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NEW DEVELOPMENTS IN INTERPOLATION FORMULAE

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ANY symmetrical interpolation formula may be written in matrix form, e.g. the six-term interpolation formula for subdivision of an interval into five parts may be expressed as

$$\begin{bmatrix} y_0 \\ y_{\frac{1}{5}} \\ y_{\frac{2}{5}} \\ y_{\frac{3}{5}} \\ y_{\frac{4}{5}} \\ y_1 \end{bmatrix} = \begin{bmatrix} c_{10} & c_5 & c_0 & c_5 & c_{10} & 0 \\ c_{11} & c_6 & c_1 & c_4 & c_9 & c_{14} \\ c_{12} & c_7 & c_2 & c_3 & c_8 & c_{13} \\ c_{13} & c_8 & c_3 & c_2 & c_7 & c_{12} \\ c_{14} & c_9 & c_4 & c_1 & c_6 & c_{11} \\ 0 & c_{10} & c_5 & c_0 & c_5 & c_{10} \end{bmatrix} \begin{bmatrix} u_{-2} \\ u_{-1} \\ u_0 \\ u_1 \\ u_2 \\ u_3 \end{bmatrix},$$

where u_t is the function to be interpolated, y_t the function giving interpolated values, and c_i the coefficients. We have

$$c_0+2c_5+2c_{10}=c_{11}+c_6+c_1+c_4+c_9+c_{14}=c_{12}+c_7+c_2+c_3+c_8+c_{13}$$

(since they each equal unity), and this is the condition that

$$c_{14}, c_{13}, \ldots, c_1, c_0, c_1, \ldots, c_{13}, c_{14}$$

are the coefficients of a summation formula expressed in linear compound form which contains the operator [5] (Mervyn Davis, *T.A.S.A.* XIX, 303). (It should be noted, however, that $c_0+2c_1+2c_2+\ldots+2c_{14}=5$, not 1 as with a normal summation formula.) Extracting the operator [5] this inverse process of summation may be repeated as many times as the above conditions are fulfilled.

Vaughan (*J.I.A.* LXXII, 482) drew attention to the very close connexion between a summation formula and an interpolation formula. He proved that the degrees of reproduction (i.e. the degree of the polynomial which is unchanged by application of the formula) of a summation formula based on the operation $[n]^r$ and of its associated interpolation formula were the same as far as degree $r-1$. In the course of his paper he showed that Woolhouse's graduation formula—the first to be expressed in summation form (*J.I.A.* XXIII, 352)—was identical with the ordinary three-term interpolation formula for subdividing an interval into five parts. Schärtlin's graduation formula

$$[3]^3\, 3^{-3}(1-\delta^2)$$

is likewise identical with the ordinary second-difference formula for subdivision into three parts.

In view of the possibility of expressing every symmetrical interpolation formula symbolically by a summation formula (Vaughan, *T.S.A.* I, 361), it is of interest to investigate the interpolation formulae which are derived from summation formulae of the type

$$k[n]^r\,[xu_0+y(u_1+u_{-1})+z(u_2+u_{-2})+\ldots].$$

Let $k[n]^r$ be described as the operator and

$$[xu_0+y(u_1+u_{-1})+z(u_2+u_{-2})+\ldots]$$

or, for brevity, $[x, y, z, \ldots]$ as the nucleus.

Consider first the operator $[5]^r$, and in order to simplify to a certain extent the numerical application take $k=10^{-r+1}$. The operator $[5]^6\,10^{-5}$ combined with a five-term nucleus produces only one summation formula correct to the fifth degree, and all others are correct either to the third degree or the first degree. Hence, by Vaughan's result, there is one derived interpolation formula with a degree of reproduction of the fifth degree (which must be the ordinary six-term interpolation formula) and all others have a maximum degree of reproduction of three. The operator $[5]^5\,10^{-4}$ combined with a nine-term nucleus can produce many summation formulae correct to the fifth degree; by Vaughan's result the associated interpolation formulae will have a degree of reproduction of four (except again the ordinary six-term interpolation formula with its degree of reproduction of five). This operator $[5]^5\,10^{-4}$ applied to the nine-term nucleus $[x, y, z, t, u]$ will therefore be considered in more detail.

The first step is to examine the conditions to be satisfied in order to obtain 'osculatory' and 'non-osculatory' interpolation formulae. It can be shown, by quite straightforward but somewhat laborious algebra, that if $y_{0,1}(p)$ is the interpolation curve derived from $[5]^5\,10^{-4}\,[x, y, z, t, u]$ applied to the interval $(0, 1)$, and $y_{-1,0}(p)$ is the interpolation curve derived from the same formula applied to the interval $(-1, 0)$, then $y_{0,1}(p)-y_{-1,0}(p)$ takes the form

$$\phi(p)\,\Delta^5 u_{-2}+\phi(-p)\,\Delta^5 u_{-3},$$

where $\phi(p)$ is a fifth-degree polynomial

$$a+bp+cp^2+dp^3+ep^4+fp^5.$$

Since

$$y_{0,1}(0)=c_{10}\,u_{-2}+c_5\,u_{-1}+c_0\,u_0+c_5\,u_1+c_{10}\,u_2=y_{-1,0}(0),$$

it follows that $a=0$. The readiest approach to obtain b, c, d, e and f is to note that $\phi(p)$ is the coefficient of u_3 in

$$\phi(p)\,\Delta^5 u_{-2}+\phi(-p)\,\Delta^5 u_{-3},$$

and thus must also be the coefficient of u_3 in $y_{0,1}(p)$, since $y_{-1,0}(p)$ contains no term u_3. Now let us write $\phi(p)$ in full for $p=0, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, 1$ so as to obtain the first five differences. On the other side we expand $[5]^5\,10^{-4}\,[x, y, z, t, u]$, take out the coefficients of u_3 and again take differences. The resulting five equations are shown in Appendix A(1). The conditions for the summation formula to reproduce polynomials of the first, third and fifth degrees and therefore for the interpolation formula to reproduce to the first, third and fourth degrees are $j=1$, $k=-5$, $m=14$ respectively (see Appendix E, in which it should be noted that the $x, y, z, \ldots$ of that Appendix are 16 times the $x, y, z, \ldots$ of $[5]^5\,10^{-4}$ $[x, y, z, \ldots]$). They are shown as conditions A, B and C respectively in Appendix A(2). The conditions for continuous derivatives at the terminal points are $b=0$, $c=0$, $d=0$, $e=0$, and are shown as conditions D, E, F and G, respectively, in Appendix A(2).

Generally, it is impossible to have seven equations satisfied with only five unknowns at our disposal. We must pick the equations which seem most appropriate for the end in view. For instance, the five conditions A, B, C, D, E determine the nucleus $[80, 48, -96, 0, 16]$ of Sprague's interpolation formula, the first osculatory six-term interpolation formula written in an equivalent form (Karup, p. 92, formula (6)). The conditions A, B, D, E, F determine the nucleus $[960, 877, -81, -609, -67]/75$ of the author's interpolation formula M3 with three continuous derivatives. This is a graduating formula, since, although it reproduces polynomials of the third degree, it does not in general reproduce the end-points of the interval. The fourth difference error is $-7\delta^4/240$.

The conditions that an interpolation formula derived from

$$[5]^5\, 10^{-4}\, [x, y, z, t, u]$$

should reproduce the end-points of the interval are obtained by expanding the formula and equating c_5 and c_{10} to zero. c_0 is, of course, unity. They are

$$381x + 730y + 640z + 510t + 370u = 10^4,$$
$$121x + 255y + 290z + 335t + 370u = 0,$$
$$x + 5y + 15z + 35t + 70u = 0,$$

which, not unexpectedly, reduce to conditions A, B, C.

It has been pointed out that the actual selection from various graduating interpolation formulae depends largely on the particular data and the amount of departure from the data permitted at pivotal points. For that reason the derivation of interpolation formulae correct to third differences with a predetermined fourth-difference error denoted by R seems to be of interest. If the formula is to have continuous first and second derivatives at the terminal points, we have to combine conditions A, B, D, E with the equation

$$z + 6t + 20u = 16(14 + 5^4R) = 224 + 10^4R,$$

since a fourth-difference error R in the interpolation formula corresponds to a fourth-difference error of 5^4R in the related summation formula (*T.A.S.A.* XXXI, 303). The solution will be

$$x = 80 + 2304R, \qquad y = 48 + 1244{\cdot}8R,$$
$$z = -96 - 3254{\cdot}4R, \qquad t = 278{\cdot}4R,$$
$$u = 16 + 579{\cdot}2R,$$

or

$$24d = 7 + 240R, \qquad 24e = -12 - 360R,$$
$$24f = 5 + 144R, \qquad b = c = 0.$$

These yield Sprague's nucleus when $R = 0$. $R = -16/579{\cdot}2 = -5/181$ determines an osculatory interpolation formula with 27 terms.

The conditions A, B, D with $t = u = 0$ yielding the nucleus

$$16\,[265, 143, -207, 0, 0]/137$$

determine a graduating tangential interpolation formula with 25 terms and the conditions A, B, C, D with $u = 0$ determine a reproducing tangential interpolation formula with 27 terms.

The conditions A, B with the sum of the squares of the terms of the nucleus $(x^2 + 2y^2 + 2z^2 + 2t^2 + 2u^2)$ minimized, yield the nucleus

$$[1744, 1544, 944, -56, -1456]/231 \quad \text{with} \quad R = -{\cdot}2432/7 = -{\cdot}035$$

nearly. Rounding off (still maintaining conditions A, B) we obtain the simpler nucleus $[10, 5, 5, -1, -6]$.

The operator $[5]^5\, 10^{-4}$ combined with a five-term nucleus produces many interpolation formulae with 25 terms and may be written as $[5]^5\, 10^{-4}\, [x, y, z]$.

Every interpolation formula having a fourth difference error R must satisfy the three conditions

$$x+2y+2z=16, \quad y+4z=-80, \quad z=224+10^4R.$$

Let us now minimize the smoothing index based on fifth differences. The square of this smoothing index will be proportionate to $x^2+2y^2+2z^2$. The nucleus $8[4, 2, -3]$ is obtained where $R=-\cdot 0248$ approximately. If, alternatively, the sum of the absolute values of the fifth differences is to be minimized, we require the minimum of

$$|760+3.10^4R|+|976+4.10^4R|+|224+10^4R|,$$

from which we derive $R=-\cdot 0244$ and the nucleus $4[14, 0, -5]$.

Let us now investigate the operator $[5]^6$ with a five-term nucleus, taking $k=10^{-6}$ for the sake of convenience. It may be written as $[5]^6\,10^{-6}\,[x, y, z]$. From Appendix E it follows that every associated interpolation formula having a fourth-difference error R must satisfy the three conditions

$$x+2y+2z=320, \quad y+4z=-1920, \quad z=6336+2.10^5R.$$

Hence

$$x=42176+12.10^5R, \qquad y=-27264-8.10^5R,$$
$$z=6336+2.10^5R.$$

Minimizing the square of the smoothing index based on sixth differences, i.e. minimizing $x^2+2y^2+2z^2$, we obtain $R=-\cdot 03456$ and the nucleus will be $64[11, 6, -9]$. If, alternatively, the sum of the absolute values of the sixth differences is to be minimized we require the minimum of

$$|10544+3.10^5R|+|13632+4.10^5R|+|3168+10^5R|,$$

from which we derive $R=-\cdot 03408$ and the nucleus $160[8, 0, -3]$.

We will now investigate interpolation formulae derived from

$$[5]^4\,10^{-4}\,[x, y, z, t, u, v, w],$$

since most of the six-term interpolation formulae published hitherto come into that category. By consideration of the terms of the expanded formula and by methods similar to those by which the conditions of Appendix A were derived, we obtain the conditions of Appendix B. We see that in a formula of the fourth degree $z=t$. In a formula of the third degree $z=t=0$; if, furthermore, the second derivative is continuous, it follows from condition O that $u=w$.

The nucleus corresponding to the operator $[5]^3\,10^{-2}$ might also be investigated without difficulty; but it seems that only one formula of that class has been published up to now, Greville's formula No. 109 (*T.A.S.A.* XLV, 202), the nucleus of which is $[0, 0, 58, -62, 0, 0, 0, 5, 5]/3$, a second-degree formula correct to first differences with first-order contact.

The four-term interpolation formula for subdivision of an interval into five parts has 19 terms. Vaughan (*J.I.A.* LXXII, 482) gives the formula corresponding to ordinary third-difference interpolation as $[5]^4\,5^{-3}(1-4\delta^2)u_x$, which, in the notation of this paper, would be written $[5]^4\,10^{-3}\,[72, -32]$. The next class is $[5]^3\,10^{-3}\,[x, y, z, t]$. The respective conditions to be satisfied may be obtained by methods similar to those previously described and are shown in Appendix C.

The four-term interpolation formula correct to second differences with second-order contact at the end-points comes from S, T, U, V. The nucleus is

$[1080, 600, -432, -208]/25$, and the formula is of the fifth degree. The condition for a formula $[5]^3\,10^{-3}\,[x, y, z, t]$ to be of a lesser degree than five turns out to be the same condition (XY) as for it to be of a lesser degree than four. Thus this class of formula does not contain any fourth-degree osculatory formula, and Jenkins's four-term interpolation formula with second-order contact is of the next class and can be written symbolically as

$$[5]^2\,10^{-2}\,4\,[61, 37, 10, -5, -8, -2]/25.$$

Lastly, let us consider an eight-term interpolation formula for the subdivision of intervals into seven parts based on $[7]^7\,7^{-6}\,[x, y, z, t, u, v, w]$. The conditions to be fulfilled, in order that it should be correct to the first, third, fifth and sixth differences, are shown as conditions (AA), (BB), (CC), (DD), respectively, of Appendix D and the conditions that it should have osculatory interpolation of the various orders are shown as conditions (EE), (FF), (GG), (HH), respectively, of Appendix D. They are derived in a like manner to the conditions of Appendix A. Similarly, if conditions (AA), (BB), (CC), (DD) are all satisfied, the end-points will be reproduced. The reproducing interpolation formula with three continuous derivatives, i.e. the solution of the conditions (AA), (BB), (CC), (DD), (EE), (FF), (GG), yields the nucleus

$$[9345, 7887, -12156, -11409, 14083, -2320, -705]/105,$$

and the graduating interpolation formula correct to fifth differences with four continuous derivatives, i.e. the solution of the conditions (AA), (BB), (CC), (EE), (FF), (GG), (HH), yields the nucleus

$$[216720, 209131, -10061, -411134, 6338, 109475, 5531]/35280$$

with a sixth-difference error of $29/5040$.

From Appendix E it may be verified that every interpolation formula derived from $[7]^8\,7^{-7}\,[x, y, z, t]$ having a sixth-difference error of R must satisfy the conditions

$$x+2y+2z+2t=1, \qquad y+4z+9t=-16,$$

$$z+6t=136, \qquad t=-5720/7+7^6R.$$

Minimizing the square of the smoothing index based on eighth differences, i.e. minimizing $x^2+2y^2+2z^2+2t^2$, we obtain $R=65871/11.7^7$ and the nucleus $[6353, 1145, -7234, 2951]/77$ and minimizing the sum of the absolute values of the eighth differences, we obtain $R=17944/3.7^7$ and the nucleus $[307, 0, -264, 112]/3$.

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APPENDIX A(1)

Six-term interpolation formulae of fifth degree or less

Operator $[5]^5\, 10^{-4}$, nucleus $[x, y, z, t, u]$

The relation between $\phi(p)$ and the nucleus

$$
\begin{aligned}
3125\,\Delta_{\frac{1}{5}}\phi(0) &= 625b + 125c + 25d + 5e + f = 10(u)/32\\
3125\,\Delta^2_{\frac{1}{5}}\phi(0) &= 250c + 150d + 70e + 30f = 10(t + 3u)/32\\
3125\,\Delta^3_{\frac{1}{5}}\phi(0) &= 150d + 180e + 150f = 10(z + 2t + 3u)/32\\
3125\,\Delta^4_{\frac{1}{5}}\phi(0) &= 120e + 240f = 10(y + z + t + u)/32\\
3125\,\Delta^5_{\frac{1}{5}}\phi(0) &= 120f = 10(x)/32
\end{aligned}
$$

The resulting five equations

$$
\begin{aligned}
625\cdot 384b &= 24x - 30y + 10z - 10t + 30u\\
25\cdot 384c &= -10x + 11y - z - t + 11u\\
5\cdot 384d &= 7x - 6y - 2z + 2t + 6u\\
384e &= -2x + y + z + t + u\\
384f &= x
\end{aligned}
$$

or

$$
\begin{aligned}
10x/32 &= 120f\\
10y/32 &= -625b + 125c - 25d + 5e + 119f\\
10z/32 &= 1875b - 125c - 75d + 55e + 93f\\
10t/32 &= -1875b - 125c + 75d + 55e + 27f\\
10u/32 &= 625b + 125c + 25d + 5e + f
\end{aligned}
$$

APPENDIX A(2)

	Condition	Equivalent to
(A)	$x + 2y + 2z + 2t + 2u = 16$	Degree of reproduction 1
(B)	$y + 4z + 9t + 16u = -80$	Degree of reproduction 3 (with A)
(C)	$z + 6t + 20u = 224$	Degree of reproduction 4 and reproduction of end-points (with A and B)
(D)	$12x - 15y + 5z - 5t + 15u = 0$	Continuous first derivative
(E)	$-10x + 11y - z - t + 11u = 0$	Continuous second derivative
(F)	$7x - 6y - 2z + 2t + 6u = 0$	Continuous third derivative
(G)	$-2x + y + z + t + u = 0$	Continuous fourth derivative
(G′)	$x = 0$	Fourth-degree formula

APPENDIX B

Six-term interpolation formulae of fifth degree or less

Operator $[5]^4\, 10^{-4}$. Nucleus $[x, y, z, t, u, v, w]$

	Condition	Equivalent to
(H)	$x+2y+2z+2t+2u+2v+2w=80$	Degree of reproduction 1
(I)	$y+4z+9t+16u+25v+36w=-320$	Degree of reproduction 3 (with H)
(J)	$z+4t+10u+20v+35w=0$	Reproduction of end-points (with H and I)
(K)	$t+4u+10v+21w=144$	Degree of reproduction 4 (with H, I and J)
(L)	$u+5v+14w=224$	
(M)	$12z-27t+20u-10v+20w=0$	Continuous first derivative
(N)	$5x-20y+27z-12t=0$	
(O)	$-10z+21t-12u+12w=0$	Continuous second derivative
(P)	$7z-13t+4u+4v+4w=0$	Continuous third derivative
(Q)	$2x+4y-13z+7t=0$	
(R)	$-2z+3t=0$	Continuous fourth derivative
(R′)	$z-t=0$	Fourth-degree formula

APPENDIX C

Four-term interpolation formulae of fifth degree or less

Operator $[5]^3\, 10^{-3}$. Nucleus $[x, y, z, t]$

	Condition	Equivalent to
(S)	$x+2y+2z+2t=40$	Degree of reproduction 1
(T)	$y+4z+9t=-120$	Degree of reproduction 2 also reproduction of end-points (with S)
(U)	$-39x+59y-30z+30t=0$	Continuous first derivative
(V)	$31x-43y+12z+12t=0$	Continuous second derivative
(W)	$-5x+6y=0$	Continuous third derivative
(XY)	$x-y=0$	Third-degree formula

APPENDIX D

Eight-term interpolation formulae of seventh degree or less
Operator $[7]^7 7^{-6}$. Nucleus $[x, y, z, t, u, v, w]$

Condition

(AA) $x + 2y + 2z + 2t + 2u + 2v + 2w = 1$
(Degree of reproduction 1)

(BB) $y + 4z + 9t + 16u + 25v + 36w = -14$
(Degree of reproduction 3)

(CC) $z + 6t + 20u + 50v + 105w = 105$
(Degree of reproduction 5)

(DD) $t + 8u + 35v + 112w = -561$
(Degree of reproduction 6, also reproduction of end-points)

(EE) $60x - 70y + 14z - 7t + 7u - 14v + 70w = 0$
(Continuous first derivative)

(FF) $-126x + 137y - 13z + 2t + 2u - 13v + 137w = 0$
(Continuous second derivative)

(GG) $232x - 225y - 15z + 15t - 15u + 15v + 225w = 0$
(Continuous third derivative)

(HH) $-21x + 17y + 5z - t - u + 5v + 17w = 0$
(Continuous fourth derivative)

APPENDIX E

$$[n]\, u_0/n = \{1 + (n^2-1)\, b/2^2 3! + (n^2-1)(n^2-9)\, b^2/2^4 5! + (n^2-1)(n^2-9)(n^2-25)\, b^3/2^6 7! + \dots\}\, u_0,$$

where $b = \Delta^2 E^{-1}$ (see H. Tetley, *Actuarial Statistics*, I, 179).

Since

$$u_r + u_{-r} = \{2 + r^2 b + r^2(r^2-1)\, b^2/12 + r^2(r^2-1)(r^2-4)\, b^3/360 + \dots\}\, u_0,$$

$$\begin{aligned}[x, y, z, t, u, v, w, \dots] &= \{(x + 2y + 2z + 2t + 2u + 2v + 2w + \dots) \\ &\quad + (y + 4z + 9t + 16u + 25v + 36w + \dots)\, b + (z + 6t + 20u + 50v + 105w + \dots)\, b^2 \\ &\quad + (t + 8u + 35v + 112w + \dots)\, b^3 + (u + 10v + 54w + \dots)\, b^4 + \dots\}\, u_0 \\ &= (j + kb + mb^2 + pb^3 + qb^4 + \dots)\, u_0, \quad \text{say.}\end{aligned}$$

Hence

$$\begin{aligned}[5]^r 5^{-r+1} [x, y, z, \dots] &= 5(1 + b + b^2/5 + \dots)^r (j + kb + mb^2 + \dots)\, u_0 \\ &= 5j\{1 + rb + r(5r-3)\, b^2/10 + \dots\}\{1 + kb/j + mb^2/j + \dots\}\, u_0 \\ &= 5j\{1 + (k/j + r)\, b + [m/j + rk/j + r(5r-3)/10]\, b^2 + \dots\}\, u_0.\end{aligned}$$

APPENDIX E (*continued*)

Respective conditions for the associated summation formulae $[5]^r\, 5^{-r}\, [x, y, z, \ldots]$ to be correct to first, third and fifth degrees are

$$j=1, \quad k+r=0, \quad m+rk+r(5r-3)/10=0,$$

i.e.
$$j=1, \quad k=-r, \quad m=r(5r+3)/10.$$

Also

$$[7]^r\, 7^{-r+1}\, [x, y, z, \ldots]$$
$$=7(1+2b+b^2+b^3/7+\ldots)^r(j+kb+mb^2+pb^3+\ldots)\,u_0$$
$$=7j\{1+2rb+r(2r-1)\,b^2$$
$$+r(28r^2-42r+17)\,b^3/21+\ldots\}\,\{1+kb/j+mb^2/j+pb^3/j+\ldots\}\,u_0.$$

Respective conditions for the associated summation formulae $[7]^r\, 7^{-r}\, [x, y, z, \ldots]$ to be correct to first, third, fifth and seventh degrees are

$$j=1, \quad k+2r=0, \quad m+2rk+r(2r-1)=0,$$
$$p+2rm+r(2r-1)\,k+r(28r^2-42r+17)/21=0,$$

i.e.
$$j=1, \quad k=-2r, \quad m=r(2r+1), \quad p=-r(28r^2+42r+17)/21.$$