$$\frac{1}{B} \simeq 117P_N.$$

Appendix 1

Let us show that, given two independent random variables X and Y, the following inequality holds:

$$\sqrt{\frac{1}{2}}\,(\sigma(X)+\sigma(Y)) \leq \sigma(X \mp Y) \leq \sigma(X)+\sigma(Y). \tag{11}$$

Since

$$\sigma(X \mp Y) = \sqrt{\sigma^2(X \mp Y)} = \sqrt{\sigma^2(X)+\sigma^2(Y)}, \tag{12}$$

and

$$\sigma(X)+\sigma(Y) = \sqrt{(\sigma(X)+\sigma(Y))^2} = \sqrt{\sigma^2(X)+\sigma^2(Y)+2\sigma(X)\sigma(Y)}, \tag{13}$$

we note that

$$\sigma(X \mp Y) \leq \sigma(X)+\sigma(Y) \tag{14}$$

(the equality holds only if $\sigma(X)$ and/or $\sigma(Y)$ are zero).

If $\sigma(X)$ and $\sigma(Y)$ are not zero, and we consider both (12) and (13), we obtain

$$\sigma(X \mp Y) = \sqrt{k}\,(\sigma(X)+\sigma(Y))\,, \tag{15}$$

with

$$k = \frac{\sigma^2(X)+\sigma^2(Y)}{\sigma^2(X)+\sigma^2(Y)+2\sigma(X)\sigma(Y)} \tag{16}$$

By letting[6] $h = \frac{\sigma(Y)}{\sigma(X)}$, we finally have

$$\sqrt{k} = \sqrt{\frac{1+h^2}{(1+h)^2}}.$$

[6] It is the same if $h = \frac{\sigma(X)}{\sigma(Y)}$.

The function $\sqrt{\frac{1+h^2}{(1+h)^2}}$, which is defined for $h > 0$, takes its minimum value $\sqrt{\frac{1}{2}}$ for $h = 1$ (i.e., $\sigma(X) = \sigma(Y)$). Further, it tends to one as h diverges (i.e., $\mid \sigma(X) - \sigma(Y) \mid \to +\infty$).

Appendix 2

Given a quadratic utility function $u(x) = x - \frac{x^2}{2B}$, which is defined for $0 \leq x \leq B$, we look for the solvency margin U satisfying

$$E\left((U+P-X) - \frac{(U+P-X)^2}{2B}\right) = 0$$

The approximation $X \simeq P + \sigma(X)Z$, together with straightforward computations, leads to

$$U^2 - 2BU + \sigma^2(X) = 0, \tag{17}$$

with roots

$${}_1U_2 = B\left[1 \pm \sqrt{1 - \left(\frac{\sigma(X)}{B}\right)^2}\right] \simeq^{7} \pm \frac{1}{B}\frac{\sigma^2(X)}{2}$$

Hence, we choose the positive root as the minimum solvency margin.

References

[1] Beard, R.E., Pentikainen, T. and Pesonen, E. (1984) *Risk Theory*, 3rd edition, Chapman & Hall, London.

[2] Campagne, C. (1961) Minimum standards of solvency of insurance firms-Report of the ad hoc Working Party on Minimum Standards of Solvency, OEEC, TP/AS(61)1.

[3] Daris, R. (1998) Un modello analitico per la minima riserva di sicurezza relativa ad un'impresa di assicurazione contro i danni, *Quaderni del Dipartimento di Matematica Applicata "Bruno de Finetti"*, n. 8/1998.

[7] The square root has been approximated by $1 - \frac{1}{2}\left(\frac{\sigma(X)}{B}\right)^2$.

[4] Daykin, C.D., Pentikainen, T. and Pesonen, E. (1994) *Practical Risk Theory for Actuaries*, 1st edition, Chapman & Hall, London.

[5] Rantala, J. (1995) A report on assessing the solvency of insurance company, CEA-Working document CP 003 (02/95).