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MODELLING AND PROJECTING MORTALITY IMPROVEMENT RATES

STEVEN HABERMAN and ARTHUR RENSHAW

Cass Business School, City University London

The Actuarial Profession Mortality and Longevity Seminar, 22 March 2011

AGENDA

- Motivation
- History
- Model Construction
- Case Study: Modelling Mortality Rates

  Modelling Mortality Improvement Rates
- Preliminary conclusions

MODELLING & PROJECTING MORTALITY IMPROVEMENT RATES: HISTORY

MR- mortality rates

Lee & Carter (1992)	Cairns, Blake, Dowd *et al.* (2008)
Brouhns, Denuit, Vermunt (2002)	Plat (2009)
Renshaw & Haberman (2006)	Haberman & Renshaw (2011)

MIR- mortality improvement rates

Willets (2004)
Richards, Kirkby, Currie (2005)
Renshaw & Haberman (2006)
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DATA

$\left(d_{xt}, e_{xt}, \omega_{xt}\right)$: age x, period t

$y_{xt} = \hat{m}_{x,t} = \dfrac{d_{xt}}{e_{xt}}$ - empirical mortality rate

$z_{xt} = 2\dfrac{\left(1 - \hat{m}_{x,t} / \hat{m}_{x,t-1}\right)}{\left(1 + \hat{m}_{x,t} / \hat{m}_{x,t-1}\right)}$ - mortality improvement rate

improving mortality (over time) $\Rightarrow \; z_{xt} > 0$

deteriorating mortality (over time) $\Rightarrow \; z_{xt} < 0$

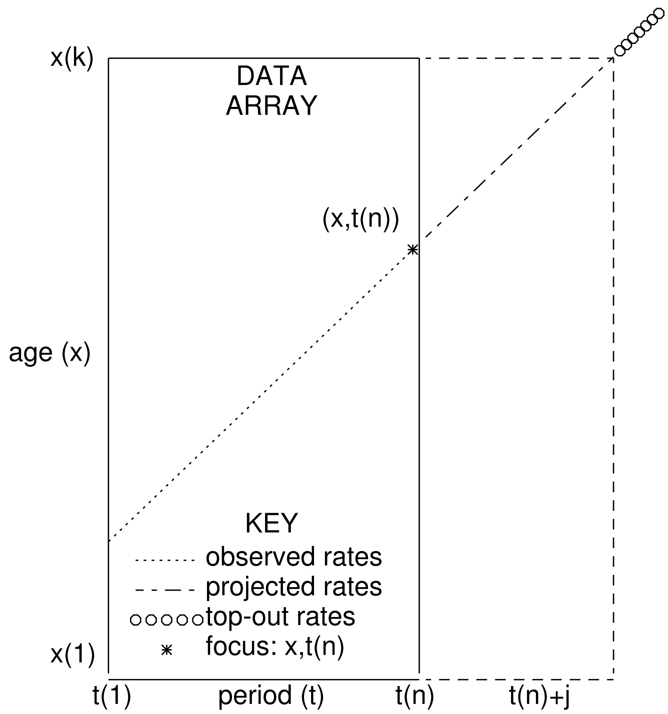
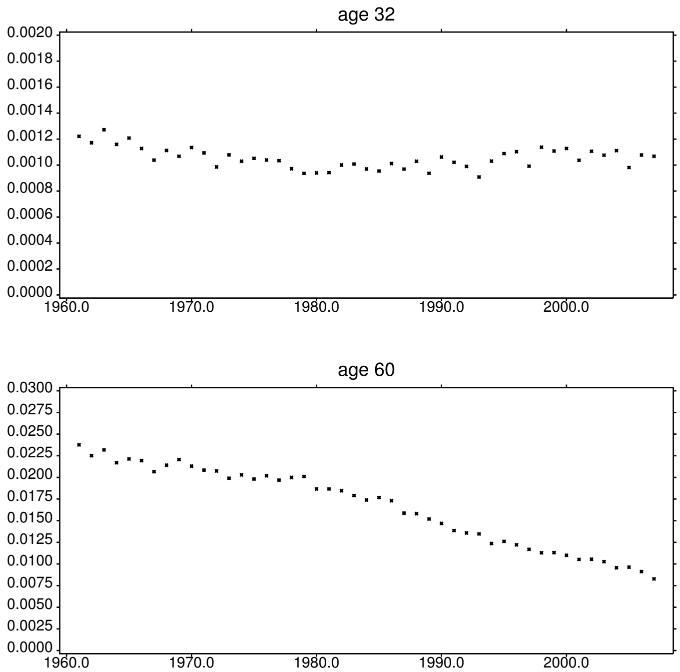
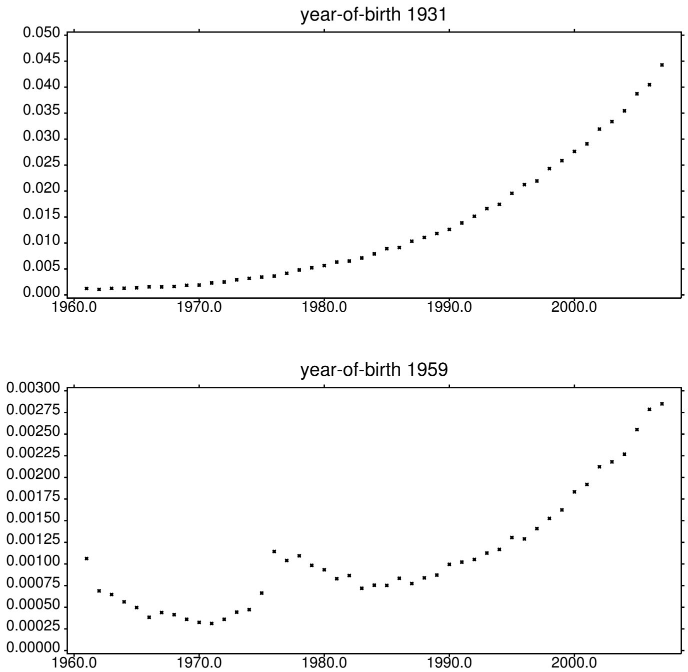


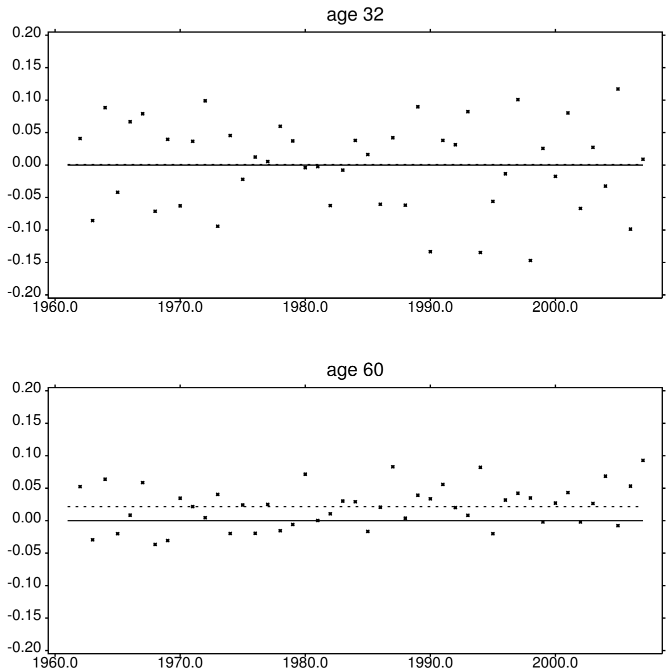
Fig 2. Schematic diagram depicting a rectanglar age-period data array with cohort trajectory, focus at age x, in peripheral year t(n), partitioned in three: comprising observed, projected and top-out mortality rates.



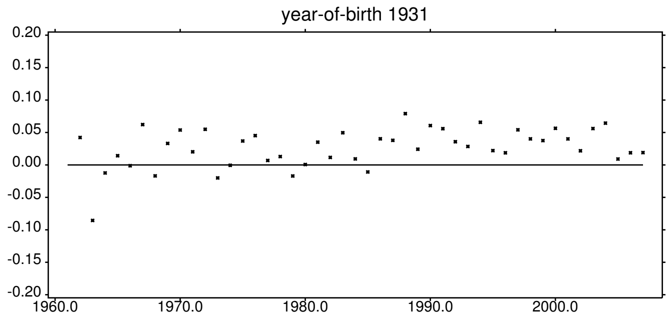
E&W 1961-2007 male mortality.
MR vs period, fixed ages: 32 or 60

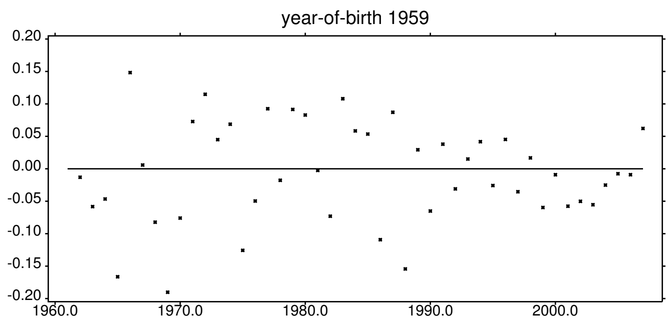


E&W 1961-2007 male mortality.
MR vs year of observation,
fixed year-of-birth: 1931 or 1959



E&W 1961-2007 male mortality.
MIR (with average) vs period,
fixed ages: 32 or 60





E&W 1961-2007 male mortality.
MIR vs year of observation,
fixed year-of-birth: 1931 or 1959

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$$\begin{tikzcd}[column sep=huge, row sep=large]
\textbf{\textit{MR}} & \hat{m}_{x,t} \arrow[r, "\textit{Model \& forecast}", "(\textbf{\textit{Route I}} \rightarrow)"'] \arrow[dd, "\text{Map (differentiate) (\&) (scale)}"'] & q_{x+j,t_n+j} \arrow[r] & \textit{Predictions} \text{ } \textit{route I} \text{ indices} \\
 & & q_{x+j,t_n+j} \arrow[r] & \textit{route II} \text{ indices} \\
\textbf{\textit{MIR}} & z_{x,t} \arrow[r, "(\textbf{\textit{Route II}} \rightarrow)", "\textit{Model \& forecast}"'] & z_{x+j,t_n+j} \arrow[u, "\text{Inverse Map (rescale) (\&) (integrate)}"'] &
\end{tikzcd}$$

$m_{x,t}$- central mortality rate (MR); $z_{x,t}$- mortality improvement rate (MIR)

q_{x+j,t_n+j}- predicted probability of death

MR predictions- ***route I***; ***MIR*** predictions- ***route II***

MORTALITY RATES (Route I) APPROACH

TARGET:

m_{xt} - central rate of mortality

PARAMETRIC PREDICTOR STRUCTURES:

$$LC: \eta_{xt} = \alpha_x + \beta_x \kappa_t$$

$$H_1: \eta_{xt} = \alpha_x + \beta_x \kappa_t + \iota_{t-x}$$

$$H_0: \eta_{xt} = \alpha_x + \kappa_t + \iota_{t-x}$$

$$M5: \eta_{xt} = \alpha_x + \kappa_t^{(1)} + \left(x - \overline{x}\right)\kappa_t^{(2)}$$

$$M6: \eta_{xt} = \alpha_x + \kappa_t^{(1)} + \left(x - \overline{x}\right)\kappa_t^{(2)} + \iota_{t-x}$$

$$M7: \eta_{xt} = \alpha_x + \kappa_t^{(1)} + \left(x - \overline{x}\right)\kappa_t^{(2)} + b\left(x\right)\kappa_t^{(3)} + \iota_{t-x}$$

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MORTALITY RATES (Route I) APPROACH cont…

$$b(x)=\left\{(x-\bar{x})^2-\frac{1}{k}\sum_{i=x_1}^{x_k}(i-\bar{x})^2\right\},\ \bar{x}=\frac{1}{k}\sum_{i=x_1}^{x_k} i$$

MODEL FITTING:

Let $D_{xt} \sim P\left(e_{xt}m_{xt}\right)$ i.i.d., with constant dispersion

$$Y_{xt}=\frac{D_{xt}}{e_{xt}},\ E\left(Y_{xt}\right)=m_{xt},\ Var\left(Y_{xt}\right)=\phi\frac{m_{xt}}{\omega_{xt}e_{xt}}$$

log link $\log m_{xt}=\eta_{xt}$, parametric predictor η_{xt}, weights $\omega_{xt}e_{xt}$,

scale parameter ϕ, variance function $V(u)=u$.

To fit: minimise the model deviance.

MORTALITY IMPROVEMENT RATES (Route II) APPROACH

MOTIVATION:

$$\log m_{xt} = \eta_{xt} \Rightarrow \frac{1}{m_{xt}} \frac{\partial m_{xt}}{\partial t} = \frac{\partial \eta_{xt}}{\partial t}$$

Notational convention: re-define symbols appropriately

TARGET:

η_{xt} - mortality improvement rate

MORTALITY IMPROVEMENT RATES (Route II) APPROACH cont…

DUAL PARAMETRIC PREDICTOR STRUCTURES:

$$LC: \eta_{xt} = \beta_x \kappa_t$$

$$H_1: \eta_{xt} = \beta_x \kappa_t + \iota_{t-x}$$

$$H_0: \eta_{xt} = \kappa_t + \iota_{t-x}$$

$$M5: \eta_{xt} = \kappa_t^{(1)} + \left(x - \overline{x}\right)\kappa_t^{(2)}$$

$$M6: \eta_{xt} = \kappa_t^{(1)} + \left(x - \overline{x}\right)\kappa_t^{(2)} + \iota_{t-x}$$

$$M7: \eta_{xt} = \kappa_t^{(1)} + \left(x - \overline{x}\right)\kappa_t^{(2)} + b(x)\kappa_t^{(3)} + \iota_{t-x}$$

LINK FUNCTION: IDENTITY

MORTALITY IMPROVEMENT RATES (Route II) APPROACH cont…

MODEL FITTING (Single stage):

Assume $Z_{xt} \sim N\left(\eta_{xt}, \sigma^2\right)$ i.i.d., constant dispersion

$$E\left(Z_{xt}\right) = \eta_{xt},\ Var\left(Z_{xt}\right) = \sigma^2 \frac{1}{\omega_{xt}}$$

identity link, parametric predictor η_{xt}, weights ω_{xt},

scale parameter σ^2, variance function $V\left(u\right) = 1$.

To fit: minimise the model deviance.

MORTALITY IMPROVEMENT RATES (Route II) APPROACH cont…

MODEL FITTING (Joint two stages):

Stage 1:

Assume $Z_{xt} \sim N\left(\eta_{xt}, \phi_{xt}\sigma^2\right)$ i.i.d., variable dispersion

$$E\left(Z_{xt}\right) = \eta_{xt},\ Var\left(Z_{xt}\right) = \sigma^2 \frac{\phi_{xt}}{\omega_{xt}}$$

identity link, parametric predictor η_{xt}, weights ω_{xt}/ϕ_{xt}

scale parameter σ^2, variance function $V\left(u\right) = 1$.

Yields: residuals $r_{xt} = z_{xt} - \hat{\eta}_{xt}$ to form Stage 2 responses.

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MORTALITY IMPROVEMENT RATES (Route II) APPROACH cont…

Stage 2:

Model squared residuals R_{xt}^2 as follows

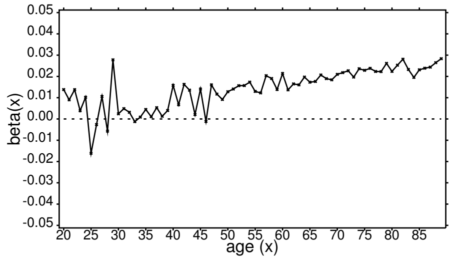
$$E\left(R_{xt}^2\right)=\phi_{xt},\ Var\left(R_{xt}^2\right)=\rho\frac{\phi_{xt}^2}{\omega_{xt}}$$

log link $\log\left(\phi_{xt}\right)=\varsigma_x$, predictor ς_x, weights ω_{xt}

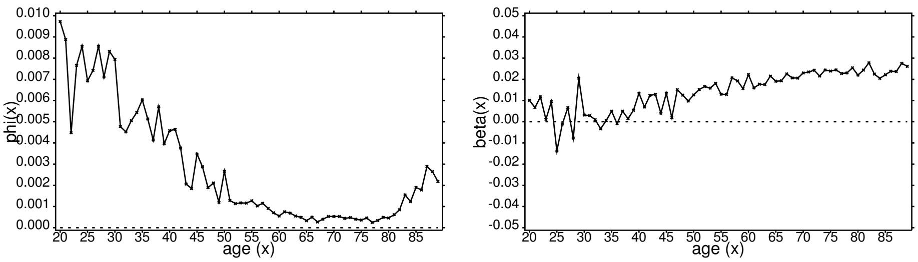
scale parameter ρ, variance function $V\left(u\right)=u^2$.

Yields: fitted values $\hat{\phi}_{xt}$ used as Stage 1 weights.

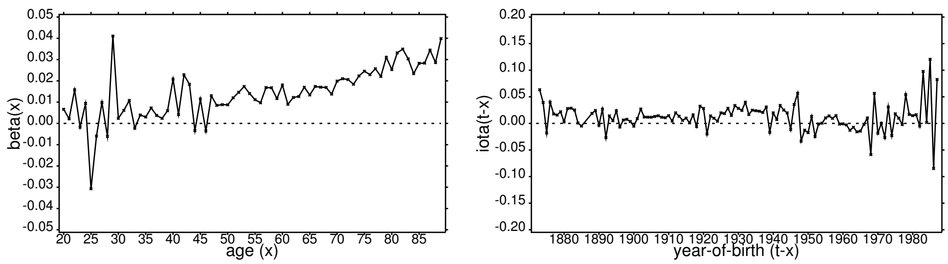
To fit: minimise the model deviances.



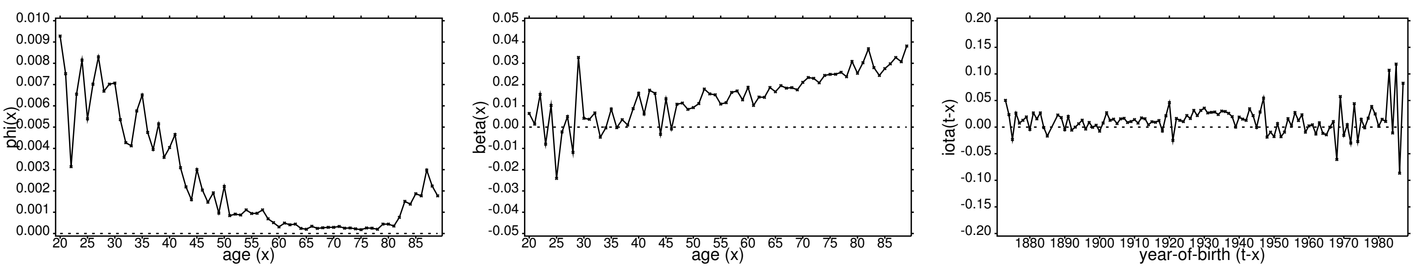
MIR LC modelling



MIR JLC modelling

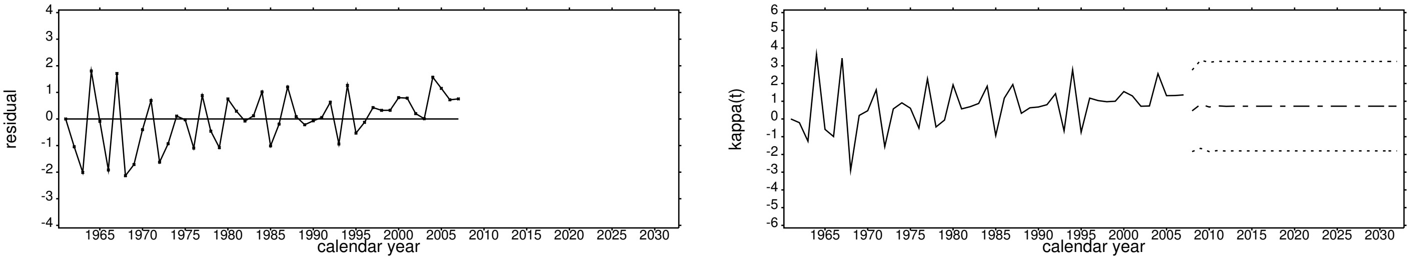


MIR H1 modelling

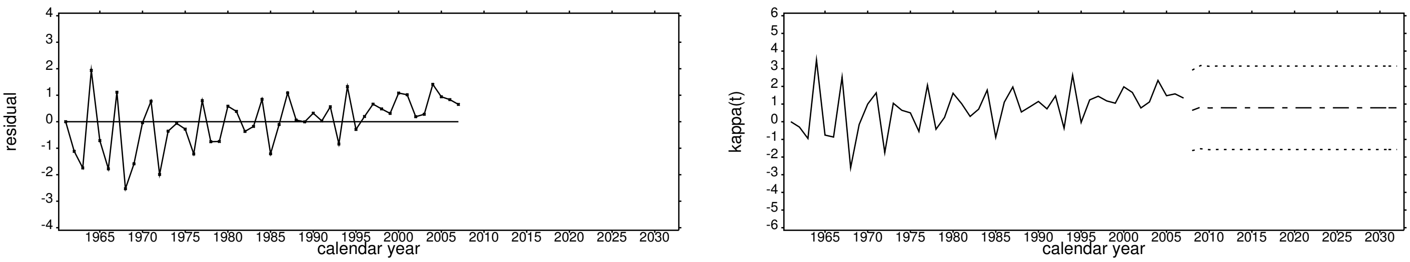


MIR JH1 modelling

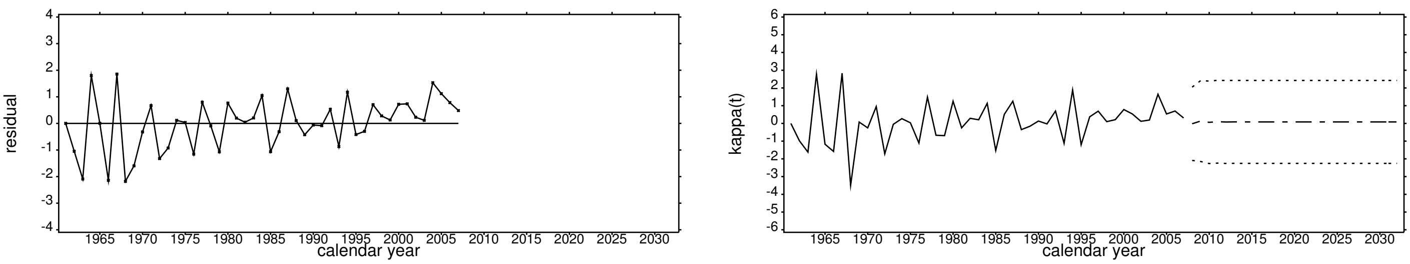
Fig 4. E&W males, ages 20-89. MIR models (rows), fitted parameters
left panels:- phi(x); centre panels:- beta(x); right panels:- iota(t-x).



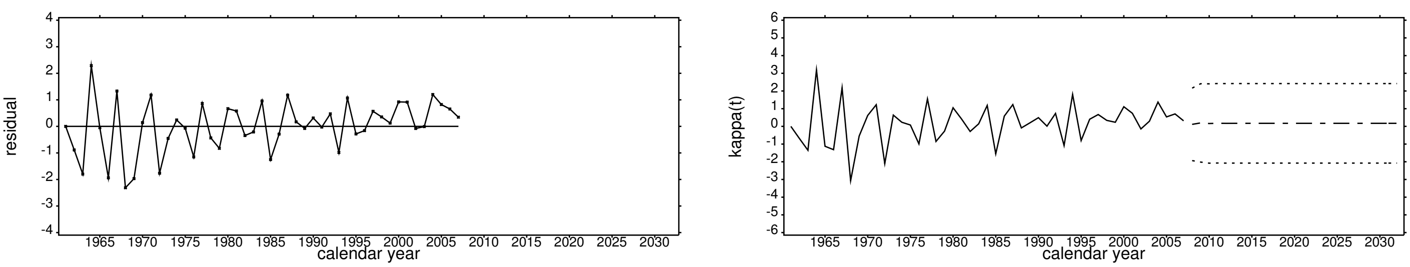
MIR LC modelling



MIR JLC modelling



MIR H1 modelling



MIR JH1 modelling

Fig 3. E&W 1961-2007 males, ages 20-89. AR(1) period index processes with forecasts (RH panels), residuals (LH panels): MIR- models (rows).

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MODEL DYNAMICS

Recall for dual modelling

$$MR : \kappa_t \mapsto MIR : \frac{\partial \kappa_t}{\partial t}$$

Consequently

$$MR : ARIMA(p,1,q) \mapsto MIR : ARMA(p,q)$$

UNIVARIATE CASE *AR*(1):

$$\kappa_t - \mu = \varphi\left(\kappa_{t-1} - \mu\right) + \varepsilon_t;\ \varepsilon_t \sim N\left(0, \tau^2\right)\ i.i.d.$$

forecasts

$$\kappa_{t_n + j} = \mu + \varphi^j \left(t_n - \mu\right);\ j = 1, 2, 3, \ldots$$

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MAPPING *MIR* PREDICTIONSTO *MR* PREDICTIONS

Compute *MIR* predictions z_{x,t_n+j} and

convert to *MR* predictions using-

$$m_{x,t_n+j} = m_{x,t_n+j-1} \frac{\left(2 - z_{x,t_n+j}\right)}{\left(2 + z_{x,t_n+j}\right)};\ j = 1, 2, 3, \ldots$$

Need starter values m_{x,t_n}.

Once converted, compute

$$q_{x,t_n+j} \approx 1 - \exp\left(-m_{x,t_n+j}\right)$$

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TOPPING OUT BY AGE

Extrapolate model projections

$$q_{x+j,t_n+j} : j = 1, 2, \ldots, x_k - x; \ (x < x_k)$$

using

$$\log\left(q_{x+j,t_n+j}\right) = a + b\left\{ j - \left(x_k - x - 1\right)\right\} + c\left\{ j - \left(x_k - x - 1\right)\right\}\left\{ j - \left(x_k - x\right)\right\};$$

$$j = x_k - x - 1, x_k - x, x_k - x + 1, \omega - x$$

requiring

$$q_{x_k-1,t_n+x_k-x-1},\ q_{x_k,t_n+x_k-x},\ q_{\omega,t_n+x_k-x}$$

to determine *a*, *b*, *c*.

INDICES OF INTEREST

Life expectancy predictions-

$$e_x(t_n) = \frac{\sum_{j \geq 1} l_{x+j}(t_n + j)\left\{1 - \frac{1}{2} q_{x+j,t_n+j}\right\}}{l_x(t_n)}$$

Fixed rate annuity value predictions-

$$a_x(t_n) = \frac{\sum_{j \geq 0} l_{x+j}(t_n + j) v^j}{l_x(t_n)}$$

ESTIMATING PREDICTION INTERVALS BY SIMULATION: MORTALITY IMPROVEMENT RATES

Generate predictions and prediction intervals.

Period index modelled as an *AR*(1) process.

(VAR(1) process if multivariate).

ALGORITHM

For simulation $k = 1, 2, ..., K$

For $j = 1, 2, ..., J$

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ESTIMATING PREDICTION INTERVALS BY SIMULATION: MORTALITY IMPROVEMENT RATES cont…

1. sample $\xi_k^{*(j)}$ from $N(0,1)$
2. compute $\kappa^{*}_{t_n+j|k} = \kappa_{t_n+j} + \sqrt{mse_{t_n+j}}\,\xi_k^{*(j)}$
3. compute $z^{*}_{x+j,t_n+j|k}$
4. compute $m^{*}_{x+j,t_{n+j|k}}$
5. compute $q^{*}_{x+j,t_{n+j|k}}$
6. apply topping-out
7. compute indices of interest.

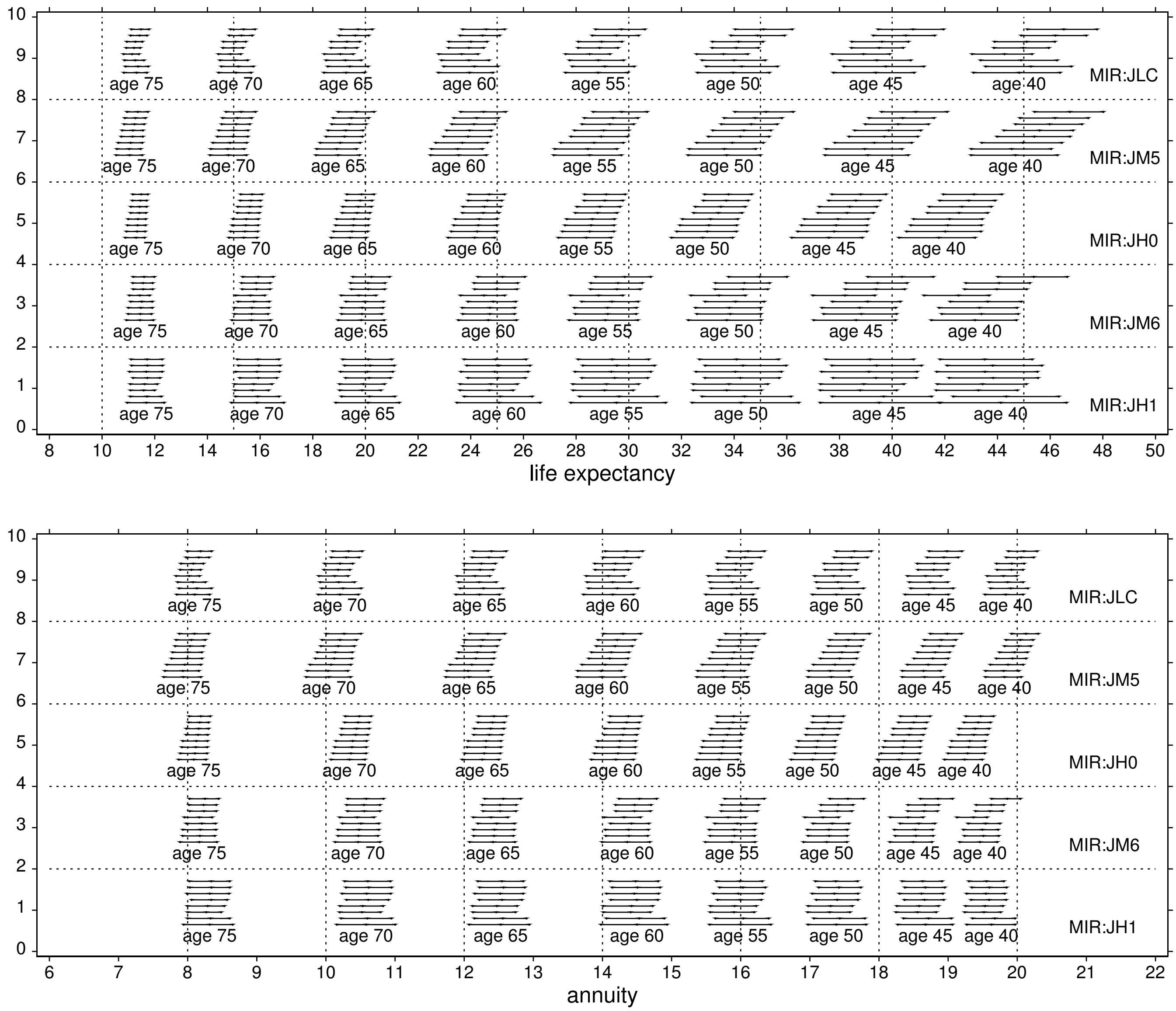

Fig 7a. E&W 1961-2007 males, ages 20-89. Year 2007 life expectancy and 4% annuity simulated 10%,50%,90% quantile predictions, subject to biennial front-end data deletions 1961(02)75 in ascending sequence, ages 40(05)75. MIR Route II approach. Structures- LC, M5, H0, M6, H1 fitted by joint modelling. Topping-out: q(109)=1

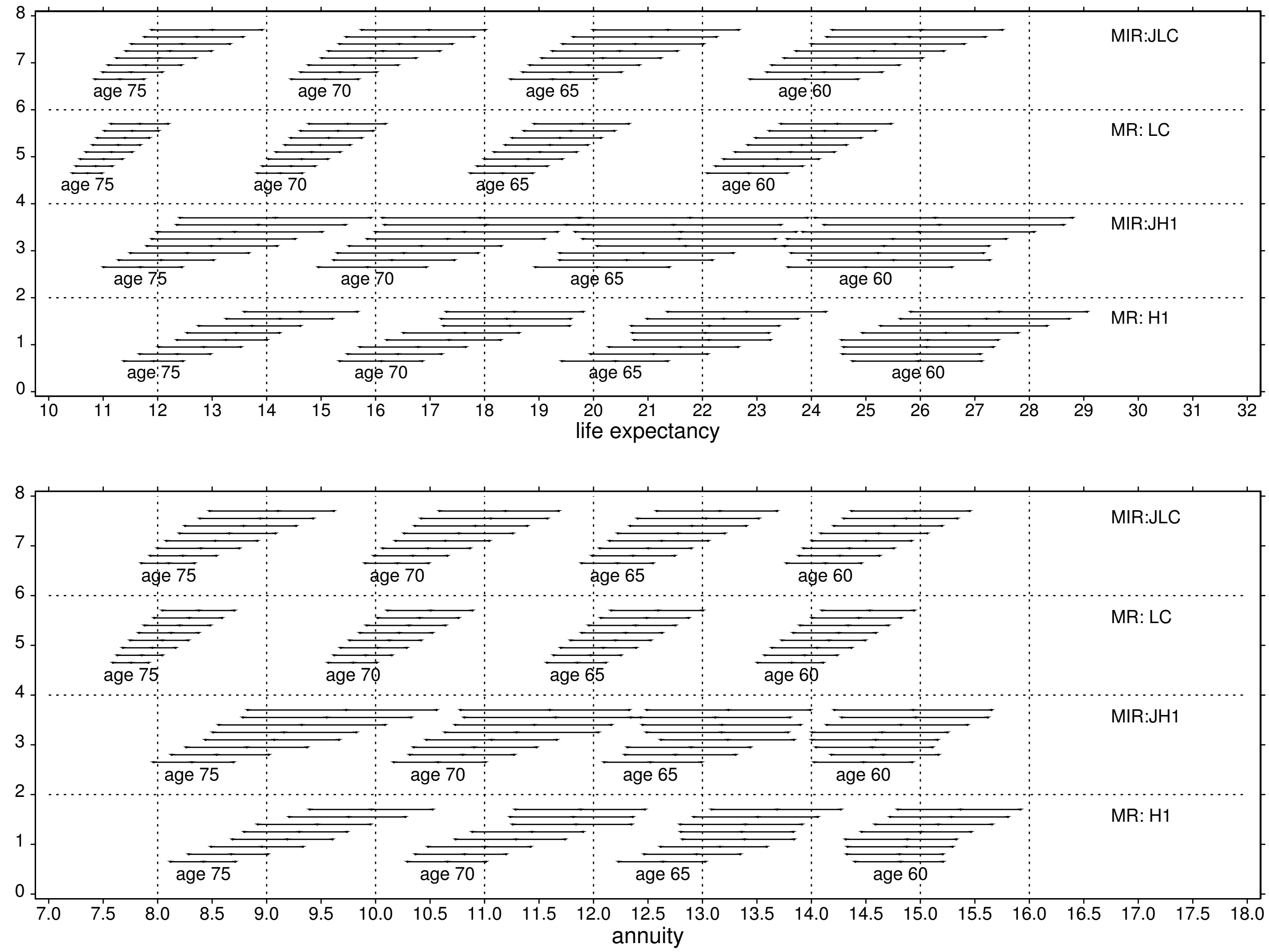

Fig 15. E&W 1961-2007 males, ages 20-89. Evolving biennial 2007(02)21 life expectancy and 4% annuity simulated 10%,50%,90% quantile predictions in ascending sequence, ages 60(05)75. Adjacent Route II and Route I approaches comparing MIR:JLC / MR:LC and MIR:JH1 / MR:H1 case structures. Topping-out: q(109)=1

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PRELIMINARY CONCLUSIONS