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ON THE NATURE OF THE FUNCTION EXPRESSIVE OF THE LAW OF HUMAN MORTALITY

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IS there a mathematical formula which will express the way in which mortality changes age by age? Many students have sought that will-o'-the-wisp but have had to confess failure. For example, Sir William Elderton says:*

> but such a course did not satisfy me, because the failure...implied that the method was artificial and that I was not getting nearer to the root of the matter. To put the matter another way—I had not reached the best form in which to express the results: I had only obtained an approximation which led no further.

2. The quest for a law of mortality is one which has had an irresistible fascination for myself, and though success has not been achieved it is hoped that the philosophical approach to the subject in this paper may lead others to success; at least the 'near misses' here recorded may save others from hours of arithmetic.

3. When the graduation of mortality statistics is being discussed it is frequently said that the mathematical formula used is of no importance, that any formula will do if it fits. Such an attitude denies any reality to the process of graduation. It assumes that the form is imposed on the data and is not inherent in the nature of the problem itself. That problem has been best stated by a poet, Richard Church:

> He will cross this desert, he will find in this age of destruction
> A motive for singing; he will see from the hills of the future
> A landscape, a history, both simple, embalmed in the past.

The poet is speaking of the historian who shall some day view our times in a perspective which is denied to ourselves who live in them. But the actuary is concerned—just as is the historian—with the observed facts of the past; he must select and arrange what is relevant in such a way as to bring out the pattern of the 'experience'. The landscape is apt to change with the viewpoint, and it is interesting to trace the way in which the viewpoint has changed through successive ages. In the actuarial world there has been a subtle change in the accepted view of the 'law of human mortality', a change of some practical importance which seems to receive insufficient recognition.

THE VIEW HELD IN THE EIGHTEENTH CENTURY

4. Throughout the eighteenth century the 'law of human mortality' was thought of in numerical terms—as a mortality table representing the number of survivors to each age out of a given number of births on the basis of certain

* *J.I.A.* LXIV, 3.

statistics. For example, the first life table computed from statistics was described by Edmond Halley* (1693) in the words:

> From these considerations I have formed the adjoyned Table, whose uses are manifold, and give a more just idea of the state and condition of mankind, than any thing yet extant that I know of. It exhibits the number of people in the City of Breslaw of all ages, from the birth to extream old age, and thereby shews the chances of mortality at all ages.

Halley's table may have been a population table rather than a life table as understood nowadays (the question was discussed by Prof. Greenwood), but the viewpoint is essentially the same.

5. The life table was given a mathematical dress by Abraham de Moivre (1725) in his hypothesis of equal decrements. Some injustice is done to the memory of a great mathematician by calling this a 'law'. The hypothesis was one of several approximations suggested by him in his book *Annuities upon Lives*, and was probably adequate for the valuation of annuities within the range of ages then commonly required in practice; de Moivre was aware that the hypothesis was defective as a representation of human mortality. For our present purpose, we may notice that the formula relates to l_x and d_x, not to q_x.

6. In later works of the eighteenth century the expression 'rate of mortality' meant a table of l_x and d_x, and not of q_x. If this appeared at all it was only as a derivative, the probability of a life aged x dying within a year.

7. The old outlook inspired the complicated formula published (1826) by Thomas Young, M.D., F.R.S.,† one of the foreign secretaries of the Royal Society. The formula was devised to reproduce d_x on the assumption of a mean mortality between the various experiences then available.

8. Joshua Milne, in *A Treatise on the Valuation of Annuities and Assurances* (1815), defined the law of mortality in a way which may indicate a new outlook. Article 143 of his book states:

> The *law of mortality* is that which determines the ratio of the number of persons who die in any period of life; and, consequently, that of the number who survive it, to the number who enter upon the same period.

However, the extensive tables at the end of Milne's book do not include any table of q_x, though tables of $\log p_x$ and $\operatorname{colog} p_x$ are given. Also, article 144 makes clear that Milne is thinking in terms of ${}_tp_x$ and $(1-{}_tp_x)$ rather than of p_x and q_x explicitly.

9. John Finlaison (1829), in his report on the mortality of government life annuitants, says:

> The basis of all questions having reference to the failure or continuance of life is well known to be the law of mortality, or the probability that a human being, who may be in any given year of age, will die in that same year.

This does not make clear that the probability relates to a whole year, but that is what Finlaison meant, and he computed his rates of mortality to seven places of decimals.

* Reprinted *J.I.A.* XVIII, 251. † Reprinted *J.I.A.* VI, 351 and VII, 14.

10. It would be interesting to chart the development of the idea of q_x as the 'rate of mortality' and of μ_x as the 'force of mortality', but this would be a digression from the main theme of the paper. It is perhaps significant that the symbol q_x does not seem to have appeared before the 1840's.

GOMPERTZ'S LAW AND ITS DERIVATIVES

11. By a historical coincidence, Benjamin Gompertz propounded his law in 1825, just one hundred years after the publication of de Moivre's hypothesis. Gompertz was trying to find some general form which would facilitate the interpolations then required in calculating complicated values dependent on several lives. He explicitly states that his hypothesis was derived from an analysis of experience, and his numerical examples were based on the Carlisle and the Northampton tables of mortality and on Deparcieux's observations. But he gave a philosophical interpretation to his hypothesis which foreshadowed the importance which was, at a later date, attained by the conception of the 'rate of mortality'. Gompertz says:

> It is possible that death may be the consequence of two generally co-existing causes; the one, chance, without previous disposition to death or deterioration; the other, deterioration, or an increased inability to withstand destruction.

Gompertz showed that his law could be understood as assuming a particular mathematical form for the average exhaustion of a man's power to avoid death.

12. We are accustomed to think of Gompertz's law as a formula for the force of mortality,

$$\mu_x = Bc^x,$$

but the force of mortality had not been named in 1825, and Gompertz, at this time, was thinking in terms of the number living at age x, l_x (he called it L_x), and its fluxion.

13. T. R. Edmonds, in his *Life Tables, Founded upon the Discovery of a Numerical Law* (1832), introduced the expression 'force of mortality' and used three Gompertz curves to represent the sections of it, up to age 9, from 9 to 55, and over age 55 years respectively.*

14. The law was given a more general application by William Matthew Makeham† (1860). Though Gompertz had thought of the causes of death as being of two kinds, chance and deterioration, he did not link them to mathematical expressions in his law. Makeham went on to do so in a further paper‡ (1867), making a rough division of causes of death between what would be approximately independent of age and what would be increasing with age. At the same time he took the opportunity to explain the term 'force of mortality', a term which, he said, had recently come into use but which was not explained in any of the standard elementary works. Thus the formula took its classic form,

$$\mu_x = A + Bc^x. \tag{1}$$

Study of the H^M experience led Makeham§ (1889) to propose the addition of a polynomial to a Gompertz curve.

* Edmonds did not acknowledge Gompertz's priority—but this is an old controversy now of little interest.

† *J.I.A.* VIII, 301.

‡ *J.I.A.* XIII, 325.

§ *J.I.A.* XXVIII, 152.

15. The fact that the Gompertz-Makeham law could not be expected to represent the mortality experience throughout life led to the investigation of formulae which might be expected to do so. It was assumed that the 'force of mortality' in fact existed and that it could be represented by a mathematical expression. Gompertz* (1860), himself, suggested a formula based on an amalgamation of several of his curves with different constants, and various combinations have been suggested from time to time, together with others of a different type. Perhaps the best illustration of the point of view is given by the mathematical formula proposed by the Danish mathematician, Thiele (1871), to express the rate of mortality throughout the whole of life—and the reference is readily available in Sprague's translation.† In Thiele's formula,

$$\mu_x = a_1 e^{-b_1 x} + a_2 e^{-\frac{1}{2} b_2^2 (x-c)^2} + a_3 e^{b_3 x}, \tag{2}$$

the last term is a Gompertz curve to represent old-age mortality and the first a decreasing Gompertz curve to represent the mortality of infancy. The middle term is a form of the normal curve of error.

16. This summary has been brief, but it seems to be clear that by the end of the nineteenth century the viewpoint had changed and the law of mortality meant the rate—or the force—of mortality, which was thought of as a separate entity, capable of having a mathematical form. It would be unfair to carry this generalization too far.

17. It will not have escaped notice that the title of this paper is identical with the first half of the title of Gompertz's paper of 1825. The purpose is much the same. What is required is a new way of looking at the problem—in some respects a return to earlier ways of thought—and this reconsideration of the nature of the rate of mortality leads to an interesting mathematical form.

FREQUENCY CURVES

18. The development of frequency curves to represent statistical distributions provided a new tool with which to tackle the problem. In this paper it is proposed to refer only to Karl Pearson's system of frequency curves. It will be recollected that his system was based on a general consideration of the shape of frequency distributions, and he assumed that

$$\frac{1}{y}\frac{dy}{dx} = \frac{x+a}{b_0 + b_1 x + b_2 x^2} \tag{3}$$

(where y is the frequency and x the variable) would represent the majority of distributions met with in practice, an assumption which has been amply fulfilled.

19. Since $\mu_x = -\frac{1}{l_x}\frac{dl_x}{dx}$, it is natural to ask whether the same expression might be used to represent a mortality experience. On this question Elderton‡ says:

> My attempts to fit $(a_0 + a_1 x)/(b_0 + b_1 x + b_2 x^2)$ to μ_x have not been successful, and I must confess that my previous attempts to fit slope relations when only parts of a distribution have been available have never satisfied me. But I am inclined to the view that for a large part of a mortality table l_x can be taken as equidistant ordinates of a frequency curve.

* *J.I.A.* XVI, 329. † *J.I.A.* XVI, 313. ‡ *J.I.A.* LXV, 19.

20. A fairly successful application of Pearson's formula to μ_x by E.L.T. No. 9 was given by Perks* (1931), who found that the force of mortality could be represented by

$$\mu_x = \frac{\cdot 014997 + \cdot 0002515x}{1 - \cdot 2999380x + \cdot 023943x^2}$$

(origin at age 54 years, unit 5 years). The denominator has no real roots and is a minimum at age 85 years ($x=6\cdot 26$); at later ages the graduated force of mortality decreases. Considering the expression as giving rise to a frequency curve (in this case l_x) it will be seen that the curve is Pearson's Type IV (see para. 47 and Fig. 3). Theoretically the curve has an unlimited range in either direction with a mode at the point (long before birth) where the numerator is zero. In fact the useful range of the curve is from ages 19 to 84 years; the graduation has effectively used only a part of one side of the frequency curve.

21. When Karl Pearson introduced his frequency curves, it was natural that he should use them to attempt a representation of the mortality curve. For this purpose he selected the function d_x in the life table as the frequency distribution and showed how it could be represented by five frequency curves, fitted successively from the oldest age backwards.†

22. The 'statistics' were the figures of d_x for E.L.T. No. 4 (Males). For old-age mortality the equation was

$$y = 15\cdot 2\left(1 - \frac{x}{35}\right)^{7\cdot 7525} e^{\cdot 2115x},$$

with the mean at age 67 years and the mode at age 71·5 years. The mortality of middle life and of youth was represented by curves which were approximately normal with means at ages 41·5 and 22·5 years respectively, and standard deviations of 12·8 and 7·8 respectively. The curve for the mortality of childhood was skew with the origin at age 2 years, the mode at age 3 years, and the mean at age 6·06 years. The curve for infantile mortality was J-shaped and was assumed to start nine months before birth.

23. Pearson's achievement is suggestive of the theory propounded, as far back as 1877, by a German, W. Lexis, whose works are not in the Institute Library. There are various references to the theory in actuarial literature, and it is discussed by Prof. Greenwood and J. O. Irwin‡ (1939). Lexis thought that the curve of deaths could be assumed to take the form of a normal distribution after a certain typical or 'normal' age had been attained, but that the mischances of life would twist the curve below that age into a skew distribution. Study of statistics relating to the older ages might reveal whether the distribution can be assumed to be normal.

24. The function d_x is (at least usually) a mathematical function, not a statistical distribution. The statistics are the exposed to risk and the deaths, and it may be possible to fit these statistics by frequency curves. This is usually possible with the exposed to risk, but the deaths may give trouble. In his Presidential address (1932) Sir William Elderton described various

* *J.I.A.* LXIII, 31.

† Reprinted in *Karl Pearson's Early Statistical Papers*, Cambridge University Press, 1948, pp. 105–7.

‡ *Human Biology*, vol. XI.

attempts he made to utilize the frequency curves for exposed to risk and deaths as a means of representing the mortality curve. These attempts* met with various difficulties—for example, when E_x and $E_x p_x$ were fitted by curves, though the fit was good the ratio of the ordinates did not give a good representation of p_x.

25. On the basis of his work with frequency curves Elderton† put forward 'an approximate law of survivorship' which stated that:

> If the population (exposed-to-risk) be expressed as a frequency curve then the survivors from that population any number of years hence will also approximate to frequency curves.

26. At about the same time Perks (supra) described some experiments in the graduation of mortality tables. He followed in Gompertz's footsteps in that he first worked out a type of formula which would fit various experiences and then attempted to provide a rationale of the method. The principal formula of the paper expressed the 'rate of mortality' (whether q_x, μ_x or the rate based on the 'central' exposed to risk) in the form

$$\frac{A+Bc^x}{Kc^{-x}+1+Dc^x}. \tag{4}$$

Perks brings out the interesting fact that such a form for the force of mortality is identical with the fundamental Pearson equation when the variable x is transformed to a geometrical variable c^x.

27. The influence of statistical ideas is apparent in the concept of 'inability to withstand destruction'. Whereas Gompertz was thinking of the individual, Perks had in mind the statistical group of the individuals exposed to risk. It is also interesting to notice that Perks comments:

> If the speculations in this paper prove to be barren we may still take refuge in the fact that μ_x is by nature a fraction.

28. Recently, Beard‡ (1948) has successfully applied Pearson's curves to a part of the d_x curve. He suggested that a mortality table might be regarded as comprised of two components, namely, an 'accident' mortality, roughly independent of age, and a 'deterioration' mortality, highly correlated with age. Beard§ (1951) has also applied Perks's formula to a wide range of mortality statistics.

EXPERIMENTAL CONSIDERATIONS

29. The author's own first experiments in curve-fitting were made on the data of King's model office, but there was no particular purpose in view and the work was discarded because the data of the model office seemed too far away from modern conditions to lead to useful results. The same feature as Elderton had noted was apparent from this work—that the frequency curves

* These attempts are different in kind from Elderton's very successful method of fitting Makeham and other formulae by the use of an assumed frequency curve which approximately represents the exposed to risk.

† *J.I.A.* LXIV, 5.

‡ *Proc. Cent. Assembly Inst. Actuaries*, II, 89.

§ *J.I.A.* LXXVII, 382.

might give a reasonable fit for the batch of new entrants, and for the survivors of that batch after 10, 20, 30 years and so on, but that the relationship between the ordinates of the several curves for a given age at entry did not give a suitable representation of the probabilities of survival.

30. The success with which Pearson's frequency curves have been fitted to a variety of statistical distributions suggests that they must have an application to the life table. The curve of deaths seemed the most suitable function to treat as a frequency curve, and the next attempts were directed to finding a suitable expression for that curve on the lines of Pearson's system but with more constants in the formula. The attempts were wholly unsuccessful.

31. Since the functions of the life table, l_x and d_x, are mathematical functions, it is permissible to use whichever function seems the most suitable for the purpose of fitting the frequency curve; the other functions of the life table will then be derivatives. On general grounds the function l_x seemed to promise more hope of success than the function d_x. In Perks's paper and the discussion on it, the suggestion was made that l_x might be brought within the range of Pearson's system if the unit of measurement were changed to c^x, x^2 and so on. Various transformations have been tried without success. With x^n as the unit of measurement the transformed l_x curve is J-shaped or bell-shaped according as n is greater or less than unity. The transformed curves took on some interesting shapes, but the main difficulty seemed to be that the middle part of the curve was too flat to be represented by a Pearson curve.

32. The work described left the author with the conviction that Pearson's frequency curves must have some application to mortality tables, though he had been unsuccessful in adapting them. It seemed that some progress might be made by a reconsideration of first principles.

33. If it were possible to observe throughout life a group of births all born within a year or other limited period the observed numbers surviving at each age, say l'_x, would constitute a frequency distribution subject to the reservation that there would be some correlation between the numbers at successive ages. It should be possible to represent this observed distribution by a frequency curve $y=l_x$, though it might be expected that the curve would take a more complicated form than Pearson's system, particularly because of the presence of correlation.

34. Some support for this view is obtainable from the approximate law of survivorship postulated by Elderton. Should any section of the distribution l'_x consisting of the part above a certain age be expressible as a frequency curve, the law (arguing backwards) would suggest that the whole distribution would also be expressible in that form—though not necessarily by the same type of curve. It seems very probable that the tail of the distribution l'_x could be represented by a frequency curve, and there seems no reason why the whole distribution should not take this form.

35. It is not usually possible to observe a group of births throughout life. More commonly we have two statistical distributions, the exposed to risk (E_x) and the deaths (θ_x), and the rate of mortality is a fraction, a ratio which expresses the relationship between the distributions. On general grounds it

might be hoped that the ratio, say $f(x)/F(x)$, could be represented by the ratio of two polynomials, thus:

$$\frac{\theta_x}{E_x}=\frac{a_0+a_1x+a_2x^2+\dots}{b_0+b_1x+b_2x^2+\dots}. \tag{5}$$

The polynomials should not be regarded as representing the separate distributions of exposed to risk and deaths; rather the whole expression represents the ratio between them.

36. By cross-multiplying and adding over the range of the distributions, it is seen that the rate of mortality can be regarded as expressing a relationship between the exposed to risk and deaths and their successive moments, thus:

$$b_0\Sigma\theta_x+b_1\Sigma x\theta_x+b_2\Sigma x^2\theta_x+\dots=a_0\Sigma E_x+a_1\Sigma xE_x+a_2\Sigma x^2E_x+\dots. \tag{6}$$

Successive multiplication of each side of the equation by x would give a set of equations in terms of the successive moments which would enable the constants to be calculated, but the method would involve high moments. For example, if three terms only on either side are retained, there are five unknowns for which five equations are required: the first equation would include the second moment, the second equation the third moment, and so on, the fifth equation including the sixth moment. It seems more reasonable to use equation (6) for successive ranges of the data, and it is clear that the best method of calculating the constants has yet to be determined.

37. The concept of the rate of mortality as a relationship between two statistical distributions and their moments should be a fruitful one. The link between the statistics and the graduated table is the postulate of a continuous force of mortality which is assumed to have operated in the observed experience. The force at any age cannot be measured directly from the statistics because it is an infinitesimal, but the continuous function, the force of mortality over the range of ages, can be obtained as a relationship between the exposed to risk and the deaths. This continuous function is the basis of the life table and all other functions of the life table are derivatives from it.

38. The postulate of a continuous function, the force of mortality, seems itself to require that the distributions of exposed to risk and deaths should themselves be approximately smooth frequency distributions.

39. The magnitude and sign of the coefficients a_0, a_1, etc., depend primarily on the moments of the exposed to risk, and those of the coefficients b_0, b_1, etc., depend primarily on the moments of the deaths. It is evident, therefore, that the fraction cannot be regarded in the same light as the fraction d_x/l_x. The formula expresses a general relationship between the exposed to risk and the deaths.

40. The distribution θ can be regarded as a selection from the distribution E. Had the selection been a random sample, the values of a_0, a_1, ... applying to the successive moments would have tended to be proportional to b_0, b_1, ..., and this process might help to capture and define the elusive quality of 'randomness'. The a_0, a_1, ..., b_0, b_1, ..., applying to the successive moments of E and θ, are obviously not proportional, and the formula (5) might be said to reflect the bias in the selection θ compared with the distribution E.

41. Should equation (6) be used to calculate the parameters, there would not usually be an equality between the total actual and expected deaths, but the difference is not of much importance.

42. The expressions have so far been developed in terms of the ordinary moments of the exposed to risk and deaths and the numerical examples are based upon those moments. It may, however, be of interest to mention that if the variable be changed to a geometric one, say $z=c^x$, the method yields a general expression which includes the formulae of Gompertz, Makeham and Perks as special cases. The change of variable alters the character of the formulae to some extent because the geometric moments are essentially positive; they cannot be negative.

43. If the change of variable be accepted, the method gives an alternative to the philosophical derivation of the Gompertz type of formula from the 'increased inability to withstand destruction'. This suggestion has not been followed up. The experiments in this paper have been confined to ordinary, not to geometric moments.

ADAPTATION OF THE PEARSON CURVES

44. Returning to the fundamental Pearson equation we may notice that the formula for $\frac{1}{y}\frac{dy}{dx}$ is a fraction, the ratio of the ordinate of a straight line to the ordinate of a parabola. Generally speaking the denominator defines the range of the curve whereas the numerator expresses the skewness; the quadratic parabola is of course symmetrical. Figs. 1–3 have been prepared to illustrate the forms of the numerator and the denominator. In looking at these diagrams it must be remembered that a large ordinate for the denominator means a small differential coefficient, and a small ordinate a large differential coefficient.

45. The formula leads to curves of three main types. When the denominator (expressed with the mean as origin) has real roots of different signs, the roots define the limits and the curve relates to the part between them. This is Pearson's Type I. In Fig. 1 the range is from P to Q.

46. When the denominator (expressed with the mean as origin) has real roots of the same sign, the curve extends in one direction to the numerically smaller root and in the other direction to infinity. The second and numerically larger root lies outside the range of the curve. This is Pearson's Type VI. In Fig. 2 the range is from Q to infinity.

47. When the denominator has no real roots the parabola does not cut the x-axis. The curve is of infinite extent in both directions. This is Pearson's Type IV and Fig. 3 illustrates this type.

48. The numerator, being a straight line, must cut the x-axis at some point m, and if that point lies within the range of the curve the differential coefficient there becomes zero; this defines the mode of the frequency curve and produces the usual 'cocked-hat' shape. If, however, the numerator cuts the x-axis at a point which lies outside the range of the frequency curve, there is no mode and the curve is J-shaped. Thus Types I and VI but not Type IV may lead to J-shaped curves. There are various transitional types of curve which are not considered here.

49. Now l_x is essentially J-shaped, and the preceding brief discussion of Pearson's curves will have shown that the fundamental formula can apply to a limited class only of such curves; the formula *needs to* be *extended.* Its

Diagrams illustrating the character of the numerator and the denominator in the expression $\frac{1}{y}\frac{dy}{dx}=\frac{x+a}{b_0+b_1x+b_2x^2}$

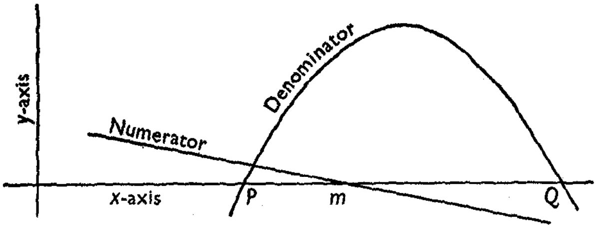


Fig. 1. Pearson Type I. Range from P to Q with mode of frequency curve at m, after which point the numerator is negative and the frequency curve decreases.

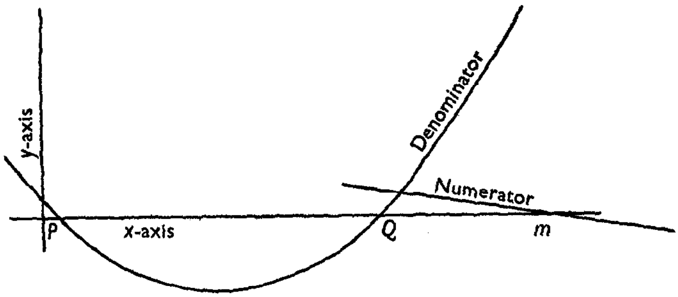


Fig. 2. Pearson Type VI. Range from Q to infinity with mode of frequency curve at m, after which point the numerator is negative and the frequency curve decreases.

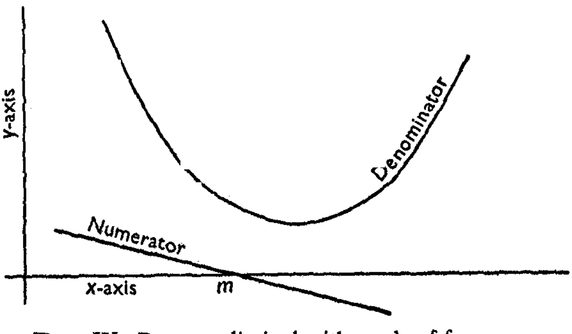


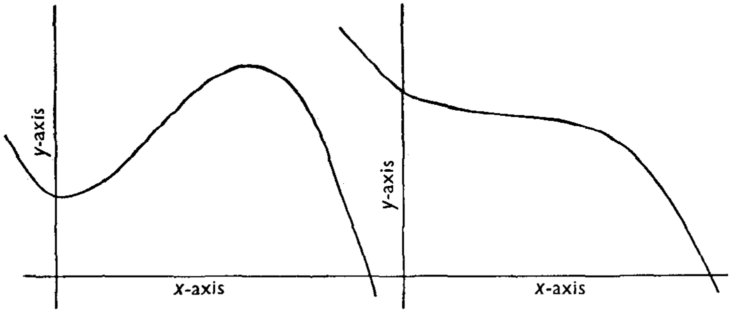
Fig. 3. Pearson Type IV. Range unlimited with mode of frequency curve at m, after which point the numerator is negative and the frequency curve decreases.

scope will obviously be much enlarged if the numerator is taken as a parabola instead of a straight line, and it is the purpose of this paper to consider the formula

$$\mu_x=-\frac{1}{y}\frac{dy}{dx}=\frac{a_0+a_1x+x^2}{F(x)}. \tag{7}$$

In this expression y has been retained as the ordinate of the 'frequency curve' which in the circumstances is l_x.

50. Since the quadratic is symmetrical the denominator of the expression must give effect to the skewness of the curve as well as the range. Should the quadratic have real roots within the range of the roots of the denominator, the range of the curve would be limited, probably in an undesirable way. In practice it has been found that the numerator takes a form which has no real roots, and it seems best to understand the formula in this sense, so that $a_1^2 < 4a_0$, and the whole of the parabola is assumed to lie above the x-axis.* The numerator is a minimum when $x = -\frac{1}{2}a_1$, and if the origin be taken at that age the numerator becomes $a + x^2$.



(*a*) (*b*)

Fig. 4*a*. Cubic with maximum and minimum points and one real root.
Fig. 4*b*. Cubic with point of inflexion and one real root.

51. So far no form has been assumed for $F(x)$, but it can be expanded and written as

$$F(x) = b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + \ldots.$$

It seems sufficient to consider the cubic and the quartic, though for a limited range the quadratic may be sufficient.

52. A cubic has either one or three real roots and must proceed from plus to minus infinity or vice versa. A quartic has two or four real roots (or none) and commences and ends on the same side of the x-axis. The useful forms for our present purpose are the cubic with one real root and the quartic with two real roots and some shapes of these curves are shown in Figs. 4*a*, 4*b* and 5.

* A possible alternative approach is to consider the expression

$$\frac{1}{y}\frac{dy}{dx} = \frac{a+x}{c_0 + c_1 x^2 + c_2 x^4}.$$

The denominator of this expression is symmetrical and would define the range. The numerator would represent the skewness and would produce a J-shaped curve when the straight line in the numerator cuts the x-axis at a point outside the range of the curve. Symmetrical curves of this type have been considered by G. H. Hansmann (1934, Biometrika, XXVI, 129). For skew curves it would be necessary to have five equations to give the origin as well as the four constants. Hansmann also refers to manuscript work by Dr David Heron on the expression

$$\frac{1}{y}\frac{dy}{dx} = \frac{a+x}{c_0 + c_1 x + c_2 x^2 + c_3 x^3}.$$

53. The curves which are likely to be useful in practice are 4*b* and 5, which have points of inflexion, but a denominator of the type shown in Fig. 4*b* could not represent decreasing mortality at the beginning of life though 4*a* could do so in suitable conditions.

54. From what has been said it seems likely on general grounds that l_x of the life table can be considered as a frequency curve with the fundamental relationship

$$\mu_x = -\frac{1}{l_x}\frac{dl_x}{dx} = \frac{a+x^2}{b_0+b_1x+b_2x^2+\dots}, \tag{8}$$

where the origin is taken at the point which makes the numerator a minimum.

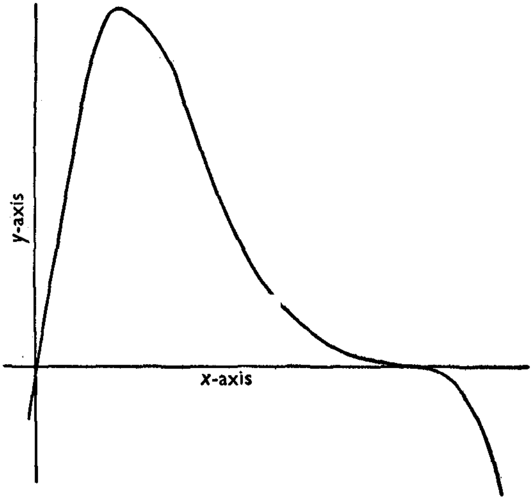


Fig. 5. Quartic with one maximum only, a point of inflexion and two real roots.

55. Having regard to the usual method of constructing a life table from a mortality experience, the formula should not be thought of as a law of mortality; the parameters of the formula express a relationship between the statistical distributions of exposed to risk and deaths, and their moments. It is assumed that b_3 and b_4 may take the value zero, particularly where the mortality experience relates to part only and not the whole of life.

NUMERICAL EXAMPLES

Assured lives, 1924–29

56. There are alternative ways of obtaining the constants of the formula: either the observed l'_x can be calculated from the ungraduated rates of mortality and can then be treated as a frequency curve, or the constants can be calculated direct from the exposed to risk and deaths by means of equation (6). The first method was tried, but was found somewhat complicated to handle. The second method is a more direct and simple approach to the problem, especially when the statistics relate to a part only of life, and the following examples are based on this method.

57. The relationship implied by equations (5) and (6) is perhaps most appropriately (though not necessarily) considered as applying to the force of mortality μ_x. Since the exposed to risk and deaths will be grouped it is necessary to make some assumption about the mean age of the group, and it has been assumed that the mean age is the central age (except for age 0); the run of the data as a whole should tend to even out the error. A more refined method would no doubt improve the results. For example, where the statistics have high contact at both ends, as is customary with assured life experiences, Sheppard's adjustments could be applied to the moments.

Table 1. Moments of mean population and deaths; British offices 1924–29 experience

Unit of measurement 10 years, durations 5 and over

Order of moment	Mean population		Deaths	
	Whole experience	Part above origin	Whole experience	Part above origin
	Whole-life assurance with profits—origin at 60			
Statistics	1,715,920	881,823	65,540	57,252
First	17,853	900,139	79,129	84,557
Second	2,792,450	1,297,265	163,042	156,145
Third	− *894,556*	2,261,795	315,899	327,891
Fourth	12,126,662	4,477,282	778,887	753,326
Fifth	−*10,652,367*	9,739,791	1,792,048	1,854,807
Sixth	—	—	5,000,875	4,830,529
	Endowment assurance with profits—origin at 45			
Statistics	4,431,574	1,872,452	28,981	19,985
First	− *711,330*	1,460,785	15,276	21,636
Second	4,440,356	1,658,628	38,781	31,275
Third	− *1,996,962*	2,322,187	42,027	53,078
Fourth	11,429,883	3,732,138	119,168	100,411
Fifth	—	—	170,067	205,658
Sixth	—	—	526,545	451,680

Notes. No adjustment was made to the endowment assurance moments for the uneven distribution of the exposed to risk and deaths over the year of age at popular maturity ages. Nor was any adjustment made for the errors in the statistics.

58. The statistics for the British Offices 1924–29 experience were the deaths at nearest integral ages, and the populations at nearest integral ages at the beginning and end of the years of exposure. Since the exposed to risk is the mean population *plus* one-half of the deaths, the mean population may be found from the tabulated exposed to risk by deducting one-half of the deaths. It is sufficient to assume that the quotient of the deaths and the mean population gives the force of mortality for the nearest integral age of the group.

59. Sufficient simultaneous equations to calculate the known constants can be obtained from (6) either by dividing the data into successive ranges or by equating successive moments for the whole range. The general procedure adopted was to form three only of such successive moments of the whole data and to use additional equations based on the moments for the part of the range above the selected arbitrary origin.

60. The moments for the mean populations and deaths are given in Table 1,

which includes all those which were calculated though they were not necessarily used.

61. These moments were substituted in equations of the form of (6). It will be understood that the form taken by θ_x/E_x in (6) only gradually emerged from the experiments. Attempts were made with the forms

$$a/(b_0+b_1x+b_2x^2+b_3x^3+x^4),$$
$$(a+x)/(b_0+b_1x+b_2x^2+b_3x^3),$$
$$(a_0+a_1x+x^2)/(b_0+b_1x+b_2x^2),$$
$$(a_0+a_1x+a_2x^2+x^3)/(b_0+b_1x+b_2x^2+b_3x^3).$$

It was clear from this work that more constants did not necessarily lead to a better fit and that the most important question was the choice of a form appropriate to the range and extent of the data. The best fit was obtained by assuming that the numerator was a quadratic without real roots and that the denominator was a cubic for whole-life assurances and a quadratic for endowment assurances.

62. The parameters for whole life assurances with profits—origin at age 60 and unit of measurement 10 years—were:

$$\begin{aligned} a_0 &= 14{\cdot}0998, & b_0 &= +701{\cdot}880,\\ a_1 &= 4{\cdot}98946, & b_1 &= -414{\cdot}557,\\ a_2 &= 1{\cdot}0, & b_2 &= +103{\cdot}095,\\ & & b_3 &= -\quad 9{\cdot}278. \end{aligned}$$

These figures were obtained by a slight variation of the procedure that has been described which would not, I think, materially affect the results.

63. The numerator is a quadratic without real roots and with a minimum point at about age 35. The denominator is a cubic which has a root at about age 114, when the curve ends, and a point of inflexion at about age 98. It is interesting to notice that the calculated force of mortality has a minimum at about age 14, though, as might be expected, the formula does not give a reasonable extension to the very young ages where there are no statistics.

64. The parameters for endowment assurances with profits—origin at age 45 and unit of measurement 10 years—were:

$$\begin{aligned} a_0 &= 10{\cdot}85629, & b_0 &= +2123{\cdot}495,\\ a_1 &= 1{\cdot}68331, & b_1 &= -1063{\cdot}7397,\\ a_2 &= 1{\cdot}0, & b_2 &= +\ 156{\cdot}31636. \end{aligned}$$

The numerator is a quadratic without real roots and with a minimum at about age 37. The denominator is also a quadratic without real roots and with a minimum at about age 79. Theoretically, the curve has an infinite extension at both ends, but its practical usefulness is limited to the range of the statistics, namely, ages 6–83. The calculated force of mortality has a minimum at about age 14, and the rates at these young ages are higher than for whole-life assurances. The statistics included some deaths at these ages whereas the whole-life assurance experience did not record any death below age 22.

Table 2. Comparison of actual and expected deaths. Whole-life assurances with profits, 1924–29 experience, durations 5 and over

Nearest integral age	Calculated force of mortality	Actual deaths	Expected deaths	Actual *minus* expected		χ^2
				+	−	
10	·002173	—	—	—	—	—
11	·002168	—	—	—	—	—
12	·002165	—	—	—	—	—
13	·002162	—	—	—	—	—
14	·002162	—	—	—	—	—
15	·002163	—	—	—	—	—
16	·002166	—	—	—	—	—
17	·002171	—	·1	—	·1	
18	·002179	—	—	—	—	
19	·002190	—	·1	—	·1	
20	·002203	—	·2	—	·2	·38
21	·002220	—	·7	—	·7	
22	·002241	4	1·9	2·1	—	
23	·002266	4	3·2	·8	—	
24	·002295	5	4·4	·6	—	
25	·002330	7	5·1	1·9	—	·78
26	·002371	9	6·0	3·0	—	1·50
27	·002418	7	6·9	·1	—	·00
28	·002471	8	8·0	—	—	·00
29	·002533	7	9·4	—	2·4	·43
						3·09
30	·002603	11	11·1	—	·1	·00
31	·002682	12	13·5	—	1·5	·30
32	·002772	14	16·3	—	2·3	·25
33	·002874	24	19·7	4·3	—	·82
34	·002988	21	24·0	—	3·0	·38
35	·003117	26	29·2	—	3·2	·31
36	·003260	28	35·1	—	7·1	1·40
37	·003421	28	41·2	—	13·2	4·11
38	·003600	55	49·2	5·8	—	1·74
39	·003801	54	58·2	—	4·2	·28
						8·59
40	·004025	69	68·3	·7	—	·01
41	·004274	78	79·1	—	1·1	·01
42	·004552	111	92·3	18·7	—	3·93
43	·004861	101	107·4	—	6·4	·34
44	·005205	122	124·7	—	2·7	·07
45	·005586	163	144·9	18·1	—	2·25
46	·006010	159	168·3	—	9·3	·48
47	·006481	207	194·0	13·0	—	·88
48	·007004	212	222·9	—	10·9	·55
49	·007584	236	255·3	—	19·3	1·42
						9·94
50	·008228	337	291·4	45·6	—	7·32
51	·008941	323	332·4	—	9·4	·25
52	·009733	385	377·4	7·6	—	·17
53	·010611	420	428·0	—	8·0	·15
54	·011585	477	485·2	—	8·2	·13
55	·012664	534	551·9	—	17·9	·59
56	·013861	632	622·8	9·2	—	·13
57	·015187	694	701·3	—	7·3	·07
58	·016657	811	786·3	24·7	—	·81
59	·018285	883	878·5	4·5	—	·03
						9·65

Table 2 (*continued*)

Nearest integral age	Calculated force of mortality	Actual deaths	Expected deaths	Actual *minus* expected +	Actual *minus* expected −	χ^2
60	·02009	1,010	982·4	27·6	—	·81
61	·02209	1,007	1,095·9	—	88·9	7·39
62	·02430	1,320	1,216·5	103·5	—	8·94
63	·02675	1,322	1,344·9	—	22·9	·40
64	·02945	1,545	1,488·5	56·5	—	2·25
65	·03244	1,635	1,656·4	—	21·4	·28
66	·03575	1,803	1,818·8	—	15·8	·15
67	·03939	1,915	1,983·9	—	68·9	2·50
68	·04341	2,025	2,164·8	—	139·8	9·46
69	·04784	2,380	2,333·1	46·9	—	·99
						33·17
70	·05271	2,429	2,483·6	—	54·6	1·29
71	·05805	2,793	2,601·0	192·0	—	15·05
72	·06391	2,750	2,698·0	52·0	—	1·07
73	·07032	2,767	2,774·7	—	7·7	·02
74	·07730	2,710	2,813·0	—	103·0	4·09
75	·08491	2,821	2,834·8	—	13·8	·08
76	·09316	2,813	2,807·3	5·7	—	·01
77	·10209	2,681	2,749·8	—	68·8	1·93
78	·11172	2,722	2,657·1	64·9	—	1·79
79	·12206	2,484	2,487·7	—	3·7	·01
						25·34
80	·13313	2,271	2,258·1	12·9	—	·09
81	·14491	1,986	2,061·7	—	75·7	3·28
82	·15742	1,885	1,889·7	—	4·7	·02
83	·17063	1,690	1,700·7	—	10·7	·09
84	·18454	1,540	1,505·1	34·9	—	1·00
85	·19910	1,327	1,271·0	56·0	—	3·08
86	·21428	1,032	1,060·3	—	28·3	·94
87	·23008	914	872·5	41·5	—	2·63
88	·24643	696	702·1	—	6·1	·07
89	·26336	548	547·1	·9	—	·00
						11·20
90	·28082	412	416·4	—	4·4	·05
91	·29883	325	306·9	18·1	—	1·51
92	·31745	251	220·7	30·3	—	5·98
93	·33670	133	157·7	—	24·7	5·98
94	·35669	130	109·5	20·5	—	6·26
95	·37754	67	67·6	—	·6	·02
96	·39945	58	44·2	13·8	—	7·40
97	·42265	30	27·4	2·6	—	·57
98	·44748	10	17·9	—	7·9	6·46
99	·47431	8	10·1	—	2·1	·75
						34·98
100	·50369	9	7·4	1·6	—	1·08
101	·53632	5	5·8	—	·8	
102	·57309	2	2·3	—	·3	·26
103	·61521	1	·9	·1	—	
104	·66432	—	—	—	—	—
105	·72275	—	—	—	—	—
106	·79387	—	—	—	—	—
107	·88274	—	—	—	—	—
108	·99747	—	—	—	—	—
109	1·1523	—	—	—	—	—
110	1·3708	—	—	—	—	—
		65,540	65,511·2	943·0	914·2	137·30

Note. The values of χ^2 were calculated from integral values of the deviations.

Table 3. Comparison of actual and expected deaths. Endowment assurances with profits, 1924–29 experience, durations 5 and over

Nearest integral age	Calculated force of mortality	Actual deaths	Expected deaths	Actual *minus* expected +	Actual *minus* expected −	χ^2
6	·002255	1	1·4	—	·4	
7	·002244	1	1·4	—	·4	
8	·002234	1	1·4	—	·4	
9	·002225	1	1·4	—	·4	
10	·002218	1	1·4	—	·4	
11	·002212	—	1·6	—	1·6	
12	·002207	—	1·8	—	1·8	
13	·002204	1	2·0	—	1·0	·00
14	·002202	1	2·1	—	1·1	
15	·002203	2	2·1	—	·1	
16	·002205	1	1·7	—	·7	
17	·002210	2	1·6	·4	—	
18	·002217	2	1·7	·3	—	
19	·002227	4	2·3	1·7	—	
20	·002239	10	4·2	5·8	—	
21	·002255	18	12·0	6·0	—	3·00
22	·002274	28	26·2	1·8	—	·15
23	·002297	52	44·7	7·3	—	1·10
24	·002325	74	69·1	4·9	—	·36
25	·002356	90	93·2	—	3·2	·10
26	·002393	114	120·8	—	6·8	·41
27	·002435	167	149·9	17·1	—	1·93
28	·002483	180	175·0	5·0	—	·14
29	·002538	203	198·0	5·0	—	·13
						7·32
30	·002599	205	222·9	—	17·9	1·46
31	·002669	262	250·1	11·9	—	·58
32	·002747	287	280·0	7·0	—	·18
33	·002835	281	311·9	—	30·9	3·09
34	·002933	341	348·6	—	7·6	·18
35	·003043	364	390·1	—	26·1	1·74
36	·003165	408	432·5	—	24·5	1·45
37	·003301	465	474·9	—	9·9	·21
38	·003453	549	517·9	31·1	—	1·86
39	·003622	520	564·6	—	44·6	3·60
						14·35
40	·003810	597	599·9	—	2·9	·02
41	·004018	691	639·5	51·5	—	4·25
42	·004250	698	678·7	19·3	—	·53
43	·004508	765	721·8	43·2	—	2·57
44	·004794	787	768·3	18·7	—	·47
45	·005112	822	805·4	16·6	—	·36
46	·005466	830	861·1	—	31·1	1·12
47	·005860	899	916·8	—	17·8	·36
48	·006297	924	975·3	—	51·3	2·68
49	·006785	990	1,031·0	—	41·0	1·64
						14·00

Table 3 (*continued*)

Nearest integral age	Calculated force of mortality	Actual deaths	Expected deaths	Actual *minus* expected +	Actual *minus* expected −	χ^2
50	·007327	1,034	1,019·6	14·4	—	·19
51	·007931	1,105	1,025·8	79·2	—	6·13
52	·008605	1,047	1,072·9	—	25·9	·64
53	·009358	1,157	1,118·4	38·6	—	1·37
54	·010196	1,154	1,150·8	3·2	—	·01
55	·011134	1,047	1,089·8	—	42·8	1·72
56	·012182	1,000	1,052·8	—	52·8	2·70
57	·013353	1,109	1,066·6	42·4	—	1·68
58	·014665	1,117	1,086·2	30·8	—	·90
59	·016131	1,116	1,079·5	36·5	—	1·29
						16·63
60	·017771	864	874·3	—	10·3	·12
61	·019606	702	724·1	—	22·1	·68
62	·021656	702	696·5	5·5	—	·05
63	·023945	706	664·6	41·4	—	2·59
64	·026497	603	621·4	—	18·4	·54
65	·029332	414	442·4	—	28·4	1·83
66	·032472	313	328·9	—	15·9	·80
67	·035937	266	282·2	—	16·2	·94
68	·039734	233	239·3	—	6·3	·16
69	·043865	200	194·0	6·0	—	·19
						7·90
70	·048320	142	139·9	2·1	—	·03
71	·053063	95	100·7	—	5·7	·38
72	·058041	83	76·0	7·0	—	·68
73	·063177	63	52·4	10·6	—	2·47
74	·068355	29	35·3	—	6·3	1·09
75	·073441	18	13·7	4·3	—	1·26
76	·078282	9	6·9	2·1	—	·63
77	·082704	5	4·8	·2	—	
78	·086547	5	2·7	2·3	—	
79	·089678	2	1·6	·4	—	·93
80	·091977	2	·9	1·1	—	
81	·093396	—	·5	—	·5	
82	·093934	—	·2	—	·2	
83	·093631	—	—	—	—	
						7·47
		28,981	28,974·0	582·7	575·7	67·67

Note. The values of χ^2 were calculated from integral values of the deviations.

65. The expected and actual deaths are compared in Tables 2 and 3 for whole-life and endowment assurances respectively. The total of χ^2 for a group of ten ages should not exceed 18·31 at the 5% probability level, but in considering the values of χ^2 the presence of duplicates in the data should be remembered; making suitable allowance for them it is clear that each of the groups of ten values of χ^2 would be acceptable at the 5% probability level. The calculated force of mortality is compared with the official graduation in Table 4.

Table 4. Comparison of the force of mortality

Age	A 1924–29	Calculated for whole-life assurances	Calculated for endowment assurances
10	·00151	·00217	·00222
20	·00235	·00220	·00224
30	·00239	·00260	·00260
40	·00377	·00402	·00381
50	·00736	·00823	·00733
60	·01894	·02009	·01777
70	·05195	·05271	·04832
80	·13240	·13313	·09198
90	·28588	·28082	—
100	—	·50369	—
110	—	1·3708	—

Notes. The minimum at age 14 was ·00216 for whole-life and ·00220 for endowment assurances. The calculated values for endowment assurances decrease after age 82 but this is beyond the range of the statistics.

Population Statistics, England and Wales 1930–32

66. Population statistics, though notoriously unreliable, are the most important of the few sources of statistics of mortality throughout life, and the next experiments endeavoured to apply this formula to E.L.T. No. 10. The results were disappointing. Various expressions were tried, but though the middle range of ages was well graduated the values at ages under 30 and over 80 were unreliable. The fault may lie in the formula, the method of fitting or the statistics themselves.

67. The moments are set out in Table 5, the parameters in Table 6 and the forces of mortality are compared in Tables 7 and 8. The calculations were made from grouped moments, so that a comparison of actual and expected deaths in similar groups would be too kind to the graduation. Calculations of the individual deviations were made but are not thought to be of sufficient interest to be included in the printed paper.

68. Probably a better graduation could be obtained by the use of grouped data adjusted by King's formula to give the central ordinate. For the preliminary work described in this paper it was felt to be undesirable to introduce any adjustment which might be held to influence the result in the desired direction.

69. The general impression left by the work on the census data is that a good result may be obtained for the adult ages though the rates at the oldest ages may be forced. It does not seem possible to reproduce the low rates experienced in early adolescence, at least without using a large number of parameters. The general shape of the falling mortality in infancy and childhood is reflected in the curve, but the exact start (where the denominator is zero and μ infinite) is difficult to find, the fall is not steep enough at first, and the rates are consequently distorted throughout childhood and beyond.

Table 5. Moments for the data of E.L.T. No. 10

Origin at age 29 last birthday, assumed $29\frac{1}{2}$, unit 10 years

Age-group last birthday	Statistics	First moment	Second moment	Third moment	Fourth moment
	Males: Census population in 1931				
0– 9	3,188,059	− *7,756,261*	19,138,725	− *47,874,701*	121,323,295
10–19	3,329,943	− *4,830,758*	7,300,414	− *11,443,029*	18,507,324
20–29	3,328,134	− *1,519,883*	966,922	− *688,437*	521,693
30–39	2,716,299	+ 1,439,652	994,365	+ 777,363	649,502
40–49	2,415,900	+ 3,729,916	5,959,945	+ 9,824,768	16,645,040
50–59	2,103,764	+ 5,319,881	13,625,186	+ 35,331,207	92,708,551
60–69	1,356,034	+ 4,748,278	16,735,310	+ 59,368,161	211,965,052
70–79	580,659	+ 2,585,168	11,552,465	+ 51,820,336	233,335,592
80–89	108,894	+ 584,657	3,145,677	+ 16,961,714	91,662,967
90 & over	5,324	+ 33,426	210,087	+ 1,321,949	8,328,237
All ages	19,133,010	+ 4,334,076	79,629,096	+ 115,399,331	795,647,253
	Males: Deaths in 1930–32				
0– 9	108,822	− *304,486*	857,554	− *2,428,330*	6,907,104
10–19	20,367	− *27,876*	39,886	− *59,563*	92,480
20–29	32,867	− *14,919*	9,488	− *6,751*	5,111
30–39	34,018	+ 19,516	14,084	+ 11,302	9,608
40–49	56,936	+ 91,288	150,982	+ 256,575	446,228
50–59	100,451	+ 260,507	683,769	+ 1,815,241	4,870,411
60–69	148,793	+ 532,312	1,916,336	+ 6,941,079	25,289,455
70–79	156,221	+ 707,135	3,213,147	+ 14,655,979	67,102,674
80–89	65,007	+ 352,466	1,915,463	+ 10,434,067	56,973,289
90 & over	5,960	+ 37,572	237,148	+ 1,498,744	9,484,439
All ages	729,442	+ 1,653,515	9,037,857	+ 33,118,343	171,180,799
	Females: Census population in 1931				
0– 9	3,124,894	− *7,602,189*	18,756,964	− *46,914,423*	118,873,340
10–19	3,311,803	− *4,788,192*	7,213,989	− *11,277,950*	18,201,068
20–29	3,523,453	− *1,605,298*	1,017,986	− *722,751*	546,441
30–39	3,142,026	+ 1,692,919	1,174,443	+ 918,590	767,128
40–49	2,801,592	+ 4,320,566	6,895,462	+ 11,352,825	19,209,856
50–59	2,346,350	+ 5,922,976	15,143,599	+ 39,202,478	102,701,005
60–69	1,571,587	+ 5,516,047	19,487,294	+ 69,293,077	247,971,935
70–79	789,955	+ 3,528,651	15,822,203	+ 71,218,531	321,807,299
80–89	194,006	+ 1,046,375	5,656,347	+ 30,646,825	166,439,928
90 & over	13,701	+ 86,344	544,831	+ 3,442,496	21,781,644
All ages	20,819,367	+ 8,118,199	91,713,118	+ 167,159,698	1,018,299,644
	Females: Deaths in 1930–32				
0– 9	84,839	− *235,993*	661,169	− *1,863,429*	5,277,954
10–19	18,786	− *25,764*	36,893	− *55,061*	85,324
20–29	31,003	− *13,627*	8,515	− *5,986*	4,493
30–39	34,247	+ 19,429	13,891	+ 11,075	9,375
40–49	48,663	+ 77,566	127,688	+ 216,218	375,054
50–59	81,080	+ 209,964	550,317	+ 1,458,955	3,909,380
60–69	126,465	+ 453,538	1,636,670	+ 5,941,980	21,698,382
70–79	164,915	+ 750,436	3,427,927	+ 15,717,858	72,339,145
80–89	97,993	+ 534,568	2,923,258	+ 16,024,940	88,064,559
90 & over	13,753	+ 87,100	552,419	+ 3,509,524	22,327,045
All ages	701,744	+ 1,857,217	9,938,747	+ 40,955,674	214,090,711

Notes. The age-groups used were 9 and under, 10–29, 30–49, 50–59, 60–69, 70–79, 80 and over. The denary age-groups are tabulated for convenience.

The moments were calculated for each individual age, on the assumption that the group for age x last birthday was concentrated at exact age $x+\frac{1}{2}$. In calculating the moments for age 0, the deaths in the first 4 weeks—about half the total—were assumed to occur at birth and the remainder at 6 months.

The fifth and the sixth moments of the deaths were also calculated.

Table 6. Parameters for the data of E.L.T. No. 10

Origin at age 29 last birthday, assumed $29\frac{1}{2}$, unit 10 years

Parameter	All	Values computed from the statistics at ages				
		1 and over	2 and over	5 and over	10 and over (a)	10 and over (b)
		Males				
a_0	13·0225	12·9259	12·7880	12·3388	12·24294	11·9184
a_1	3·8151	3·4223	2·8619	1·0351	·6719	·8455
a_2	1·0	1·0	1·0	1·0	1·0	1·0
b_0	1336·532	1324·216	1306·647	1249·370	1237·980	1206·362
b_1	−149·50548	− 182·36954	−229·24741	− 382·07494	− 412·45260	−384·10557
b_2	−104·66084	− 86·61093	− 60·86421	+ 23·07325	+ 39·75823	+ 32·28951
b_3	+ 24·67376	+ 21·51249	+ 17·00320	+ 2·30234	− ·61992	—
b_4	− 1·48168	− 1·29649	− 1·03235	− ·17118	—	—
		2 and over (a)	2 and over (b)	2 and over (c)	10 and over	
		Females				
a_0	—	49·253	14·3342	52·685	41·6537	—
a_1	—	5·697	7·9187	6·0	3·7	—
a_2	—	1·0	·9253	1·0	1·0	—
a_3	—	0	1·0	0	0	—
b_0	—	5705·7600	1545·923	6100·760	4817·981	—
b_1	—	−1170·9726	−573·14220	−1261·78471	−1113·45802	—
b_2	—	− 321·6976	−240·78606	− 346·54285	− 203·91501	—
b_3	—	+ 105·7345	+ 10·22027	+ 114·69903	+ 76·81939	—
b_4	—	− 7·5555	+ 1·44911	− 8·14343	− 5·60728	—
b_5	—	0	0	− 0·01910	0	—

CONCLUSION

70. To the reader who has ventured so far it will be apparent that the methods discussed in this paper have yielded only a partial solution to the problem, but they have introduced what is believed to be a fresh way of looking at it which may prove to be fruitful.

71. The suggestion is that the rate of mortality should be viewed simply as a functional relationship between the exposed to risk and the deaths. As such it should not be thought of as having a mathematical form, dependent on the various causes of death. The mathematical form of the rate of mortality springs naturally from the functional relationship between the exposed to risk and the deaths in the mortality experience.

72. In this paper the relationship has been assumed to be between the statistics of exposed to risk and deaths and their moments. Such a relationship leads to a differential equation similar to the one adopted by Karl Pearson for the graduation of frequency curves, which suggests that l_x has some of the characteristics of, and a mathematical form similar to, Pearson's frequency curves—but the form is more complicated because, usually, l_x is not a true frequency curve.

73. Pearson's system of frequency curves is of most general application to distributions of the 'cocked-hat' type, though it also deals with some J-shaped and twisted J-shaped distributions. The system can be extended to a wide range of twisted J-shaped distributions by making the numerator a quadratic without real roots, that is to say

$$\frac{1}{y}\frac{dy}{dx}=\frac{a+x^2}{b_0+b_1x+b_2x^2+\ldots},$$

where the origin is at the age at which the numerator is a minimum. The numerator is positive for all values of x, so that the differential coefficient does not change sign within the range of the curve, and the curve itself continuously decreases—or increases. The range is defined by the real roots of the denominator. At these points the denominator becomes zero and the differential coefficient infinite.

74. Pearson integrated his differential equation to find the equation for the frequency curve which he fitted to the statistics by using the observed frequencies and their moments. The equivalent process would be to integrate (8) to find the mathematical form for l_x and to fit this curve to the values of l'_x computed from the ungraduated rates of mortality. Though this process has been tried, it was not convenient, particularly in the early stages when the precise mathematical form had not been determined.

75. In this paper the differential equation for the force of mortality has been fitted directly to the statistics by assuming that it springs from the equation connecting the exposed to risk and the deaths and their moments.

76. In practice it is found that the denominator has one or more real factors, possibly combined with a quadratic without real roots. Hence the differential equation can be developed by partial fractions and integrated; in effect this produces l_x by integration of μ_x. The process is not shown here because it follows exactly the procedure described in Elderton's *Frequency Curves and Correlation*, 3rd ed. pp. 46–48.

77. Though the approach to the problem is thought to be consistent and logical, no exclusive claims can be made for any method; the method must take its place beside the others of all kinds. But it is of some interest that the same philosophical approach can lead to curves of the Gompertz type by changing the variable to a geometric one.

78. It is hoped that the J-shaped frequency curve may have applications to other statistics than mortality.

79. Members will understand that the investigations described in this paper depend upon a mass of arithmetical calculation which could not have been tackled without the work of a team of helpers. In the earlier work which attempted to represent the ungraduated l_x by a frequency curve much of the arithmetical work was done by R. H. Storr-Best. The division of the arithmetic described in this paper was in general as follows: the moments were computed by C. M. O'Brien, F.I.A.; the author's share of the work was the solution of the simultaneous equations; the graduated values and the comparisons of actual and expected deaths were calculated by G. E. Wallas, A.I.A.; the application of the χ^2 test was made by G. V. Bayley, F.I.A. Their assistance is gratefully acknowledged.

Table 7. Comparison of μ_x computed from the data of E.L.T. No. 10

MALES

Age	μ_x computed from the statistics at ages						Official (central rate)
	All	1 and over	2 and over	5 and over	10 and over		
					(a)	(b)	
$4\frac{1}{2}$	·00530	·00415	·00332	·00231	·00222	·00226	·00360
$9\frac{1}{2}$	·00314	·00292	·00269	·00228	·00223	·00225	·00162
$19\frac{1}{2}$	·00251	·00251	·00250	·00248	·00248	·00248	·00302
$29\frac{1}{2}$	·00325	·00325	·00326	·00329	·00330	·00329	·00335
$39\frac{1}{2}$	·00538	·00538	·00538	·00537	·00537	·00537	·00532
$49\frac{1}{2}$	·01037	·01037	·01036	·01035	·01034	·01035	·01063
$59\frac{1}{2}$	·02266	·02267	·02267	·02269	·02269	·02268	·02250
$69\frac{1}{2}$	·05597	·05596	·05594	·05587	·05585	·05592	·05664
$79\frac{1}{2}$	·14567	·14578	·14595	·14665	·14684	·14737	·14364
$89\frac{1}{2}$	·29600	·29514	·29385	·28856	·28730	·27537	·31529
$99\frac{1}{2}$	·44013	·41133	·37359	·23211	·25465	·22323	·61735

Notes. The denominators for 'all ages' and for 1, 2 and 5 and over are quartics; for 10 and over the denominator is (*a*) a cubic, (*b*) a quadratic. Since the population and deaths have been used without adjustment, the figures are strictly speaking of m_x, not μ_x, and comparison has been made with the central rate at the integral age last birthday by the official graduation.

Table 8. Comparison of μ_x computed by various types of formula from the data of E.L.T. No. 10

FEMALES

Age	μ_x computed from the statistics at ages				Official (central rate)
	10 and over	2 and over			
	Type of formula				
	Quadratic / Quartic	Quadratic / Quartic	Cubic / Quartic	Quadratic / Quintic	
$4\frac{1}{2}$	·00263	·00294	—	·00294	·00336
$9\frac{1}{2}$	·00231	·00241	(·00311)	·00241	·00143
$19\frac{1}{2}$	·00230	·00231	·00292	·00231	·00261
$29\frac{1}{2}$	·00288	·00288	·00309	·00288	·00315
$39\frac{1}{2}$	·00433	·00433	·00426	·00433	·00424
$49\frac{1}{2}$	·00769	·00769	·00761	·00769	·00766
$59\frac{1}{2}$	·01631	·01630	·01641	·01630	·01640
$69\frac{1}{2}$	·04146	·04148	·04115	·04145	·04119
$79\frac{1}{2}$	·11323	·11299	·11711	·11348	·11470
$89\frac{1}{2}$	·27229	·27355	·2581	·32708	·26727
$99\frac{1}{2}$	infinite	infinite	·2062	infinite	·529

Notes. The '10 and over' graduation assumes that the numerator is a minimum at age 11; an earlier age would have been a better choice. The cubic and the quartic in the second of the '2 and over' graduations both have roots at about age $13\frac{1}{2}$; the additional parameter has worsened the fit, not improved it, and the roots suggest that the ratio of a quadratic to a cubic would do as well. The last of these graduations, in which the denominator is a quintic, assumes that the numerator is a minimum at age $-\frac{1}{2}$. Since the population and deaths have been used without adjustment the figures are strictly of m_x, not μ_x, and comparison has been made with the central rate at the integral age last birthday by the official graduation.

ABSTRACT OF THE DISCUSSION

Mr R. G. Barley, in opening the discussion, referred to the statement at the beginning of paragraph 3 of the paper:

> When the graduation of mortality statistics is being discussed it is frequently said that the mathematical formula used is of no importance, that any formula will do if it fits. Such an attitude denies any reality to the process of graduation.

Those words seemed to him to imply a sharp division between the scientific sheep, concerned with the reality which might be attributed to the process of graduation, and the unscientific or practical goats, concerned only to secure, for example, a workable table of premiums. That was a distinction which he did not believe existed in fact, and he thought that it was that statement which was at the bottom of some of the difficulties which he had found in reading the paper.

When a graduation was being made for a specific practical purpose there were always considerations which removed the result to some degree from that which might otherwise be regarded as ideal. It was essential to take into account whether it would be safer for such a practical purpose to over-estimate or to under-estimate mortality, and whether the trouble and expense of refined methods were justified by either the size of the experience or the use to which the results were to be put.

He would assume for the purposes of the discussion that considerations of that kind might be entirely disregarded. There were still left, however, at least two possible attitudes towards the graduation of mortality data. The hypothesis that 'any formula will do if it fits' was meaningless without some definition of the word 'fit'. A test for graduation was usually undertaken with the idea that the observations were a sample and that the purpose of the graduation was to remove random variations due to sampling. One attitude, therefore, might be that if the random variations had been removed from a mortality experience so far as could be judged the object had been achieved, regardless of whether the graduation method used would be suitable for another experience. He did not think that that attitude need deny reality to the graduation process. The other attitude was that of seeking something which was common to more than one mortality experience, and naturally involved the consideration of common features of graduation formulae which had been successful and the attempt to construct formulae which would be successful in more than one instance. The idea of a law of mortality necessitated that more searching attitude to the problem.

He was sure that the use of a quotation from Gompertz as the title of the paper was more than a mild literary conceit, in spite of the modest statement in paragraph 77 that

> the method must take its place beside the others of all kinds.

The title, it would be noted, postulated that there was a law of mortality, and it distinguished human mortality. In what sense were they to understand 'law'? The author pointed out that there had been a change in attitude with regard to the function expressive of the postulated law, that in the early tables attention had been directed primarily to the number of survivors, and that it was only later that the law had been thought of in terms first of a rate, and then of a force operating on the number alive at a given age. During the same period, ideas had changed as to the meaning of 'law'. To Gompertz, a natural law must have seemed a much more definite statement than it did in the middle of the twentieth century when people were more cautious in interpreting natural phenomena than the men of the early nineteenth century had been. If, however, they were to continue to use the word 'law', they must admit that they were seeking in some sense a generalization, even if they did not expect to find one so simple, so sweeping in its embrace, and in such definite terms as a man of Gompertz's day might have hoped. If they were of the opposite opinion, and did not admit of the possibility of generalization beyond the observations of one experience, they must adopt the first attitude which he had mentioned, namely, that each graduation was an entirely separate attempt at removing random variation from a particular experience.

When the results of a mortality investigation were applied as a measure of future mortality it was assumed that generalization was permissible. It was possible to go further and to make a generalization which he thought was incontestably true. The late Lord Keynes had once said 'In the long run we are all dead'. He submitted that that was undeniably a law of mortality, and there was no *prima facie* reason for believing that no attempt should be made to elaborate it. Each successive development of that primary law would approach nearer to the point where, with the parameters computed for a given mortality experience, the law was expressed in a form applicable only to that experience. That was the light in which to regard the search for a law of mortality, which should be continued even though the immediate practical advantages of any particular step might appear to be limited. He wished that the author had found it possible to expand the opening sections of the paper and so to give his readers the benefit of his undoubtedly valuable observations on the meaning of the words 'law of mortality'.

In the main part of the paper the author set out most clearly the derivation of his suggested formulae. His method of adapting the Pearson system was one which had most interesting possibilities. There were two points on which the speaker wished to comment.

First, reference was made, in paragraphs 33–35, to the idea of examining an actual observed life table and, like so many investigators, the author was content to examine that idea very briefly. He was strongly of opinion that that was something which had to be tackled if appreciable progress were to be made. The ordinary mortality experience was a section cut across a consecutive series of surviving lives of a number of generations. The author asked in paragraph 17 for a return to earlier ways of thought. The speaker believed that the earlier investigators had instinctively been right in using the concept of a life table as a description of mortality, although it might not be such a simple description as it had appeared to be before the effect of secular change was so fully appreciated. Apart from the formal correlation between the numbers living at successive ages in the life table, there were also on the human scale disturbances caused by secular events, such as dramatic advances in medical science on the one hand and a major war on the other. That suggested that it might be necessary to study the relation between life tables in the sense of true survivorship lines and the cross-section which they normally observed in a human mortality experience. Unless that cross-section was instantaneous, which it was not in practice, an element of correlation could not be entirely absent. Although the cross-section was normally their only method of investigating a particular mortality for a particular purpose, they would limit their achievement in the solution of the law of mortality problem if they regarded it as the primary form of the statistics.

Secondly, he did not fully understand paragraph 55, where the author said that his formula 'should not be thought of as a law of mortality'. It was not clear whether there was in the author's opinion any fundamental reason why a law, or at any rate part of a law, should not be expressed as (in his own words) 'a relationship between the statistical distributions of exposed to risk and deaths, and their moments'. As he saw it, in discussing the nature of the function expressive of the law of human mortality the author introduced two ideas. Passing over the analysis of lines of survivorship, he represented the relationship θ to E by a fraction, postulating that the numerator and denominator need not respectively represent the separate distributions of θ and E. That was the first idea, of which the second was a development, but not, he thought, a necessary development.

The second idea emerged from the parallelism between the fundamental Pearson equation and the form chosen to illustrate the first idea, and the result was an extension of the Pearson system to bring mortality data within its scope. He thought that the author had at that point refrained from any attempt to construct a formula the parameters of which might be considered to be derived philosophically from a consideration of the nature of death, and that seemed wise in the then existing state of knowledge, but it still appeared that the author's reasoning was intended to be an alternative approach to a law of mortality, even though he might have been travelling hopefully rather than arriving. He therefore sought enlightenment on paragraph 55, probably because it was not easy to grasp in what sense the author used the word 'law'.

Turning to the last part of the paper, that gave the numerical results of what were, presumably, the more successful of the experiments undertaken by the author's indefatigable team of computers. In paragraph 66 it was suggested that the reason for the disappointing results of graduating population statistics could lie in the statistics themselves. That, he thought, was the kernel of the matter. There were there two experiments, one with assured lives, where the fit was good, and one with population statistics, where the fit was not so good. It had always seemed to him that a substantial advance towards a law of mortality would not be made by graduating and regraduating assurance statistics. For one thing, they applied principally to ages where the progression of mortality was relatively uncomplicated, and also to a somewhat special body of lives. Far more attention ought to be paid to the difficult parts of the mortality curve, the parts where assurance office data were either non-existent or scanty. That was why he had referred at the beginning of his remarks to the elimination of immediate practical considerations. The fact that a formula successfully graduated assurance statistics was less weighty evidence that it represented a real advance towards a law of mortality than would be a measure of success with some other kind of mortality statistics in addition to that with assurance statistics. The author recognized that by making his experiments cover more than one kind of statistics, but it should be a matter of course in the presentation of all attempts to seek a law of mortality that the reasons for the choice of the particular data for experiment be given.

What was most urgent, he believed, was a search for statistics rather than formulae. Formulae tested only on current or recent data of assured lives or national population were in serious danger, whatever their merits, of being regarded as mere additions to the possible methods of obtaining a premium basis; the discussion of laws of mortality went far beyond that.

Perhaps they would have before long comparatively easy access to electronic computing machines. It might be a wise expenditure of energy to pause from computing by more laborious methods—and the vast amount of computation prevented many ideas from being fully worked out—and reconsider general principles. He had often wanted, although he did not seem to have had the opportunity, to go through all the literature and write down, not necessarily in mathematics, how much was really known about mortality. How much further could they go with reasonable certainty beyond saying 'In the long run we are all dead'? He thought that it was an over-simplification, again in the author's words, to imagine 'a mathematical formula which will express the way in which mortality changes age by age', if the mortality were human mortality. Such a formula might conceivably be found to express the course of mortality in given standard conditions for some form of non-human life and, if found, would assist in extending the knowledge of human mortality. If, however, the word 'law' was to be used in relation to human mortality, it was necessary to envisage, not a single formula or even a group of formulae leading to a simple progression age by age, but rather an organized system of knowledge, some of it expressed mathematically and some of it more conveniently expressed in other terms, indicating how a particular experience might best be examined. Matter was less complex than matter plus life, and that in turn was less complex than matter plus life plus intelligence. One could not expect a law of human mortality to be as simple in expression as a law of physics.

Mr William Phillips referred to paragraph 5, where the author said that some injustice was done to a great mathematician by calling de Moivre's hypothesis a 'law'. He concurred, and thought a great injustice had been done by putting de Moivre's name to it at all. In A.D. 230 the Roman Empire introduced, under the Falcidian law, what was later called estate duty. Life incomes being then a favourite form of bequest, it was necessary to get the capitalized value, and to do that the rate of interest was fixed at 4%, and Ulpian provided a mortality table of equal decrements with ω at 70. From the records it was not possible to be certain whether it was equal decrements of l_x or of N_x, but that did not matter much if the user were satisfied to take the annuity value as that of an annuity-certain. After the French lost their hold upon Italy in 1814 the Ulpian mortality table was again brought into use for estate-duty purposes, so that it seemed likely it had been in use until the French occupancy; thus when de Moivre wrote in 1725 he was probably doing no more than quoting the recognized system of his age.

The normal curve of errors, enunciated by de Moivre in 1733, attracted little attention, and twenty-two years later Simpson (1755) suggested that the curve of errors was an equilateral—though presumably he meant an isosceles—triangle. Now clearly, if mortality were to be represented by a triangle, one with marked negative skewness was required. Such a triangular curve of deaths was presented to the Institute eight years after its foundation. William Orchard had then died at the early age of thirty, and Peter Gray (1856, *J.I.A.* VI, 181) went through his papers and discovered that Orchard had suggested a triangle with the base extending from 0 to 96, truncating it at 20, however, with the apex at 80. Peter Gray thought so much of the suggestion that he calculated interest functions from it at various rates to five places of decimals. He then found that it corresponded, with remarkable closeness he said, to Davies's Equitable Table.

In paragraph 15 the author referred to Thiele's formula of 1871. On Monday, 31 January 1870, two papers were read to the Institute, the first by Prof. Oppermann of Copenhagen, the second by Woolhouse. Woolhouse's paper was printed in the *Journal*; the other was not, but a summary of it, and of the discussion upon it, appeared in the *Insurance Record*, 11 February 1870. Prof. Oppermann said he had sought for, and failed to find, a law which would apply throughout life, but he had devised a formula which fitted the first twenty years of life. No such formula had previously been devised. He gave numerical values to his parameters, but his formula could be stated in the form

$$\mu_x = a/\sqrt{x} + b + c\sqrt{x}.$$

Sprague, in the discussion, said that it appeared to him to be 'a very curious and valuable formula'.

Thiele was unable to use Oppermann's formula in its precise form because he could not make it join his own curve for middle life, and therefore he generalized the formula, with full acknowledgments to Oppermann. Thiele pointed out that

> the structure of the formula contains a definite intimation that with respect to the first years of life we shall make more progress by taking the square root of time, instead of time itself, as the independent variable of the mortality

and added that it was his conviction that Prof. Oppermann had gained for himself lasting credit by the formula. So far as the speaker knew not a word had ever been heard of it since. How wrong Thiele was!...Or was he?

Mr H. A. R. Barnett felt that someone who did not know a great deal about the actuarial profession, or possibly even a student, would get the impression on reading paragraph 71 that it had been the usual practice for the profession to consider the various causes of death, and that the author thought that that practice should be discontinued. In fact, although ideas in the past had been based on assumptions as to the results of certain causes of death, no such practice had ever been usual, and no thorough investigation of such causes, so far as he was aware, had ever been made, although Thiele hinted that he might have made one which could lead to a law of mortality with a philosophical explanation supported by facts.

With that consideration in mind, though in ignorance of Thiele's paper, he had, when the Registrar General's *Statistical Review of England and Wales* for 1949 was published, investigated the male rates of mortality from different causes in broad groups. He knew that a number of limitations had to be placed on the data, the main one being that the populations were only estimates and the last Census was then 18 years old; but he thought that such an investigation might indicate the shapes of the component parts of the rate of mortality.

As a starting point, he separated as two broad groups the two causes for which the mortality rates did not increase regularly throughout adult life, namely, malignant tumours (or, as the 1950 *Review* called them, malignant neoplasms) and tuberculosis. He kept accidental or violent deaths separate, but he found that they were so few as to be relatively unimportant, besides giving increasing rates throughout life. He found, on subsequently testing the possible subdivisions, that he obtained the best results by dealing separately with tuberculosis of the respiratory system combined with the malignant group affecting the respiratory system. The mortality rates from these two groups of respiratory diseases fell, over almost the whole of life but ignoring ages beyond 85, into the form of a section of the normal curve of error, with a maximum at about age 64 and an almost symmetrical fall around that age. He found that mortality rates over age 30 from all other causes fell approximately into the form of a Gompertz curve, although a better curve could be fitted if he excluded all malignant neoplasms and all respiratory diseases of any kind. He subsequently came to the conclusion that if most of the deaths from the non-respiratory neoplasm group around the middle ages were added to the respiratory group, and the remainder to the other causes group, the two resulting sets of rates fell quite naturally into the shapes of a normal curve of error and a Gompertz curve. A similar conclusion applied to each year since 1945.

He added that ever since he had started that investigation he had thought that there was some message carried in those annual Reviews, and particularly in the table of causes of death according to sex and age groups; he was now convinced that that message was signed 'Thiele'. He therefore thought, in opposition to the suggestion in paragraph 71, that there was much to be learned from investigating causes of death, and that the Continuous Mortality Investigation data should contain some information on that vital matter. He suggested that the deaths should be recorded in four groups, described as (i) accidental or violent, (ii) malignant neoplasms, (iii) tuberculosis, and (iv) all other causes. In groups (ii), (iii) and (iv) that information should be supplemented by a record of whether or not the site of the disease was the respiratory system. He did not think that the keeping of those data should be an arduous task.

In his view, the reason why one of the parts of the curve of mortality rates (as distinct from death frequencies) was of the shape of a normal curve of error, which Thiele also suspected, might be that there was an underlying frequency of persons who reached each year of life having become likely subjects for one or other of the diseases in question. Having already become such subjects, their chances of death within a year were approximately constant irrespective of age, and therefore the mortality rates from those diseases combined would be similar in shape to the frequency curve.

He was convinced that, by a suitable division of deaths, it would be easy to fit to the national male mortality rates above age 30 a curve of the form $B_1 c_1^x + B_2 c_2^{-x^2}$, that being another form of two terms of Thiele's formula, but it would be almost impossible to find trial values for c_1 and c_2 if the details of causes of death were not available. Similar details of deaths of assured lives should not only facilitate the fitting of such a curve with a philosophical background, but be invaluable for examining secular progressions and indicating the precise effects of selection, which in any form was probably merely the process of partial elimination of the likelihood of a number of causes of death. There was a strong case for such an improvement in the data, even though when it came to the point of graduating a standard table they would probably continue to graduate by a Perks curve or, perhaps, by Ogborn's method.

That led him to ask: 'What had the author's method achieved which could not have been achieved by a Perks formula?' The author had applied his method successfully to assured lives data but not successfully to national data. Beard had had a similar experience with the extensive application of the Perks family of curves. In other words, the author had succeeded where others had succeeded, he had failed where others had failed, and he had failed with many more parameters than the 4 or 5 of the Perks curve; so that what at first promised to be a great step forward had proved to be little more than a great step sideways.

The method must, as the author himself had said, take its place beside others. The author had not made out a case for discarding previous lines of thought; indeed, they should rather be developed.

Mr W. Perks suggested that there was some danger of confusion between at least four different processes covered by the idea of graduation. First, there was the construction of tools for practical actuarial purposes, standard tables or *ad hoc* tables for particular problems. Then there was the manipulation of the data of past experience with the view to extrapolation into the future, much of which seemed to him to be misguided, because it assumed, without justification, a steady continuity of change. To be worth while at all, that technique should provide useful properties for practical application. The third of the four processes was pure graduation or curve-fitting in the old theoretical sense, complete with probability tests. It was the easiest thing in the world to confuse that with the fourth process, the search for a satisfactory theory of mortality.

Like others before him, including the speaker himself, the author had not escaped that confusion, and the paper was really about curve fitting; at any rate, he could not himself agree that the process of fitting Pearson frequency curves, even if successful, would take anyone a fraction of a step nearer to a satisfactory theory of mortality, unless some more or less satisfactory 'explanation' could be provided as well. The mere reproduction of the essential mathematical features of the data did not constitute a theory or provide a law in any but the narrowest sense of the word 'law'. From that point of view, the paper brought out once again the point that modern adult mortality needed four or five parameters to provide a close mathematical representation, although in stable conditions three might be found sufficient.

While a formula suitable for the adult ages might be taken back satisfactorily to the infantile ages by including terms with two or three more parameters, he doubted whether that would mean much. The period of growth and the period of decay were so essentially different that a curve to cover the whole range must have the character of two or more curves joined together. To go back beyond birth to, say, minus 9 months involved false ideas in two respects. First, it ignored the fact that age was measured from birth and not from conception, and that the period of gestation varied by as much as 3 months. Secondly, parturition was a major shock to the child, and the discontinuity at birth was an essential feature of mortality data. There was no practical analogue of the mathematical concept of μ_0, even as an approximation.

Provided the formula adopted was capable of expressing the main features of adult mortality rates—i.e. provided the mathematical form gave positive values and, at any rate after age 30, was steadily increasing—the fit might be expected to depend broadly on the number of parameters introduced. His own formula, Beard's incomplete-gamma-function technique, Starke's way of using the logistic and other S-shaped curves, and Ogborn's ratios of parabolas, cubics, etc., all showed that three parameters were not enough, and that more than five would take one very little further than would five. All those forms provided reasonable fits if the data were amenable at all to curve-fitting. It did not matter much which formula was used, though he might, perhaps, be forgiven for thinking that his own was often a shade better than the others. Put simply, if Makeham's formula fitted the middle of the adult age range, an additional parameter was needed for the older ages and another for the younger, making five altogether. If there was a hump on the rates in the 20's, two more parameters might be needed. An example of that was given in the discussion on Vaidyanathan's paper on the Oriental experience.

If anybody asked him what practical purpose was served by curve-fitting he would find it difficult to say, because he firmly believed that hypothetical tables were the best kind of tools in practical actuarial work; but curve-fitting provided a first-class training in arithmetical computation and gave students an extremely valuable insight into the way in which figures behaved from the points of view both of mathematical law and of statistical fluctuations. He did not believe that actuaries would be fully trained unless they had the sense of numbers which that sort of work gave.

For a theory of mortality, as distinct from mere curve-fitting, they had not progressed much beyond the ideas of Gompertz. Mixed populations and changing environments were sufficient to account for the need to modify Makeham's formula to fit present-day data. With those disturbances, not much progress could be hoped for merely by studying the data of human mortality; instead, there was need to attack the problem from two other points of view: first, they needed to build up probability models by analogy, and secondly, they needed to study the data of simpler populations in conditions of as complete homogeneity, constitutional and environmental, as possible.

For the probability models they could start with the classical example of the shooting problem. Suppose that a large number of targets were being steadily shot at—at random—and when a target had been hit n times it was treated as a 'death'. That model gave a Type-III distribution, as had been mentioned before by Mr Beard. The link with his Centenary Paper, where he used the Type-III form, was obvious, but there was a theoretical difficulty in that link, because the origin of his curves was at the wrong end—the skewness was in the wrong direction.

The shooting problem also provided a link with the Makeham formula if the shooting were allowed to increase in time in geometrical progression, but there was a theoretical difficulty there also because the frequency curve in the shooting problem was $(\mu l)_x$ whereas l_x was the frequency curve in the case of Makeham's formula. With the introduction of the idea of continuous shooting and infinitesimal contributions to the fatal number of hits, and also the idea of recovery from the hits, he suggested that that model might be found to be associated with Rich's theory, which remained so tantalizingly unpractical for such a promising idea.

For models of that kind they must make up their minds whether they regarded $(\mu l)_x$ or l_x as the frequency curve, whether they were thinking in terms of a distribution of deaths over the ages and stretched out in time or of a census of the age distribution of a stationary population at a point of time. If the death curve was a Pearson distribution, the l_x curve would not usually be a Pearson distribution. He fancied that the author had not avoided that confusion. However, there was some support for his idea of a ratio of polynomials as the form for μ_x, if they thought of μ_x $(=(\mu l)_x/l_x)$ as the ratio of the initial ordinate to the tail area of a frequency curve. In the *Journal of the Royal Statistical Society*, Series B (XIV, 158), Dr K. D. Tocher mentioned that

> nearly all the usual and difficult transcendental functions required in analysis could be obtained from Gauss's hypergeometric continued fraction.

Dr Tocher gave for an example the tail area of the normal curve as the product of the initial ordinate and a continued fraction in terms of the variable x. The continued fraction could be approximated by a ratio of polynomials, and perhaps more information on the idea might give some support to the author. It might also be of interest to mention that the ratio of the initial ordinate to the tail area of a frequency curve could often be approximated by the form

$$(A+Bc^x)/(1+Dc^x).$$

For the life tables of simpler populations than human beings a great deal of data could be obtained from other sciences. The simplest model of all was a lump of radioactive material. The emission of α-particles was equivalent to death and the force of emission was a constant. There were other classical examples in the literature, such as the life-distributions of electric bulbs, which could be fitted by Pearson Type-III curves. There were life tables for various populations of fruit fly which were not unlike Makeham curves. There must be a great deal of other data in the biological field, including microscopic populations, and it gave an opportunity for some young actuary to extract all that information and study it for its bearing on the general mortality problem. It seemed to the speaker that those probability models and the data of simple populations were the only ways in which an advance would be made in the development of a satisfactory theory of the life table. Without a sound theory for simple populations in homogeneous environments they could not hope to achieve anything more than mere curve-fitting in the more complicated field of human mortality.

Mr. H. W. Haycocks assumed that the author would prefer speakers to discuss the philosophy in his paper rather than the arithmetic. The arithmetic was plain to everybody, but he had found the author's speculations very mysterious and difficult to understand. His own remarks might seem so obvious to most people as to be regarded as not worth making, but after reading some of the passages in the paper and listening to Mr Phillips, he felt that they ought to be made.

It was essential for the teacher of a science to hold a scientific philosophy; otherwise it was quite impossible to put practice and theory in their proper places. The author used the word 'philosophy', but it would seem that he either misused it or used it in a very trivial sense to express his own attitude to Pearson's curves. A philosophy of actuarial science would make interesting reading, and it was fairly obvious from the remarks they had just heard that there was one member of the Institute present who would enjoy himself in writing it. Such a philosophy would analyse the concepts of force of mortality, life table, service table, salary scale, valuation and so on and, what was much more important, it would analyse the uses to which those concepts were put. It was interesting that attempts had been made in recent years to put even such difficult concepts as the principles of investment and of valuation in their proper places. Those concepts were parts of the various theoretical tools and models used by the actuary to make certain kinds of inferences. Some of the models, as for instance the mortality table, had the appearance of a law in the sense that one variable was related to another, and it was not necessary that such a law should be expressed in a mathematical formula.

His main point was that such a law could be understood only in the context of its use. That had been brought out by various speakers. They all found fault with the author because they wanted to put their various laws to different uses. Mr Perks had pointed out that such use might be aesthetic, and it was interesting to notice that the author began by quoting a poet as expressing his basic point of view.

The conclusions about the world formulated by scientists were not derived from inquiring into the nature of the laws which they used, but were derived in accordance with such laws; in other words, the conclusions were inferred by applications of the laws. Words such as 'true', 'false' and 'probable' were applicable not to the laws themselves but to the statements which scientists made when applying them. The actuary did not require a law of high predictive value; he was only required to make a reasonable assumption about mortality. As a result of experience he found rational ways of doing his business. The predictive content of the mortality law lay in the word 'reasonable'. The actuary should have good grounds for thinking it reasonable. It should be added that the secular trend in mortality was perhaps a more important uniformity than the age trend.

Once a mortality table was accepted, together with a rate of interest, it did have the force of a law in the sense that all inferences were exact. The net premium and policy values must be what they were because of the mortality and interest used, and in that sense there was a law for the actuary; i.e. the scope of the law was sufficiently wide for the purposes of the actuary; he could make a number of inferences from it which served his purposes. A biologist, however, would find the scope of a mortality law of that kind extremely limited, and that must be the case, because age was the only variable taken into account. It was unusual for a paper on mortality in non-actuarial literature to treat age as the important and only factor.

From the actuary's point of view, an increase in scope could be obtained by choosing a graduation formula which would aid subsequent calculations; that was the main reason for choosing a Makeham curve or the formula suggested by Mr Beard. It was clearly essential in the philosophy which he had outlined that the model should not be confused with the observations; inferences from models were exact, but the comparisons with observations were not. It was always necessary to have in mind the amount of permissible error. The author seemed to tend to confuse the two levels of discourse: for example he seemed to think that when talking about a frequency distribution one must always be thinking about an empirical distribution.

The modern tendency was to formulate a statistical hypothesis. He was not sure whether the author had meant to do this or not; no mention was made of a statistical hypothesis, but the author did apply the χ^2 test and discussed the effects of duplicates, which implied a statistical model. Assuming that it was decided to formulate such an hypothesis, it would seem best to try to represent the death curve, preferably for a generation, by some formula. He did not understand the statement in paragraph 24 that

> The function d_x is (at least usually) a mathematical function, not a statistical distribution.

The basic probabilities were d_x/l_0, which implied the setting up of an hypothesis as to the distribution of d_x, or d_x/l_0. In the first case there was a hypothetical frequency distribution and in the second a hypothetical probability distribution. He could only assume that the author was mixing the model with the actual observations.

In practice it was seldom possible to observe a generation for any length of time, and it was necessary to estimate probabilities from a mixture of generations;

but it should be noted that in departing from the generation as the basic data they were making a large assumption by combining the resulting probabilities. It could be justified only because the actuary did not require a law of high predictive value.

His own philosophy led him to the conclusion that the method of graduation adopted was not very important, and he would be inclined to say that any formula would do, without even adding the proviso that it should fit. Like Mr Perks, he thought that graduation had intrinsic value for the mathematician, but hardly for the actuary in his professional work, in which the simplest methods would serve. But he did not want to be misunderstood; actuaries should by all means study the subject of curve-fitting. They might find a simple formula which would benefit subsequent arithmetical work; or in the long run it might be that they would want to adopt a more complicated model than they had at present.

Sir William Elderton said he had a faint recollection that somewhere about 1780 William Morgan did a good deal of work on mortality and compared expected and actual deaths, and from some of his work on that subject Davies made his table. He did not see any relation to l_x, but it came near to the conception of q_x. He believed that references had been made in recent years to Oppermann (perhaps by Steffensen) showing that the formula mentioned by Mr Phillips gave a fairly good approximation.

He thought it possible that some of their difficulties in approaching the use of a mathematical formula to express mortality lay in finding the ideal statistics to which it should be applied. In a community where the population was approximately constant and the rates of mortality did not vary much over a long period it might be possible to see a mathematical law which was invisible in the statistics which actuaries were bound to use; for those statistics were cross-sections with regard to time and to several other variables.

He would like to extend to the author his most sincere sympathy; he had made a great many attempts at many of those things and the result had always been just a little disappointing. He gathered from what had been said by Mr Perks and other speakers that they had found the same thing—'The little more and how much it is; the little less and what worlds away!' It might be a fruitless search, but to find a way of expressing something in a mathematical form was always interesting and, whatever philosophy they might make up, he was sure that actuaries and statisticians would go on doing it and would enjoy even their disappointments.

Perhaps it was all a waste of time. He thought it was quite true that, if it were merely to graduate a table in order to use it for practical purposes, any method would serve; but that would not lead anywhere or satisfy anyone, and so they would go on trying. He felt that if the author had had a little more luck things might have so come out that everybody would have felt that it was a wonderful show. They would all feel, however, that it was a good show anyway.

Mr R. E. Beard claimed to speak as one of those who had made a number of attempts to find satisfactory mathematical expression for mortality statistics, most of which had failed; indeed, he believed that all the attempts he had so far made had really failed by the standards to which he considered they should be subject. The ground which the author had covered was territory which anyone who was experimenting with or endeavouring to find alternative mathematical expressions for mortality statistics would have to cover, but there were grounds for believing that the approach would be unprofitable in terms of the title of the paper. To his mind the experiments fell, as Mr Perks had said, in the sphere of curve-fitting, and he did not think that they were likely to afford much further insight into the nature of the problem of finding a law of mortality. If it were assumed that mortality was subject to the operation of physical laws there was some justification for searching for the laws which operated, but it was first necessary to examine the problem more closely.

It would seem reasonable to assume that a life and death process was determined by the interplay of a number of factors, possibly varying with age, which led to a probability distribution for age at death. If proper observations were available it might be expected that a curve of deaths could be fitted by some form of probability distribution or frequency curve; but it seemed to him that there was no special reason why such a curve should be expressed by one of the Pearson group, the form being conditioned by the life and death process and probably leading to a more complicated distribution than a Pearson curve.

If, however, that approach were examined closely it would be found that no observations existed which could be used to test directly any particular life and death process. It was unlikely that a curve of deaths calculated from observations restricted to a few calendar years would be described by the natural law, and certainly a generation curve of deaths would be affected by secular variations in mortality. It followed that even if a mathematical expression were found for the particular curve of deaths it might still not be possible to work back to a life and death process. For those reasons it seemed to him that any approach restricted to trying to discover a law of mortality by the empirical method of finding a mathematical expression to represent a given mortality experience was unlikely to be successful.

Leaving those generalities and coming to the author's experiments, it seemed to him that the effect of endeavouring to find expressions to represent assured-lives data or population data was essentially an exercise in curve-fitting to find how much of the finer structure of mortality could be accounted for by the particular mathematical form adopted for μ_x. Since the method had probably no significance from the point of view of finding a mathematical law, the only consideration was how completely a given form would reproduce the statistics. Such an exercise was interesting, but did not appear to lead to any other useful end.

In paragraph 30 the author stated that Pearson curves 'must have an application to the life table'. He could not agree with the use of the word 'must', and would at best accept the word 'may'. The Pearson system could be regarded as a mathematical form fulfilling certain conditions, and other curves could be devised which might be more appropriate to mortality statistics.

In paragraph 33 there was a question of the correlation existing between values of l'_x in the table. It had already been pointed out that it must be kept clearly in mind whether a model or observations were being used. In the former case there would, of course, be no correlation.

He could not follow the argument in paragraph 34 and doubted whether it was valid.

He would like to refer to the extension of the Pearson system which the author had used. In equation (3) of the paper the introduction of a term in x^2 in the numerator was equivalent to adding a constant to the right-hand side of that equation. Integrating for y would show that the resulting form was equivalent to the Pearson form multiplied by an exponential. In his own experiments to fit the curve of deaths with Pearson-type frequency curves he had found that an exponential term seemed to be called for, and he had included one, but the results were still not entirely satisfactory. It was interesting to arrive at the same form by different approaches; the function resulting was the confluent hypergeometric function, which had come into the limelight in recent years in certain problems in theoretical statistics, and from that point of view it was not a new curve. It was, however, interesting to notice that in certain circumstances it had the appropriate shape for a curve of deaths from 0 to ω, but the attempt to fit it was never successful over the whole range. That was, perhaps, not surprising, because even if it were the physical law describing the mortality process it would not necessarily fit any observed data without the addition of more parameters.

Mr Barnett had mentioned his investigations into mortality by cause of death, and had suggested that respiratory and neoplasm deaths should be aggregated. That seemed an interesting link with various suggestions that had been made during the past twenty years that there was a reciprocal relationship between deaths from cancer and tuberculosis.

Mr J. B. H. Pegler, in closing the discussion, remarked that the paper might be roughly divided into two parts. The first was a general discussion of the author's objects, including an historical survey of the search for a law of mortality; the second was an account of the author's own researches and results. He was certainly not competent to pass judgment on the technical aspects, and he therefore proposed to confine his remarks to the earlier section of the paper. As the historical survey indicated, there had been many attempts to fit curves to mortality data. The author was not the first to try to do more than merely find the best fit by a mechanical process of trial and error, and the speaker agreed with Sir William Elderton in thinking that he would not be the last.

The author's professed aim had been to find the function expressive of the law of human mortality, and in giving such a title to his paper he had run into trouble from the very start. The phrase 'law of mortality' might be interpreted in a number of different ways, and it might be that there was some sense in which the pattern of the experience or something inherent in the data justified the reference to a law of mortality, though a number of speakers seemed to have some doubts on that score. The discussion of that topic led into very deep water in which he did not propose to try to swim. There were some who contended that it was not justifiable to speak of a law of mortality in any sense, but he thought that that was going too far. Perhaps 'law' was not the most satisfactory word to use, but he would agree with the author in condemning the view that any formula would do if it fitted.

Nevertheless, he felt that in discussing graduation in the context of a law of mortality it was essential to make clear the objects aimed at. In his reference to the processes covered by the idea of graduation Mr Perks had provided a criterion by which to judge the broad aims and methods of the paper. The author had traced with scholarly care the changes over the last 250 years in the 'viewpoint' from which the life table had been regarded. It was clear that he attached a great deal of importance to the choice of the particular life table function—i.e. l_x, d_x, or μ_x—as the one to express his conception of the law of mortality. Moreover, he evidently regarded as of importance for that purpose the question whether a particular function was a mathematical function or a statistical distribution. The speaker was afraid that the author had not convinced him—and, if he had understood Mr Beard correctly, he had much the same view—that there was any fundamental distinction there. The choice of the function to be graduated and fitted depended on the purpose of the graduation and not on any quality inherent in the data or in the form in which they were originally observed.

The title of the paper referred to 'the function expressive of the law of human mortality'. He considered that the most that could be hoped for was to find the best function; to find *the* function—by implication the only one—would be impossible. In looking for the best function, however, it was essential to select the best available data to which to fit it. Mortality data derived in the manner used for the production of the A1924–29 Table and the English Life Tables contained lives which had entered at very widely separated dates and had been exposed to a very different experience over the years. Sir William Elderton had suggested that something of that sort might obscure the law of mortality which was being sought. The data were far from homogeneous, and any formula which was used must allow for that fact. Perhaps the ideal might be a basic formula applicable to homogeneous mortality, with some additional parameters included which could be varied to help to fit the function to the heterogeneous vicissitudes which had undoubtedly left their mark on the experience; or perhaps, by a further generalization on the lines suggested by Mr Perks, a basic formula could be devised applicable to all simple, homogeneous populations exposed to the risk of progressive breakdown and eventual death, and that could be modified, first of all to fit it to a particular class, such as the human population, and then still further to take into account heterogeneity in the data. In his own view it was very doubtful whether a polynomial or the ratio of two polynomials would yield any helpful results in the search.

A considerable part of the paper was devoted to a discussion of the Pearson frequency curves, and the author had stated twice, and emphatically, that those curves must have some application to mortality data. He did not give any very solid grounds for that conviction, and after stating it he found it necessary to make some fairly substantial modifications to the Pearson formula before doing his curve-fitting. The fact that the modified form, when it was applied, gave (in the author's own words) results which 'were disappointing' was eloquent testimony to the author's intellectual honesty, which had prevented him from going back and adjusting his convictions to bring them more into line with his results.

To a layman like himself it would appear from the paper that the author had produced further evidence, none the less valuable for being, so to speak, negative, that the approach to fitting mortality data *via* Pearson frequency curves was not a very hopeful one. Some time previously the author had referred to a minor fallacy of investment principle as having been 'Peglerized'. He might now return the compliment and suggest that that avenue of mortality curve-fitting had been 'Ogbornated'.

The President (Mr W. F. Gardner), in proposing a vote of thanks to the author, said he was sure that members would wish to express their appreciation of the paper both to the author and, as he was sure the author would wish, to his team of helpers also. The early part of the paper must have represented considerable and diligent historical research over a long period, and it was valuable to the Institute to have this record in so compact a form. It might be a little too compact for some of the enthusiasts, but the President believed that in its present state it would be appreciated by the generality of members, amongst whom he numbered himself.

Those who professed a severely practical outlook—and many of them were not present that evening—might wonder whether all that research and all that calculation were really worth while, and that was a point on which one or two speakers had already touched. He thought very definitely that the work was worth while, for two reasons. First, the so-called practical outlook often depended upon an unacknowledged mass of backroom work; secondly, as in the games which they all played, both at school and, he hoped, subsequently, on the field of sport, there was the attraction and the irresistible fascination of striving towards a perfection which they knew in their hearts they could never hope to attain. The value was in the striving and not in the attaining.

Reference had been made to the electronic computer and, although he thought that the parallel was not true, it did bring to his mind the thought that it was one thing to fly over Mount Everest in an aeroplane, as had been done, and quite another to climb it. He found it very appropriate and natural that among the membership of the Institute there were those who were anxious and willing to pursue what the author had called a will-o'-the-wisp. He hoped that it would always be so.

One added virtue of the paper was that it had given members the pleasure once more of hearing in their discussions the voice of Sir William Elderton.

Mr M. E. Ogborn, in reply, thanked the President for his kind remarks. It was refreshing, he said, to come to an oasis after a severe bout of criticism.

He did not know many of the things which Mr Phillips had told them, and they would be a valuable addition to the history of the matter. The investigation of a mortality experience by comparing the expected and actual deaths was introduced by Price in 1775 and used by Thiele a little later on. Whether Price was the first to use that comparison he did not know, but he believed that he was and that, as Sir William Elderton had said, came near to the conception of q_x.

The attempts to represent d_x by a mathematical function really derived, he thought, from the work of Thomas Young, who was referred to in paragraph 7 of the paper. Young was a well known man outside the actuarial profession, and at the South Bank Exhibition his portrait had been shown together with that of Newton. He was primarily a medical man, but became medical adviser to one of the early companies and interested himself in mortality figures, giving a rather complicated law of the type in question in 1826. A life of Young was about to be published, and he hoped that, with the permission of the Editors, a review of that book would appear in due course in the *Journal*.

He thought that a good deal of the criticism that evening was due to a misunderstanding of the paper. It had arisen partly from the fact that he had a fairly clear conception of the philosophical background of mortality in his mind, which he might have enlarged on had the arithmetic been more successful; but he had restricted the paper because he had felt that otherwise it would be putting too large a superstructure upon a few arithmetical results. He would emphasize, however, that the philosophical approach had been arrived at first and the arithmetic done afterwards, and that was the justification for putting it in that form. In July 1950, he had come to the conclusion—which eventually proved to be wrong, as so many *a priori* conclusions did—that the force of mortality was simply the reciprocal of a polynomial, and that it was only necessary to take enough parameters to reproduce any expression; but in fact it did not work out in that way.

He had in mind by 'law' a law perhaps in the physical sense, trying to represent the rate of mortality by a mathematical expression which would be valid in different experiences, although, of course, with different parameters. The theme of the paper was to suggest that there was no such law, or that the subject should not be approached in that way, and then to see what could be said about mortality in the absence of a law of that kind.

The piece of poetry in paragraph 3 was not simply decoration, but expressed what to his mind was the problem, namely the endeavour to see what the data could reveal. In looking at the problem in that way it seemed to him that the data could be made to reveal their secrets if a relationship were assumed between the exposed to risk and the deaths, and their moments, so linking up with the conception of graduation by a frequency curve where, as it were, a distribution was made to reveal its secrets in terms of the statistics and their moments. The core of the argument was on pages 7 and 8. Considering the exposed to risk at any given age, it was wrong to think of there being one rate of mortality applicable to each individual within the exposed to risk. They were in various states of health; the exposed to risk concealed a distribution according to the state of health. If the exposed to risk were followed through age by age it would be found that the distribution became heaped up towards the end where the impaired lives were. He had once startled one of his brother actuaries at a dinner by asking him why it was that the rate of mortality increased so regularly with age. The reason seemed to be clear; the rate of mortality was really an average, and the steady increase of the rate of mortality with age arose simply from the heaping up of the exposed to risk towards the impaired end. In equation (6) in paragraph 36—the simple relationship of exposed to risk and deaths, and their moments—the parameters could be regarded as expressing the way in which that transfer of lives from healthy to impaired occurred, so that the distribution was heaped up towards the end.

One point which had interested him was that, as he had said earlier, his preconceived ideas about the shape which ought to be obtained were all wrong. They invariably did prove to be wrong. In paragraph 61 he gave some of the shots which he had tried. The arithmetical work really consisted of trying one thing after another to see what did fit, and it did seem that the best results were obtained by taking the quadratic in the numerator and letting the denominator vary, so that there was perhaps more to be said for it than some of the speakers would admit, and the formula as given was one which could represent mortality with some claim to be based on a logical foundation, although it was not a law of mortality in the ordinary sense. It could not be called a law, for one reason because it did link the mathematical expression with the form of the exposed to risk. If the form of the exposed to risk were altered, would the ultimate expression be altered? That was an interesting question, and it might well be that it would.

The use of χ^2 had been criticized. He did that for lack of better inspiration. He recognized that the use of χ^2 did not logically follow from the earlier part of the work, but he did not feel himself equal to developing any new tests, and he thought that the old one would appeal to members.

He thought that there would be an easier application for the method to mortality other than human mortality. The restriction to human mortality was merely to restrict the amount of arithmetic. He had wondered whether to try to get the help of other statistics, but he had not felt equal to the task of doing the arithmetic which would have been involved in such an extension of the work.

In following the 'will-o'-the-wisp' he had been reminded of a metaphor used by Jeans which appealed to him. Jeans said that the pursuit of truth was rather like following a river winding over a plain. Following the course in one direction, the explorer felt sure that the ocean of reality lay that way, but then came a bend in the river and the ocean of reality seemed to be in the opposite direction. After many disappointments, the explorer gave up thinking at every turn that he was at last in the presence of the 'murmers and scents of the infinite sea'!

The following written contributions have since been received:

Mr Barnett writes:

I am grateful to Mr Penman for having pointed out to me, after the discussion, that in one of the early investigations of assured lives' data made by the Institute, the details on the cards included 'cause of death'. The causes were not particularly well recorded, and the information was not used. This fact does not invalidate my remarks that no thorough investigation has been made according to cause of death.

Mr Phillips has written:

The following are further references for the historical summary of the efforts prior to 1931 to reduce the mortality table to a mathematical curve.

Peter Gray, in his paper upon Orchard's formula (1856, *J.I.A.* VI, 181), said that

the desirableness of being able to express the curve of mortality by means of an equation...is sufficiently testified by the fact of many eminent mathematicians having given their attention to the subject

and he gave the references for formulae for l_x, by Lambert, Duvillard, and Thomas Young, and set out Babbage's formula, which he copied from a letter written by Babbage to Baily, the generalized form of which is:

$$l_x = a - bx + cx(x-1)/2.$$

The so-called de Moivre and Babbage formulae are referred to again in the *Journal* (1873, XVIII, 59), where Lambert's formula is given:

$$l_x = a[(96-x)/x]^2 - b[e^{-x/c} - e^{-x/d}]$$

and Littrow's:

$$l_x = a - bx + cx^2 - dx^3 + \dots$$

and Moser's very curious formula, which may be best understood in the form:

$$l_x = l_0[1 - x^{\frac{1}{4}}(a - bx^2 + cx^4 - \dots)].$$

The author's paragraph 14 and its formula (1) seem to suggest that Makeham used the symbol μ_x for the force of mortality in 1867, or even that he invented it. On the contrary he used the symbol F_x, whereas Woolhouse, if not the first and only begetter of μ, was certainly using it as early as February 1863 (*J.I.A.* XI, 321). As the author points out, Gompertz was in 1825 thinking in terms of l_x, not of a rate of mortality or a rate of survival; but from the very outset in 1860 Makeham was stating the law of mortality thus:

the probabilities of living, increased or diminished in a certain constant ratio, form a series whose logarithms are in geometrical progression.

The author has given us two Makeham *Journal* references, namely VIII (1860) and XIII (1867). There are two other references between these, *J.I.A.* IX, 361, and XI, 236; and of course many more since, notably *J.I.A.* XVII, 305 with important corrections at 445, while the student should not miss Woolhouse's neat summary of Gompertz and Makeham in *J.I.A.* XXVIII, 481.

I do not think any historical summary in which Makeham rightly figures so largely should fail to observe, first, that 1860, when Makeham put forward his hypothesis, was before he had passed any of the examinations of the Institute, and second, that from that day until his death he never wrote or spoke of his 'laws' otherwise than as Modifications of Gompertz. Indeed the last paper he wrote for the Institute, twenty-nine years after his first, was entitled *Further improvements of Gompertz's law* (1889, *J.I.A.* XXVIII, 152).

I think, also, we should have a further and better reference to Karl Pearson's work, for which the author very naughtily gives only a 1948 reference. *The Chances of Death and other Studies* was published in January 1896; but it appears that a lecture delivered to the Leeds Philosophical and Literary Society in January 1895 was the basis of the paper in which Pearson commenced his d_x curve nine months before birth, much to the annoyance of Mr Perks.

Mr Ogborn writes:

The paper did not attempt a complete history of the subject but with the assistance of Mr Phillips I am adding some further references and remarks.

De Moivre would not, I think, have claimed originality for his hypothesis; it was simply a convenient numerical approximation similar to, though simpler than, others that had been suggested, e.g. by Johan de Witt in the seventeenth century. De Moivre found that the simplest hypothesis, that d_x was constant, was sufficient over the range of ages that usually concerned him.

The transition from the attempt to represent d_x to the attempt to represent q_x or μ_x by a mathematical function took place over a period of years which cannot, I think, be dated with precision. The beginnings can be seen in the 1820's with the work of Gompertz and Finlaison. A late example of the earlier concept is the formula propounded by William Orchard and published by Peter Gray in 1856 (*J.I.A.* VI, 181). The formula assumes that d_x is triangular in shape and yields the following values:

Function	Ages between 20 and 80	Ages over 80
d_x	x	$5(96-x)$
l_x	$3840-x(x-1)/2$	$2{\cdot}5(96-x)(97-x)$
μ_x	$\dfrac{2x-1}{(88{\cdot}1-x)(87{\cdot}1+x)}$	$\dfrac{5(96{\cdot}5-x)}{(96-x)(97-x)}$

The form of μ_x shows that these expressions are both segments of Pearson curves for l_x. The first formula for l_x is a particular example of the general form given by Babbage, which was, in fact, a Pearson curve though the formula did not necessarily use the whole range of the curve.

My knowledge is insufficient to do justice to continental work. The formulae of Oppermann (1870) and of Wittstein (1883), referred to by Mr Phillips, appear to have been derived from a consideration of the slope of the curve of the rate of mortality. They are mathematical expressions for, rather than mathematical interpretations of, the rate of mortality. Oppermann's use of the square root of the age to represent mortality at the infantile ages may be compared with the work of I. A. Christensen and Bj. Drachmann (*Grundlag for Livrenter*, 1928), They applied the Gompertz formula to annuitant mortality by transforming the variable to the square-root of the age, and including a Makeham constant, thus:

$$\mu_x = a + bc^{\sqrt{x}}/\sqrt{x}.$$

In the early years of life when $e^{x^{-\frac{1}{2}}} < c$ the use of the square root of the age leads to a decreasing rate of mortality by the formula; at the oldest ages the use of the square root of the age retards the rate of increase in mortality by the formula.

Oppermann's formula is a special case of yet another group of formulae which can be derived from the concept of the 'continuous relationship' by transforming the variable to the square-root of the age. If $z = x^{\frac{1}{2}}$, we have

$$dz = \tfrac{1}{2}x^{-\frac{1}{2}}\,dx \quad \text{and} \quad f(z)\,dz = \tfrac{1}{2}x^{-\frac{1}{2}}f(x^{\frac{1}{2}})\,dx.$$

The general form of the force of mortality becomes

$$\mu_x = \frac{a_0 + a_1 x^{\frac{1}{2}} + a_2 x + \dots}{b_0 x^{\frac{1}{2}} + b_1 x + b_2 x^{\frac{3}{2}} + b_3 x^2 + \dots}.$$

It may well be that advance can be made along these lines because the formula does not require such high moments for its solution by the method of paragraph 36.

Yet another approach to the problem is the *Généralisation de la loi de Makeham* by M. A. Quiquet (1889), a recent discussion of which by P. J. Richard is in No. 200 of the French *Bulletin Trimestriel*. Quiquet solved the general problem of the form taken by the rate of mortality on the condition that a group of N lives of different ages may be replaced by a smaller group of n lives without alteration to the probabilities of survival. There are alternative solutions leading either to a polynomial or to a multiple Gompertz curve with a Makeham constant.

While appreciating the quality of the work which has been expended on the representation of the rate of mortality by a mathematical formula, it is permissible to hold the view that the concept of such a formula is artificial, that the rate of mortality has no 'natural properties', and that the concept of a statistical relationship is a sounder and more logical approach to the problem. It goes rather deeper than the mere removal of random variations suggested by Mr Barley, and does not, I think, involve the confusion referred to by Mr Perks and other speakers.

Those critics who made so much play with the aside in paragraph 24 would, I feel, have understood better what I meant had they had more experience of curve-fitting and paid more attention to paragraph 33.

I agree with Mr Beard that the argument of paragraph 34 is weak—perhaps my heart over-ruled my head.

The fact that the mortality experience may be heterogeneous, possibly a cross-section of different generations, does not seem to me to be material. The life-table rests on the assumption of a continuous mathematical function μ_x. Formally, as is suggested in paragraph 38, the sufficient condition for the existence of the continuous mathematical function μ_x is that the distributions of exposed to risk and deaths shall be mathematically smooth frequency curves. Other considerations affect the use, not the formal existence, of the function μ_x. Heterogeneity in the data may make the mathematical function more complicated, but not necessarily so—it might conceivably simplify the function.

Probability models do not seem to me to throw any light on the nature of the mortality function though they have their place, of course, in the theory of graduation tests. To my mind, the life table is simply a convenient process akin to commutation columns, for combining the rates of mortality for practical purposes. The problem is to find a suitable link between the statistics and the finished life table. Not much attention seems to me to have been paid to this problem—I personally find the link in the postulate of a continuous relationship between the exposed to risk and deaths.

For any given age-group the exposed to risk consists of lives in various states of health; if we had sufficient knowledge of the individual lives we could form the frequency distribution according to the state of health, and since the chances of life are related to the states of health the same or a similar frequency distribution would show the range of mortality variations within the age-group. Thus the observed rate of mortality is an average of a wide distribution of chances of death.

With advancing age, after an initial period, the health distribution shifts more and more to the end which represents the impaired lives. This shift in the health distribution produces the increase in the average mortality over the group as age advances. So it may be said that the increase in the average mortality with advancing age is merely the outward effect of a more fundamental process, the impairment of health. The various parameters of the suggested formulae can be regarded as the expression of the trend of this more fundamental process.

The explanation I have given is in no sense new or original but it is more satisfying to me than can be found by considering the rate of mortality as a probability. As Mr Perks said, on those lines 'they had not progressed much beyond the ideas of Gompertz'; but, to me, Gompertz's words merely clothe our ignorance in verbal majesty.