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MORTALITY TABLES GIVING THE SAME POLICY VALUES

By G. V. BAYLEY, F.I.A.

Assistant Actuary, The Equitable Life Assurance Society

In a translated note entitled *Mortality Tables giving the same Policy Values*, [*J.I.A.* LXII, 109 (1931)] Dr S. Dumas investigates the conditions which must be satisfied if different mortality tables are to produce identical policy values. A number of theorems are deduced including the well-known one:

> The necessary and sufficient condition in order that two mortality tables shall give the same whole-life policy values is that the values of q_x satisfy the relation
>
> $$q'_x = q_x + \frac{k}{v\ddot{a}_{x+1}}.$$

In the last section of the note he investigates continuous policy values and his last theorem states:

> When the continuous methodi s used, it is impossible to find two mortality tables producing the same policy values either for whole life or endowment assurances.

This theorem is not, I think, correct. The question came to light in the course of preparation of the new text-books and it was suggested that the subject was worth investigation. When the continuous method is used it is in fact possible to find two mortality tables giving the same policy values for whole-life assurances.* Dr Dumas's conclusion in the case of endowment assurances is valid if qualified by the condition that the 'special' force of mortality is finite throughout the duration of the assurance.

His argument commences with the formula

$${}_n\bar{V}_{x:\overline{m}|} = 1 - \frac{\bar{a}_{x+n:\overline{m-n}|}}{\bar{a}_{x:\overline{m}|}}$$

and he continues:

> In order that two tables I and II may produce the same policy values it is necessary that
>
> $$\frac{\bar{a}^{\mathrm{I}}_{x:\overline{m}|}}{\bar{a}^{\mathrm{II}}_{x:\overline{m}|}} = \frac{\bar{a}^{\mathrm{I}}_{x+n:\overline{m-n}|}}{\bar{a}^{\mathrm{II}}_{x+n:\overline{m-n}|}} = (1+k) \qquad \text{(A)}$$
>
> and so $$\int_0^{m-n} [v^t l^{\mathrm{I}}_{x+n+t}/l^{\mathrm{I}}_{x+n} - (1+k) v^t l^{\mathrm{II}}_{x+n+t}/l^{\mathrm{II}}_{x+n}]\, dt = 0. \qquad \text{(B)}$$
>
> This integral cannot be identically zero unless we have for all values of t
>
> (22) $$l^{\mathrm{I}}_{x+n}/l^{\mathrm{II}}_{x+n} = l^{\mathrm{I}}_{x+n+t}/l^{\mathrm{II}}_{x+n+t}\,(1+k),$$

* See for example Simonsen, W. (1948). *On changes in policy values caused by alterations in the basis of valuation. Proceedings of the Institute of Actuaries Centenary Assembly*, vol. II, p. 195.

and in particular
$$\begin{cases} l^{\mathrm{I}}_{x+n}/l^{\mathrm{II}}_{x+n} = l^{\mathrm{I}}_{x+n+1}/l^{\mathrm{II}}_{x+n+1}\,(1+k) \\ l^{\mathrm{I}}_{x+n}/l^{\mathrm{II}}_{x+n} = l^{\mathrm{I}}_{x+n+2}/l^{\mathrm{II}}_{x+n+2}\,(1+k) \end{cases}$$

whence

(23) $$l^{\mathrm{I}}_{x+n+1}/l^{\mathrm{II}}_{x+n+1} = l^{\mathrm{I}}_{x+n+2}/l^{\mathrm{II}}_{x+n+2}.$$

Conditions (22) and (23) are compatible only when k is zero, that is to say when the two mortality tables are identical.

Condition (22) is a *non-sequitur*. In order that policy values shall be equal, the necessary condition which follows from the equations (A) is that

$$\bar{a}^{\mathrm{I}}_{x+n:\overline{m-n}|} - (1+k)\bar{a}^{\mathrm{II}}_{x+n:\overline{m-n}|} = 0$$

for all values of n.

It follows that the first derivative of this expression must also be zero for all values of n, so that

$$(\mu^{\mathrm{I}}_{x+n} + \delta)\bar{a}^{\mathrm{I}}_{x+n:\overline{m-n}|} - 1 = (1+k)\left[(\mu^{\mathrm{II}}_{x+n} + \delta)\bar{a}^{\mathrm{II}}_{x+n:\overline{m-n}|} - 1\right], \qquad \text{(C)}$$

and hence
$$\mu^{\mathrm{II}}_{x+n} = \mu^{\mathrm{I}}_{x+n} + \frac{k}{\bar{a}^{\mathrm{I}}_{x+n:\overline{m-n}|}} \qquad \text{(D)}$$

for all values of n.

It is not correct simply to equate to zero the integrand of expression B for all values of t. The definite integral is a function of the variable n and must be differentiated with respect to n before being equated to zero.

If μ^{II}_{x+n} is to remain finite when $n=m$, expression C will be true only provided $k=0$ and hence, from D, $\mu^{\mathrm{II}}_{x+n} = \mu^{\mathrm{I}}_{x+n}$ for $0 \leqslant n \leqslant m$, a conclusion also reached by Simonsen.* On the other hand, if no restriction is placed upon μ^{II}_{x+n}, expression D gives the necessary and sufficient condition for equal policy values. In this case, the special mortality basis assumes that μ is infinite at the maturity age and therefore that there are no survivors to the end of the term of the assurance. However, the really heavy mortality appears only at the limit of the term as the numerical example below shows. Since condition D has transformed what was formerly an endowment assurance into a whole-life assurance, Dr Dumas's theorem is, in a sense, correct: it is not possible to find two different mortality tables giving the same *endowment assurance* policy values, when the continuous method is used.

For whole-life assurances the necessary and sufficient condition is given by an obvious modification of expression C. There is no special requirement in this case for making k zero so that alternative mortality tables may be found which give identical policy values.

A numerical example is shown below using English Life Table No. 8 (Males) 3% for a ten-year endowment assurance commencing at age 40.

The table shows values of $\bar{a}^{\mathrm{II}}_{x:\overline{50-x}|}$ which have been calculated by constructing the special mortality table (II) from the relation

$$\mu^{\mathrm{II}}_{x} = \mu^{\mathrm{I}}_{x} + \frac{\cdot 01}{\bar{a}_{x:\overline{50-x}|}}.$$

The result is shown in col. (9) and may be compared with $\bar{a}^{\mathrm{II}}_{x:\overline{50-x}|} = \dfrac{\bar{a}^{\mathrm{I}}_{x:\overline{50-x}|}}{1\cdot 01}$ in col. (10).

* *Ibid.* p. 193.

Comparison of $\bar{a}^{\text{II}}_{x:\overline{50-x}|}$ on the special mortality basis calculated by two different methods

(Basis I: E.L. No. 8 (Males), 3% interest)

x (1)	μ^{I}_x (2)	$\bar{a}^{\text{I}}_{x:\overline{50-x}|}$ (3)	μ^{II}_x (4)	l^{II}_x $(l^{\text{II}}_{40}=l^{\text{I}}_{40})$ (5)	$D^{\text{II}}_x = v^x l^{\text{II}}_x$ (6)	$\bar{D}^{\text{II}}_x$ (7)	$\bar{N}^{\text{II}}_x - \bar{N}^{\text{II}}_{50} = \Sigma \bar{D}^{\text{II}}_x$ (8)	$\bar{a}^{\text{II}}_{x:\overline{50-x}|} = \frac{\bar{N}^{\text{II}}_x - \bar{N}^{\text{II}}_{50}}{\bar{D}^{\text{II}}_x}$ (9)	$\frac{\bar{a}^{\text{I}}_{x:\overline{50-x}|}}{1\cdot 01}$ (10)
40	·007927	8·2725	·009136	716,727	219,718	215,502	1,799,613	8·1906	8·1906
41	·008370	7·5713	·009691	710,016	211,321	207,206	1,584,111	7·4962	7·4963
42	·008866	6·8465	·010327	702,950	203,124	199,104	1,376,905	6·7786	6·7787
43	·009406	6·0970	·011046	695,483	195,113	191,180	1,177,801	6·0365	6·0366
44	·009985	5·3211	·011864	687,568	187,274	183,421	986,621	5·2683	5·2684
45	·010611	4·5172	·012825	679,143	179,592	175,807	803,200	4·4724	4·4725
46	·011289	3·6833	·014004	670,107	172,041	168,308	627,393	3·6468	3·6468
47	·012014	2·8173	·015563	660,302	164,586	160,873	459,085	2·7893	2·7894
48	·012771	1·9167	·017988	649,392	157,152	153,371	298,212	1·8976	1·8977
49	·013574	·9786	·023793	636,333	149,506	144,841	144,841	·9688	·9689
50	·014453	·0000	∞	0	—	—	—	—	—

Col. (5) l^{II}_x has been found by a method involving the approximate integration of μ^{II}_x.

Col. (7) $\bar{D}^{\text{II}}_x = \frac{D^{\text{II}}_x + D^{\text{II}}_{x+1}}{2} - \frac{1}{12}(\mu^{\text{II}}_x + \delta) D^{\text{II}}_x + \frac{1}{12}(\mu^{\text{II}}_{x+1} + \delta) D^{\text{II}}_{x+1}$ for $x = 40$ to 48;

$\bar{D}^{\text{II}}_{49}$ was calculated in two stages by approximate integration of the expressions:

$$\text{colog}_e\, {}_tp^{\text{II}}_{49} = \int_0^t \left(\mu_{49+t} + \frac{\cdot 01}{\bar{a}^{\text{I}}_{49:\overline{t}|}}\right) dt,$$

$$\bar{D}^{\text{II}}_{49} = D^{\text{II}}_{49} \int_0^1 v^t\, {}_tp^{\text{II}}_{49}\, dt.$$