NEW DEVELOPMENTS OF THE LEE-CARTER MODEL FOR MORTALITY DYNAMICS

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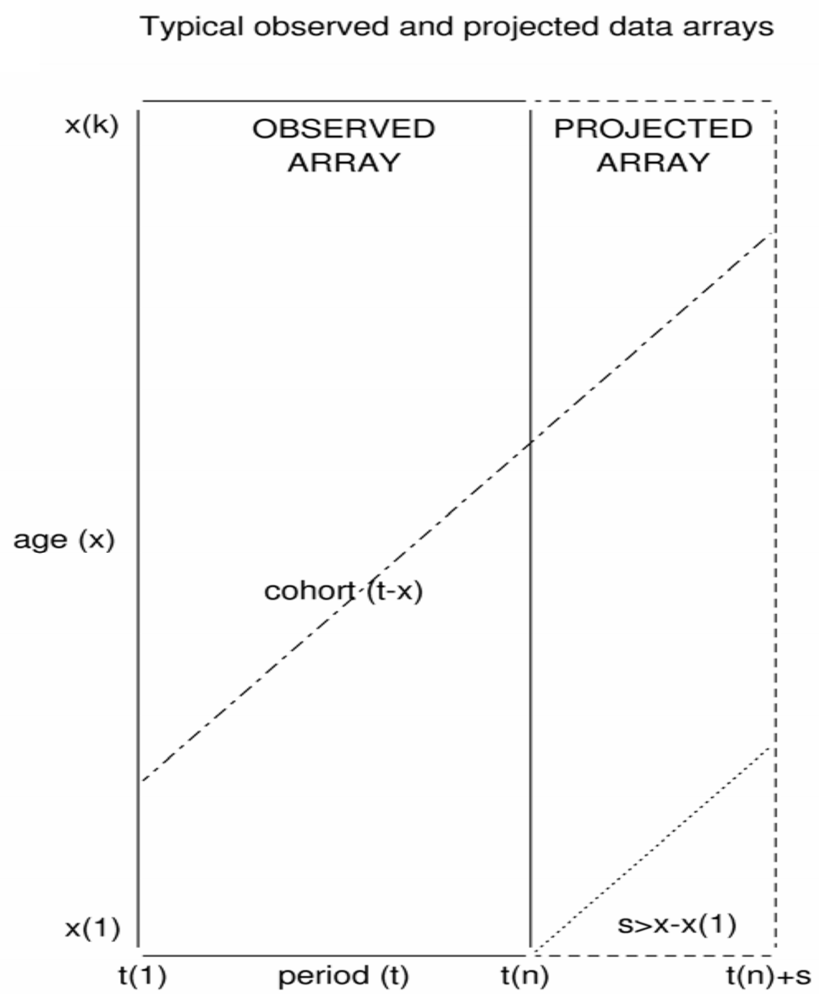
AGENDA

- Data and Notation
- Basic Lee-Carter Model
- Age-Period-Cohort Model
- Risk Measurement
- Concluding Comments

ILLUSTRATION OF DATA CONFIGURATION

Typical rectangular data array and targeted projected array.



NOTATION

Data: (d_{xt}, e_{xt})

d_{xt} = number of deaths at age *x* and time *t*

e_{xt} = matching exposure to risk of death

with empirical central mortality rate

$\hat{m}_{xt} = d_{xt} / e_{xt}$.

Random variables:

D_{xt} = number of deaths at age *x* and time *t*

Y_{xt} = response (generic)

LEE CARTER MODELS: BASE VERSION

STRUCTURE

One of the benchmark demographic models used for mortality modelling and projections in many countries. Lee and Carter (1992) proposed:

$$\ln m_{xt} = \eta_{xt} + \varepsilon_{xt}, \quad \eta_{xt} = \alpha_x + \beta_x \kappa_t,$$

where the ε_{xt} are IID $\mathrm{N}(\mathbf{0}, \sigma^2)$ variables.

This is a regression framework with no observable quantities on the RHS.

STRUCTURE (cont)

Structure is invariant under the transformations

$$\{\alpha_x, \beta_x, \kappa_t\} \mapsto \{\alpha_x, \beta_x / c, c\kappa_t\}$$

$$\{\alpha_x, \beta_x, \kappa_t\} \mapsto \{\alpha_x - c\beta_x, \beta_x, \kappa_t + c\}$$

and is made identifiable using the following constraints (which are not unique):

$\sum_{t=t_1}^{t_n} \kappa_t = 0, \ \sum_x \beta_x = 1,$ and which imply the least squares estimator

$$\hat{\alpha}_x = \frac{1}{t_n - t_1 + 1} \sum_{t=t_1}^{t_n} \ln \hat{m}_{xt}$$

INTERPRETATION OF PARAMETERS

α_x: 'average' of $\log m_{xt}$ over time t so that exp α_x represents the general shape of the age-specific mortality profile.

κ_t: underlying time trend.

β_x: sensitivity of the logarithm of the hazard rate at age x to the time trend represented by κ_t.

ε_{xt}: effects not captured by the model.

FITTING BY SVD

(Lee and Carter (1992))

A two-stage estimation process: estimate α_x as above. Estimate κ_t and β_x as the 1st right and 1st left singular vectors in the SVD of the matrix

$$[\log \hat{m}_{xt} - \hat{\alpha}_x].$$

Thus

$$\log(\hat{m}_{xt}) = \hat{\alpha}_x + s_1 u_1(x) v_1(t) + \sum_{i>1} s_i u_i(x) v_i(t)$$

where

$s_i,\ u_i,\ v_i$ = (ordered) singular values and vectors

and

$$\hat{\beta}_x \hat{\kappa}_t = s_1 u_1(x) v_1(t)$$

subject to the constraints on κ_t and β_x.

Finally, $\hat{\kappa}_t$ are adjusted so that

$$\sum_{all,\, x} d_{xt} = \sum_{all,\, x} \hat{d}_{xt} \ \forall\ t. \qquad \text{where } \hat{d}_{xt} = e_{xt} \exp(\hat{\alpha}_x + \hat{\beta}_x \hat{\kappa}_t).$$

FITTING BY WEIGHTED LEAST SQUARES (GAUSSIAN)

Perform the iterative process

Set starters $\hat{\alpha}_x, \hat{\beta}_x, \hat{\kappa}_t$; *compute* $\hat{y}_{xt}$

$\downarrow$

update $\hat{\alpha}_x$; *compute* $\hat{y}_{xt}$

update $\hat{\kappa}_t$, *adjust s.t.* $\sum_{t=t_1}^{t_n} \kappa_t = 0$; *compute* $\hat{y}_{xt}$

update $\hat{\beta}_x$; *compute* $\hat{y}_{xt}$

compute $D(y_{xt}, \hat{y}_{xt})$

$\downarrow$

repeat; stop when $D(y_{xt}, \hat{y}_{xt})$ *converges*

Where $y_{xt} = \log \hat{m}_{xt},\ \hat{y}_{xt} = \hat{\eta}_{xt}, D(y_{xt}, \hat{y}_{xt}) = \sum_{x,t} w_{xt}(y_{xt} - \hat{y}_{xt})^2$

with weights

$w_{xt} = d_{xt}$ (or = 1).

For a typical parameter, we use the updating algorithm:

$$updated(\theta) = \theta - \frac{\partial D}{\partial \theta} \Bigg/ \frac{\partial^2 D}{\partial \theta^2}.$$

POISSON BILINEAR MODEL

$$Y_{xt} = D_{xt}, \; \mathrm{E}(Y_{xt}) = e_{xt} \exp(\alpha_x + \beta_x \kappa_t), \; \mathrm{Var}(Y_{xt}) = \phi \mathrm{E}(Y_{xt})$$

with log-link and non-linear predictor

$$\eta_{xt} = \log e_{xt} + \alpha_x + \beta_x \kappa_t .$$

Perform iterative process with

$$y_{xt} = d_{xt}, \; \hat{y}_{xt} = \hat{d}_{xt} = e_{xt} \exp(\hat{\alpha}_x + \hat{\beta}_x \hat{\kappa}_t)$$

$$D(d_{xt}, \; \hat{d}_{xt}) = \sum_{x,t} 2 w_{xt} \left\{ d_{xt} \log\left(\frac{d_{xt}}{\hat{d}_{xt}} \right) - \left(d_{xt} - \hat{d}_{xt} \right) \right\}$$

with weights

$$w_{xt} = \begin{matrix} 1, \; e_{xt} > 0 \\ 0, \; e_{xt} = 0 \end{matrix}$$

LEE CARTER: BINOMIAL

$$\eta_{xt} = \alpha_x + \beta_x \kappa_t \qquad \sum_x \beta_x = 1,\ \kappa_{t_n} = 0$$

and link functions $\eta_{xt} = g\left(q_{xt}\right)$

Possible choices of g are:

I. complementary log-log link $\qquad \eta_{xt} = \log\left\{-\log\left(1-q_{xt}\right)\right\}$

II. log-odds link $\qquad \eta_{xt} = \log\left(\frac{q_{xt}}{1-q_{xt}}\right)$

III. probit link $\qquad \eta_{xt} = \Phi^{-1}\left(q_{xt}\right)$

DIAGNOSTICS

1) Proportion of the total temporal variance explained by the 1st SVD component:

$$\frac{s_1^2}{\sum_{all,\, i} s_i^2} \times 100\%.$$

(Not a good indicator of goodness of fit.)

2) Standardised deviance residuals

$$\hat{\varepsilon}_{xt} = \mathrm{sign}(y_{xt} - \hat{y}_{xt})\sqrt{\mathrm{dev}\left(x,t\right)/\phi}$$

(we could also use standardised SVD residuals)

3) Plot differences between actual total and expected total deaths for each time period, *t*.

PROJECTIONS

Time series (ARIMA)

$$\{\hat{\kappa}_t : \mathrm{t} \in [t_1, t_n]\} \mapsto \{\dot{\kappa}_{t_n+s} : s > 0\}.$$

Construct mortality rate projections

$$\dot{m}_{x,t_n+s} = \hat{m}_{xt_n} \exp\{\hat{\beta}_x(\dot{\kappa}_{t_n+s} - \hat{\kappa}_{t_n})\},\ s > 0$$

by alignment with the latest available mortality rates.

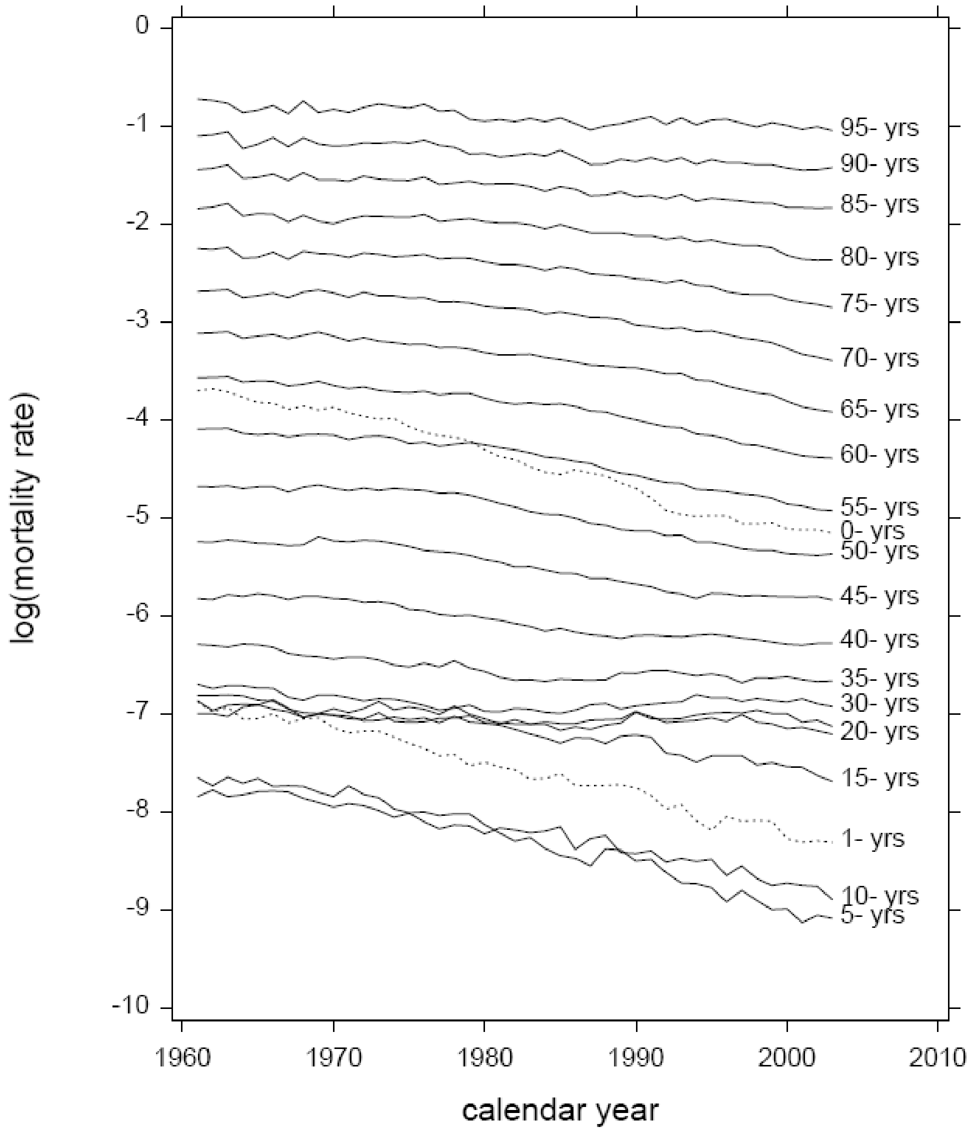
Note

$$F(x, t_n + s) = \exp\{\hat{\beta}_x(\dot{\kappa}_{t_n+s} - \hat{\kappa}_{t_n})\},\ s > 0$$

is a mortality reduction factor, as widely used in the UK and elsewhere.

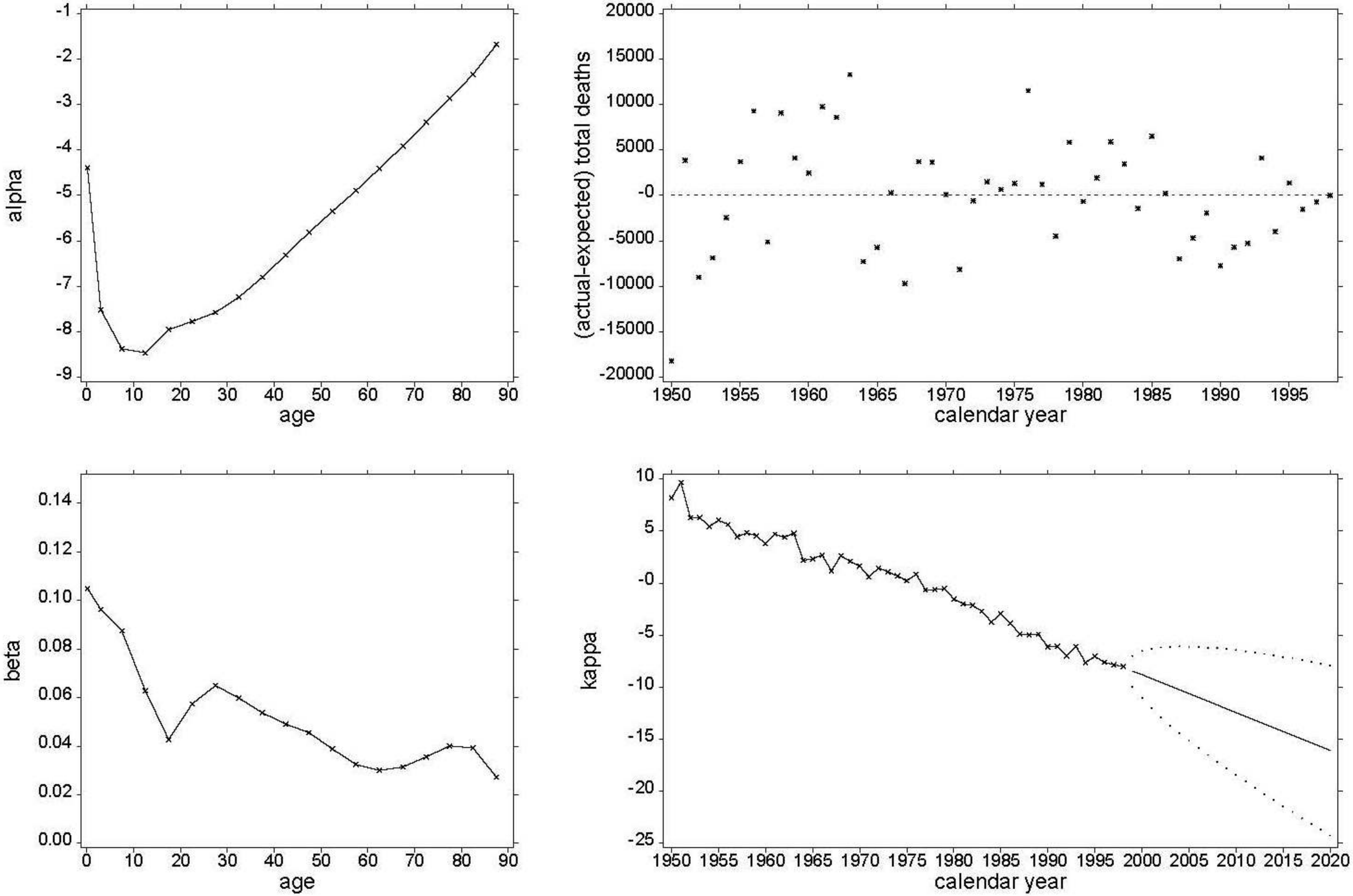
For ARIMA(0,1,0) with drift parameter λ

$$F(x, t_n + s) = \exp(\hat{\beta}_x \hat{\lambda} s),\ s > 0,$$

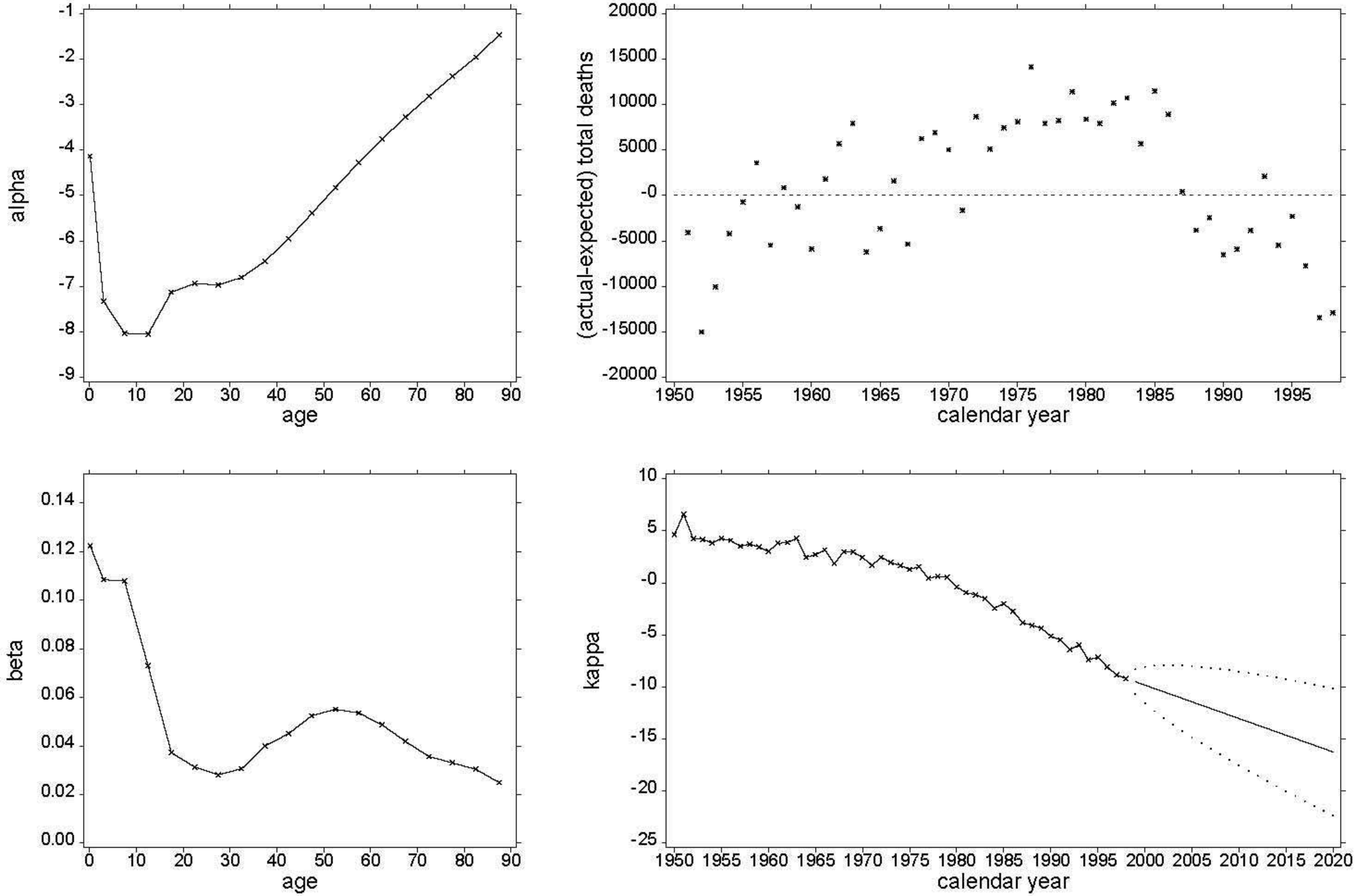


E+W 1961 – 2003 male mortality experience. Log crude mortality rates against period, for grouped age (0-,1-,5-,10-, …, 90-, 95-).

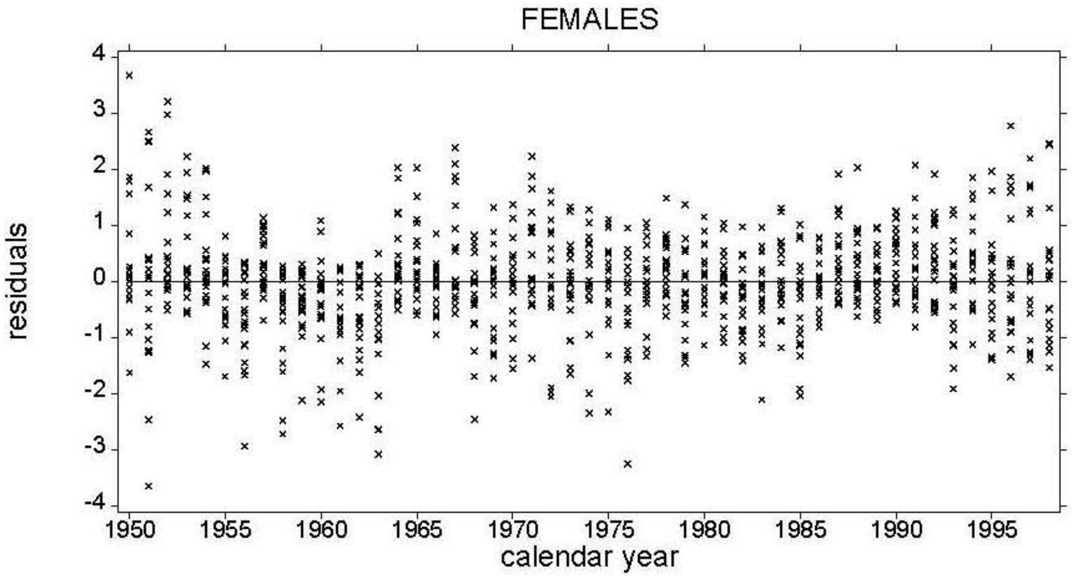
E+W female mortality experience (LC)



E+W male mortality experience (LC)



LC model: E+W mortality experience


FEMALES
residuals
calendar year
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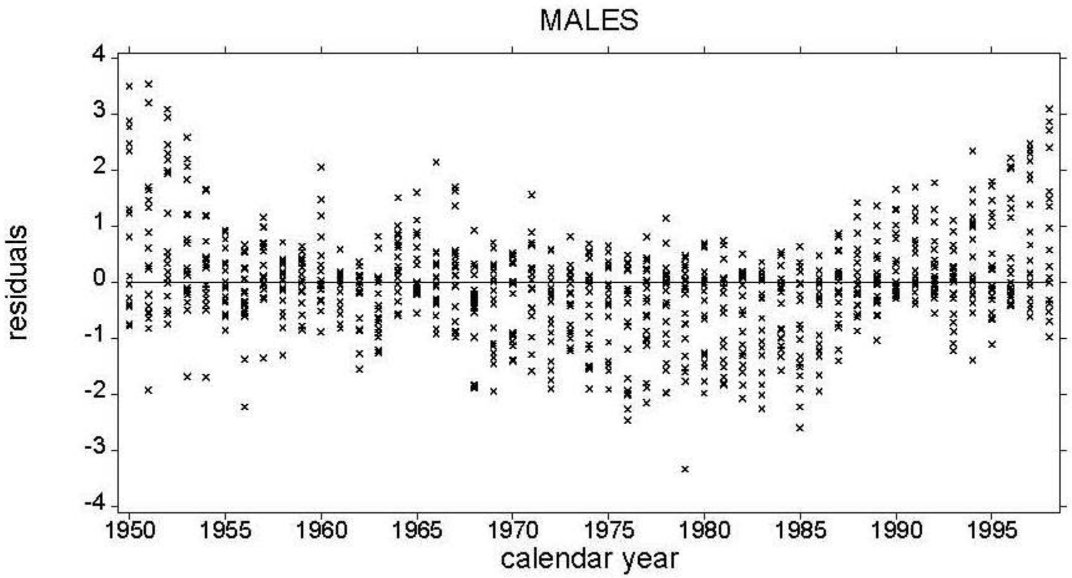
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INTERIM CONCLUSIONS

- Basic LC model fits England and Wales females mortality experience fairly well; but poor fit for male experience

- Enhancements

  - optimize choice of fitting period

  - add a second factor: $\alpha_x + \beta_x^{(1)}\kappa_t^{(1)} + \beta_x^{(2)}\kappa_t^{(2)}$

  - allow for a cohort effect

AGE-PERIOD-COHORT MODELS IN LC FRAMEWORK

UK: Strong cohort effect for those born in period 1925 – 1945 (also US, Japan, Germany Sweden)

The Lee-Carter structure may be expanded to incorporate a cohort effect:

$$\text{APC: } \eta_{xt} = \log e_{xt} + \alpha_x + \beta_x^{(0)} \iota_{t-x} + \beta_x^{(1)} \kappa_t$$

under the Poisson setting.

The age-cohort substructure
AC: $\beta_x^{(1)} = 0$
is also of interest, while we recall that for the standard model
LC: $\beta_x^{(0)} = 0$.

FITTING AGE-PERIOD-COHORT MODELS

Fitting is problematic because of the relationship
cohort = (period – age) or *z* = *t* – *x*
between the three main effects.

We resort to a two-stage fitting strategy, in which α_x is estimated first, according to the original Lee-Carter SVD approach, thus

$$\hat{\alpha}_x = \frac{1}{t_n - t_1 + 1} \sum_{t=t_1}^{t_n} \ln \hat{m}_{xt}$$

The remaining parameters can then be estimated subject to the parameter constraints

$$\sum_x \beta_x^{(0)} = 1, \ \sum_x \beta_x^{(1)} = 1 \ and \ \iota_{t_1 - x_k} = 0 \ or \ \kappa_{t_1} = 0.$$

Effective starting values are obtained by setting $\beta_x^{(0)} = \beta_x^{(1)} = 1$ and fitting a restricted version of the model to generate starting values for ι_z and κ_t.

PROJECTIONS

Use separate ARIMA time series

$$\{\hat{\kappa}_t : t \in [t_1, t_n]\} \mapsto \{\dot{\kappa}_{t_n+s} : s > 0\},$$

$$\{\hat{\iota}_z : z \in [t_1 - x_k, t_n - x_1]\} \mapsto \{\dot{\iota}_{t_n - x_1 + s} : s > 0\},$$

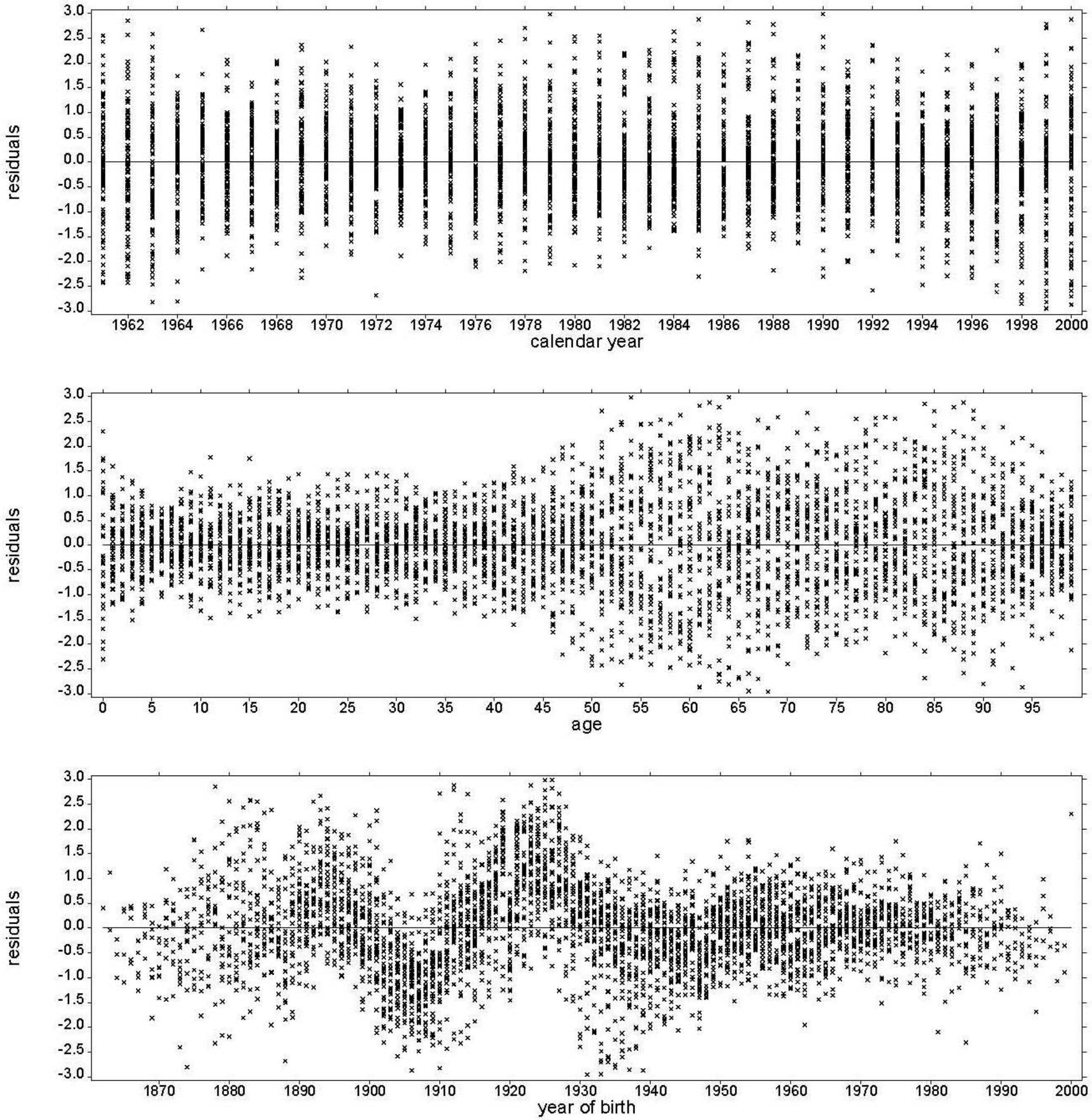
then

$$F(x, t_n + s) = \exp\{\hat{\beta}_x^{(0)}(\tilde{\iota}_{t_n - x + s} - \hat{\iota}_{t_n - x}) + \hat{\beta}_x^{(1)}(\dot{\kappa}_{t_n+s} - \hat{\kappa}_{t_n})\},\ s > 0$$

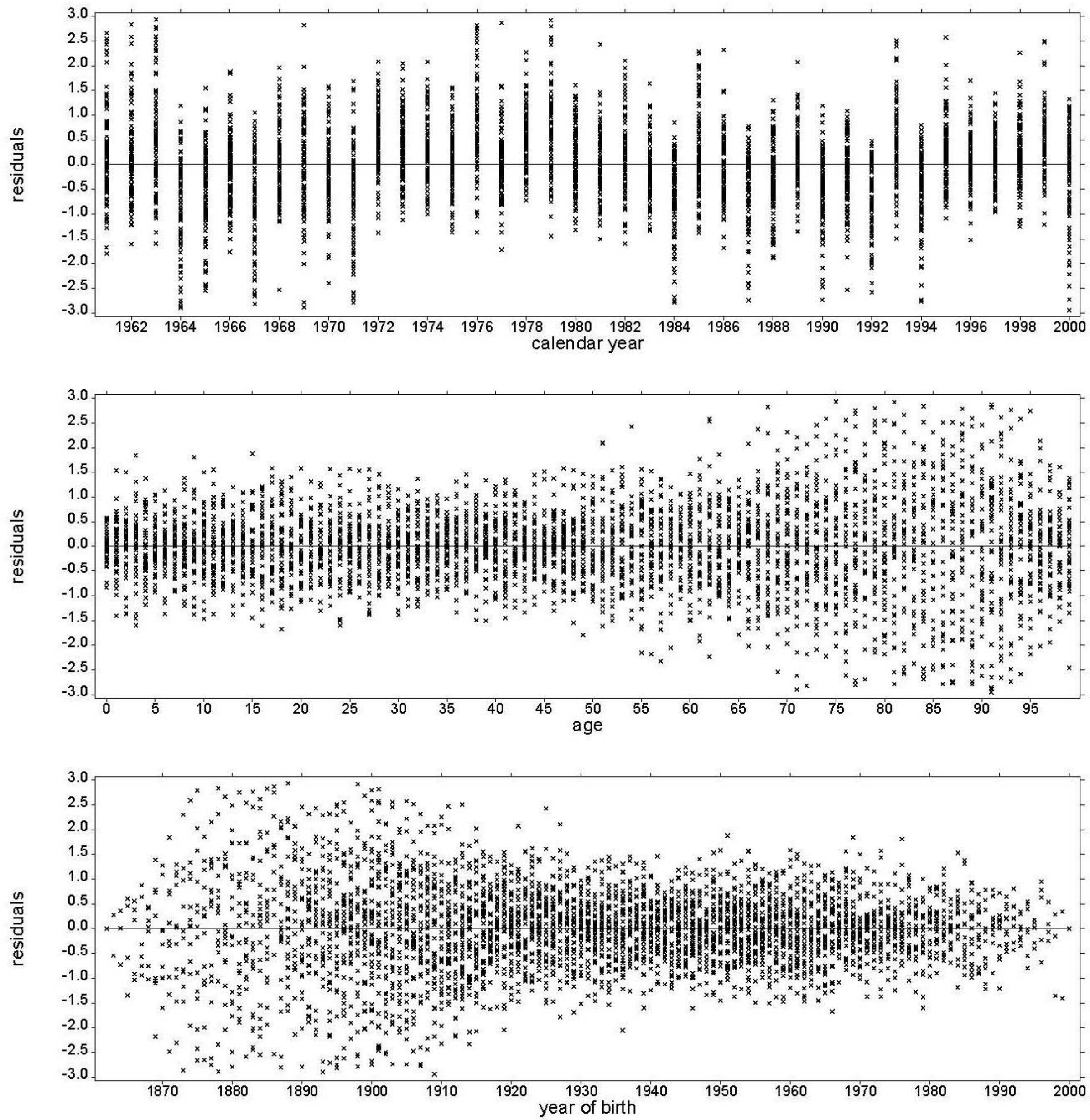
where

$$\tilde{\iota}_{t_n - x + s} = \begin{matrix} \hat{\iota}_{t_n - x + s},\ s \le x\text{-}x_1 \\ \dot{\iota}_{t_n - x + s},\ s > x\text{-}x_1 \end{matrix}.$$

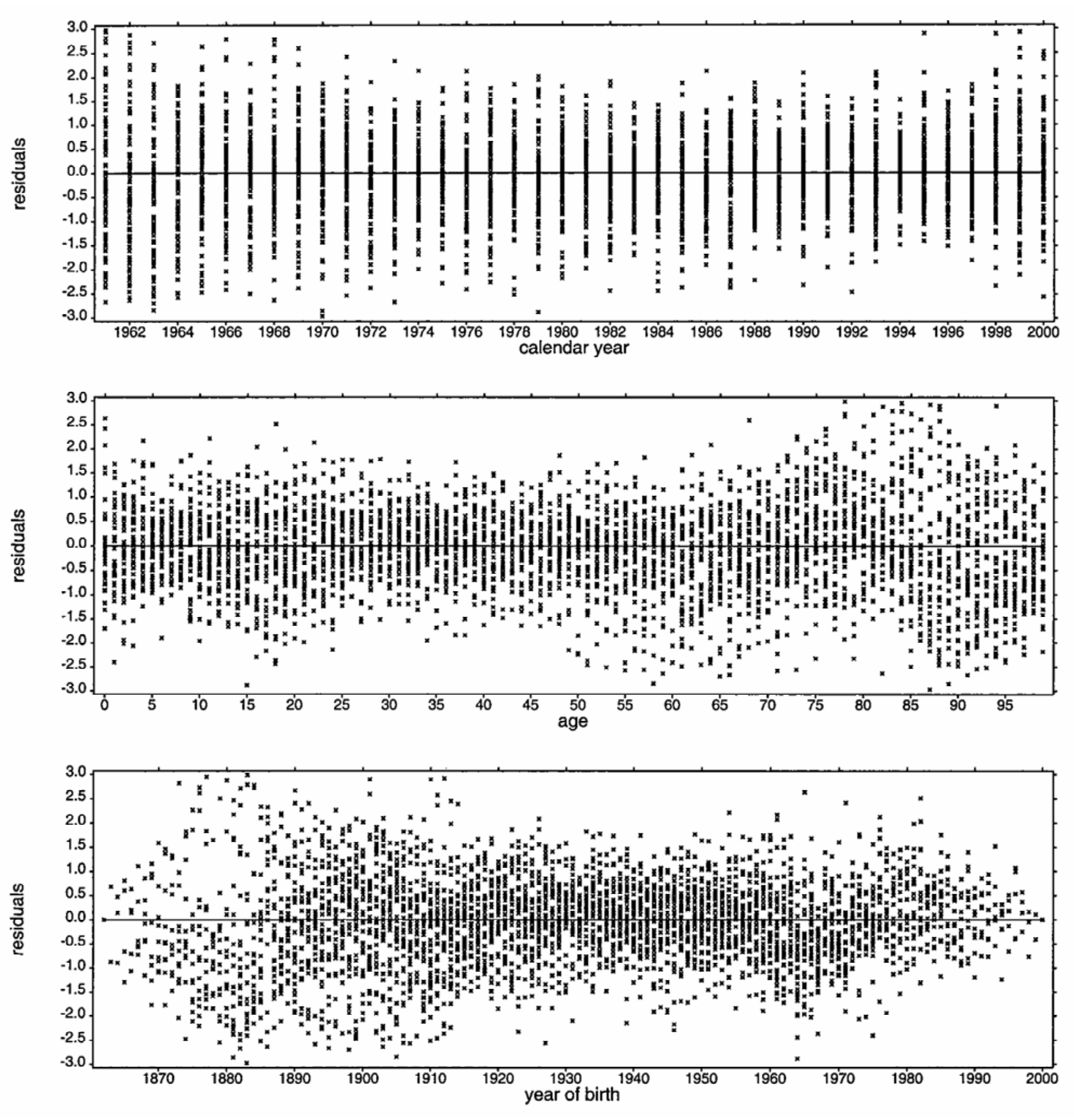
UK female mortality experience (LC) – residual plots



UK female mortality experience (AC) – residual plots



UK female mortality experience (APC) – residual plots



APPLICATIONS

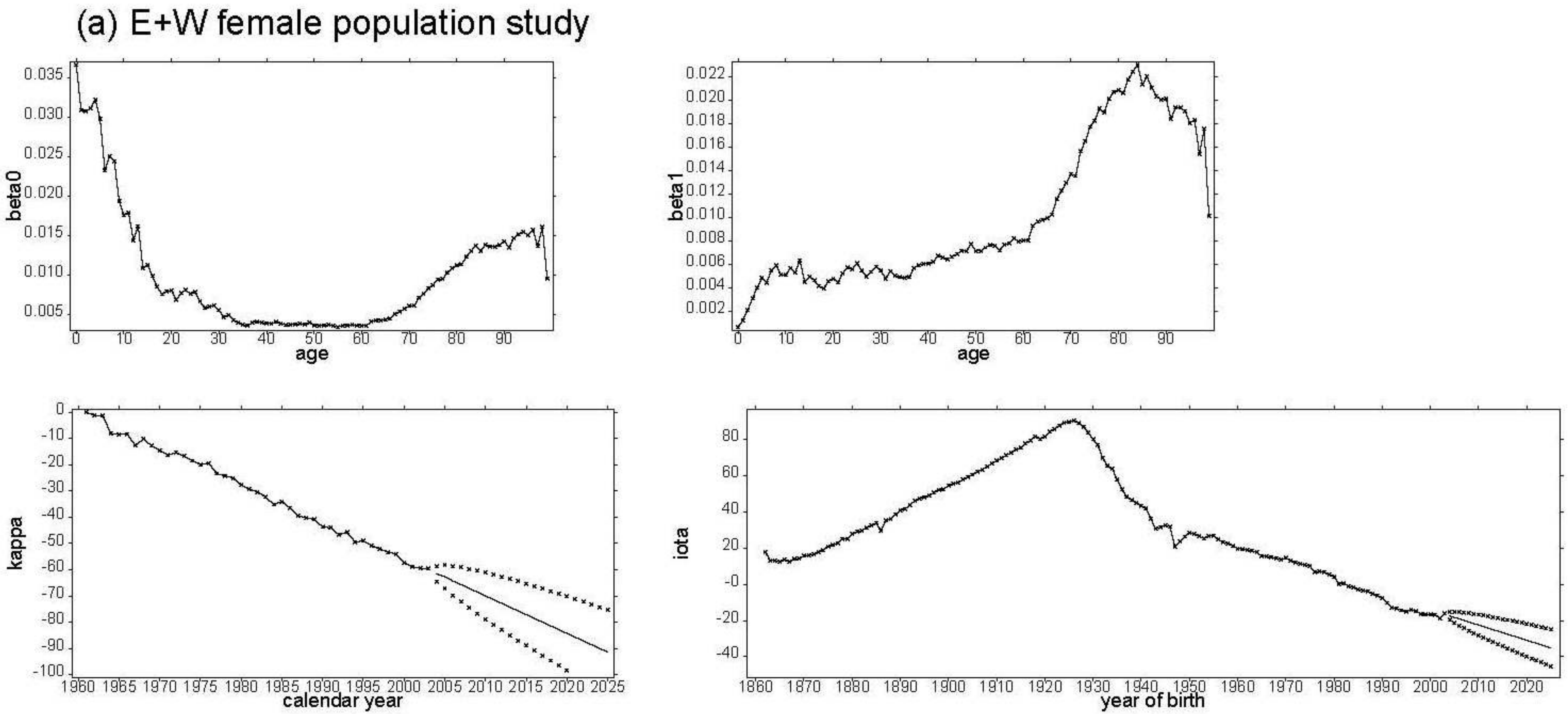
Time Series Forecasts

- For $\kappa_t : y_t = a_0 + a_1 t + \sum_{i=1}^{p} \phi_i y_{t-1}$ with p=2

  $= \kappa_t - \kappa_{t-1}$

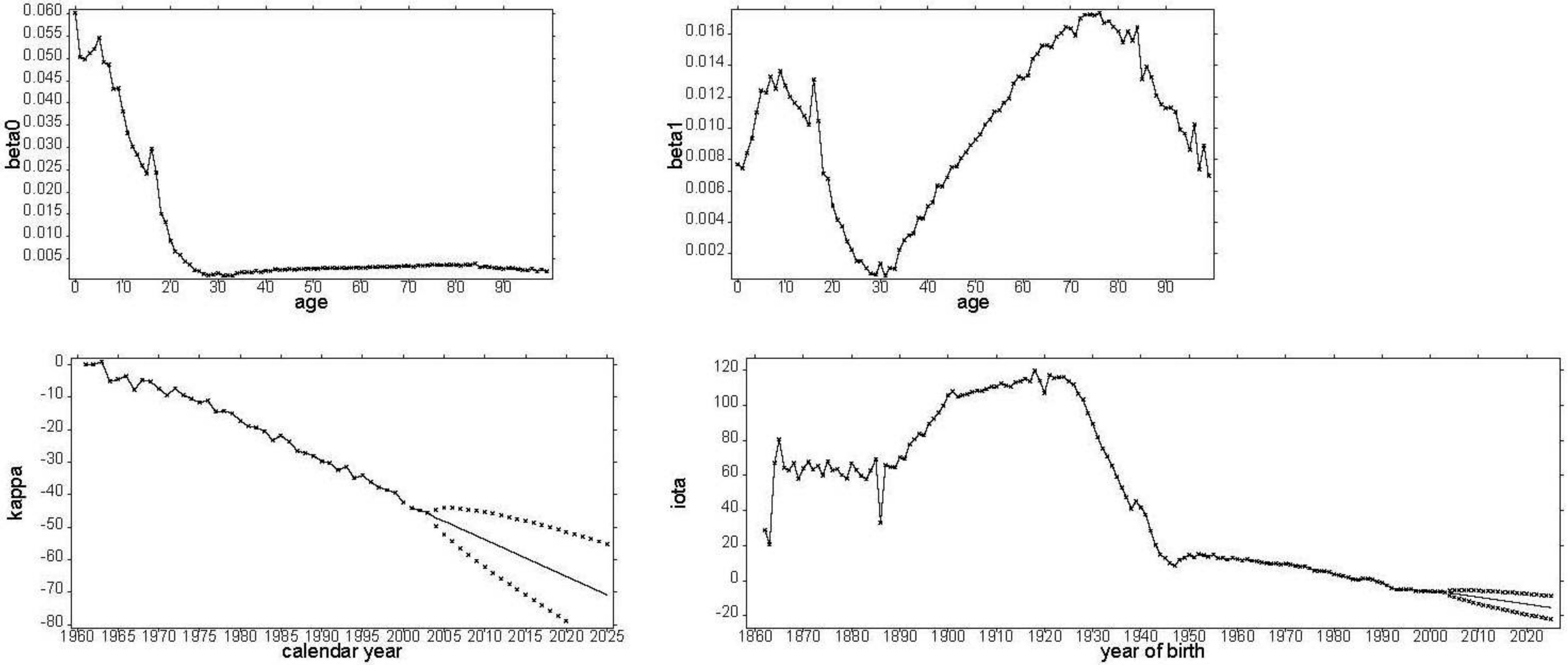
($a_1 \neq 0$ for males; $a_1 = 0$ for females)

- For ι_z ARIMA (1,1,0)

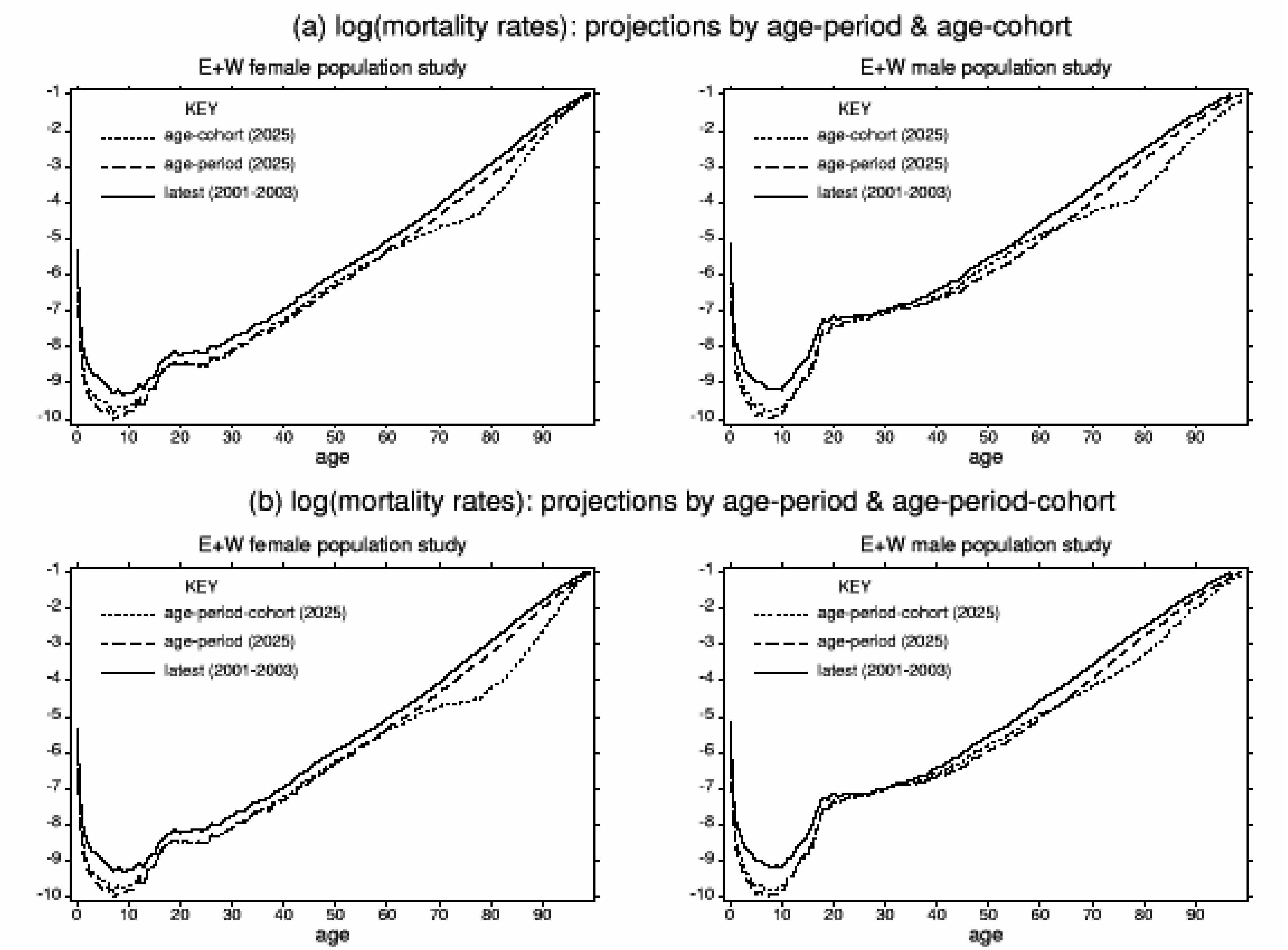


England and Wales population, parameter estimates, APC model: (a) females; (b) males

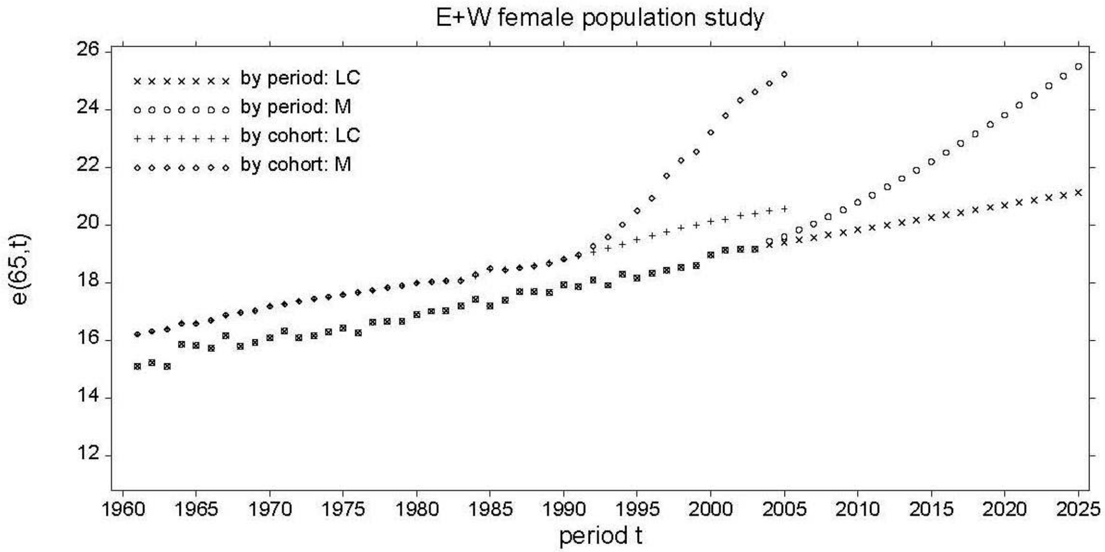
(b) E+W male population study

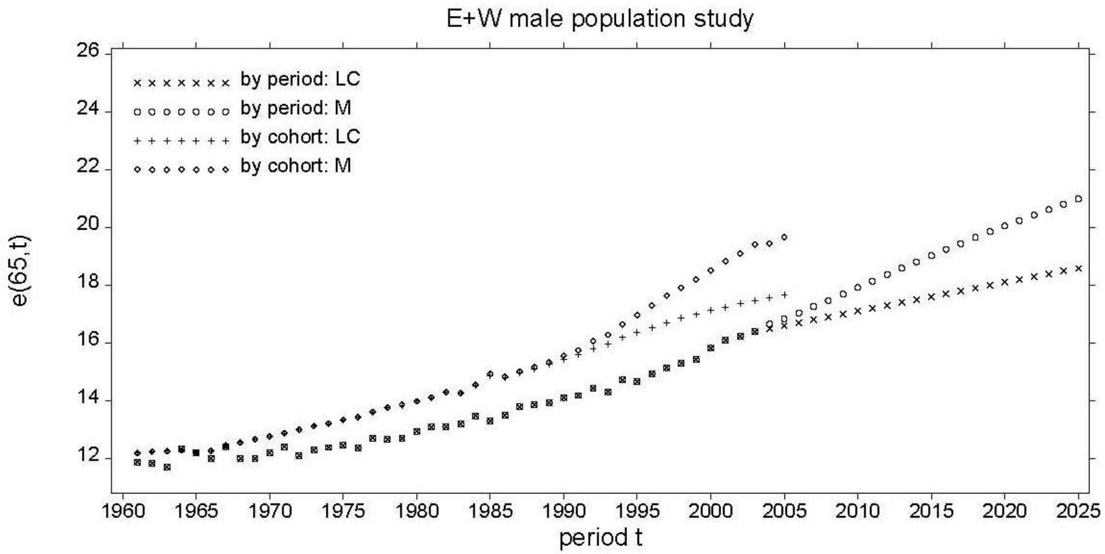


England and Wales population, parameter estimates, APC model:
(b) females; (b) males



Latest and projected $\log \mu_{xt}$ age profiles:
(a) LC and AC modelling; (b) LC and APC modelling





Life expectations at age 65 for a range of periods, computed by period and by cohort under age-period (LC) and age-period-cohort (APC) modelling

RISK MEASUREMENT AND PREDICTION INTERVALS

- uncertainty in projections needs to be quantified i.e. by prediction intervals

- but analytical derivations are impossible

- 2 different sources of uncertainty need to be combined

  - errors in estimation of parameters of Lee Carter model
  - forecast errors in projected ARIMA model

- indices of interest (e.g. hazard rates, annuity values, life expectancies) are complex non linear functions of $\alpha_x, \beta_\chi, \kappa_t$ and ARIMA parameters.

DIFFERENT SIMULATION STRATEGIES

A) Semi-parametric (Poisson) Bootstrap: generates new data sets

Let $\hat{d}_x$ be fitted number of deaths.

Simulate response $d_x^{(j)}$ from Poisson ($\hat{d}_x$)

Compute $\mu_x^{(j)}$

Fit model: obtain estimates of $\alpha_x^{(j)}, \beta_x^{(j)}, \kappa_t^{(j)}$

Compute $\dot{\kappa}_{t_n+k}^{(j)} \left[= \hat{\kappa}_{t_n} + k\hat{\theta}^{(j)} \text{ for ARIMA } (0,1,0) \right]$

Repeat for $j = 1, \ldots, \mathrm{N}$

DIFFERENT SIMULATION STRATEGIES **(Continued)**

C) <u>Residuals Bootstrap: generates new data sets</u>

Let r_x be the deviance residuals

Sample with replacement to get $r_x^{(j)}$

Map from $r_x^{(j)}$ to $d_x^{(j)}$ for each x

Compute $\mu_x^{(j)}$

Fit Model: obtain estimates of $\alpha_x^{(j)}, \beta_x^{(j)}, \kappa_t^{(j)}$

Compute $\dot{\kappa}_{t_n+k}^{(j)} \left[= \hat{\kappa}_{t_n} + k\hat{\theta}^{(j)} \text{ for ARIMA (0,1,0)} \right]$

Repeat for $j = 1, \ldots, \mathrm{N}$

DIFFERENT SIMULATION STRATEGIES (Continued)

B) Parametric Monte Carlo Simulation: generates new parameter estimates from fitted parameter estimates

Simulate $\boldsymbol{e}^{(j)}$ vector of $\mathrm{N}(0,1)$ errors
Let $\boldsymbol{C}$ be the Cholesky factorisation matrix of the variance- covariance matrix (needs to be invertible)

Compute simulated model parameters

$$\theta^{(j)} = \hat{\theta} + \sqrt{\varphi}\ \boldsymbol{C}\ \boldsymbol{e}^{(j)}$$

where φ is optional scale parameter

Compute $\dot{\kappa}_{t_n+k}^{(j)}$

Repeat for $j = 1, \ldots, \mathrm{N}$

JOINT MODELLING

Attempt to model variable dispersion parameter (rather than fixed φ)

2 stage process (LG model)

1. Model D_{xt} as independent Poisson response

   Define $R_{xt} = \omega_{xt} \dfrac{\{D_{xt} - \mathrm{E}(D_{xt})\}^2}{\mathrm{E}(D_{xt})}$ the resulting squared Pearson residuals

2. Define R_{xt} as independent gamma responses

   $$\mathrm{E}(R_{xt}) = \phi_{xt}, \ \mathrm{Var}(R_{xt}) = \tau \frac{\mathrm{V}\{\mathrm{E}(R_{xt})\}}{\omega_{xt}}; \ \mathrm{V}(u) = u^2$$

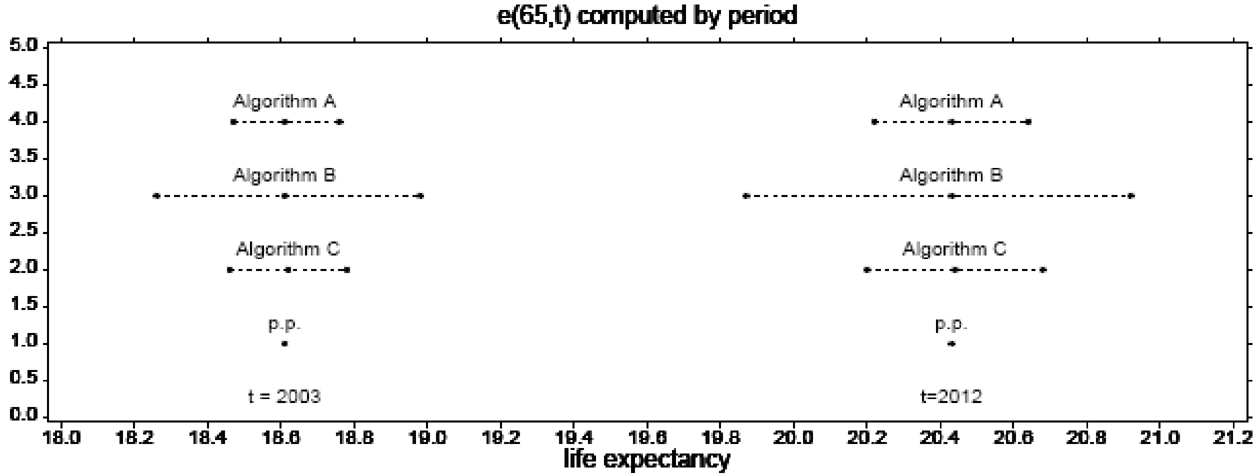
   and log link and linear parametric structure in age.

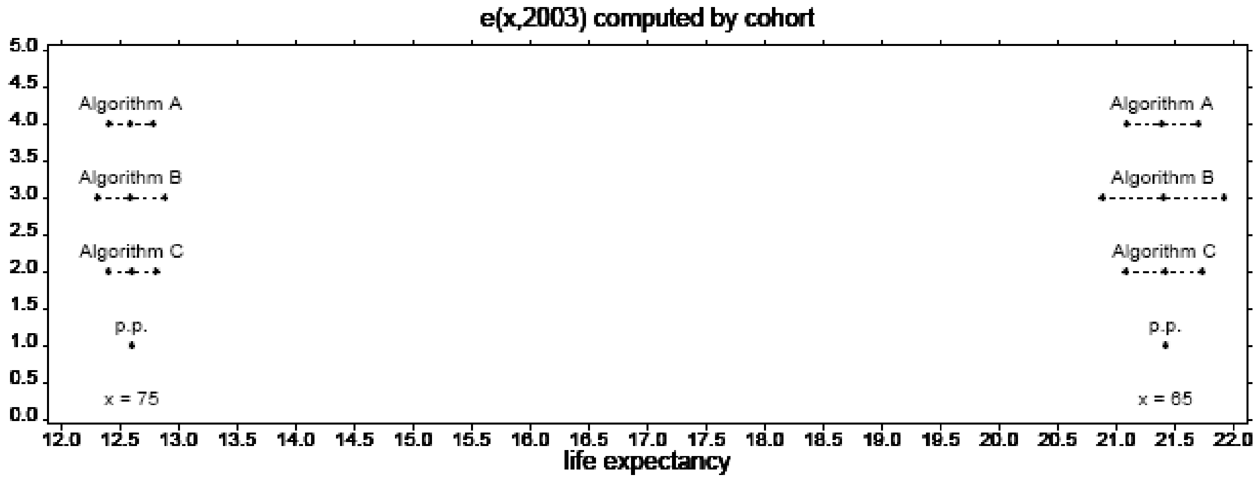
NEGATIVE BINOMIAL MODELLING

Extend Poisson model (with no scale parameter $\varphi = 1$)

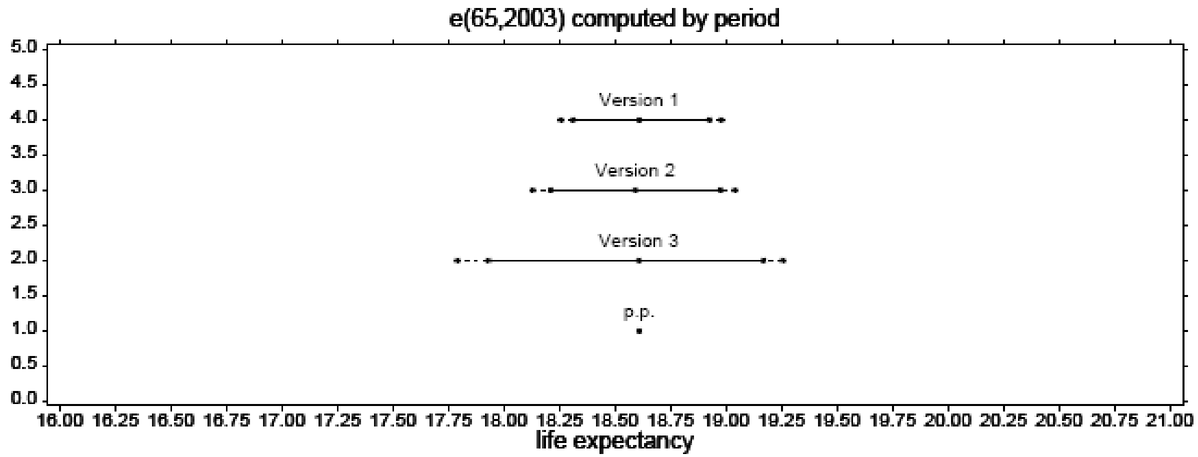
Variance function in GLM becomes:

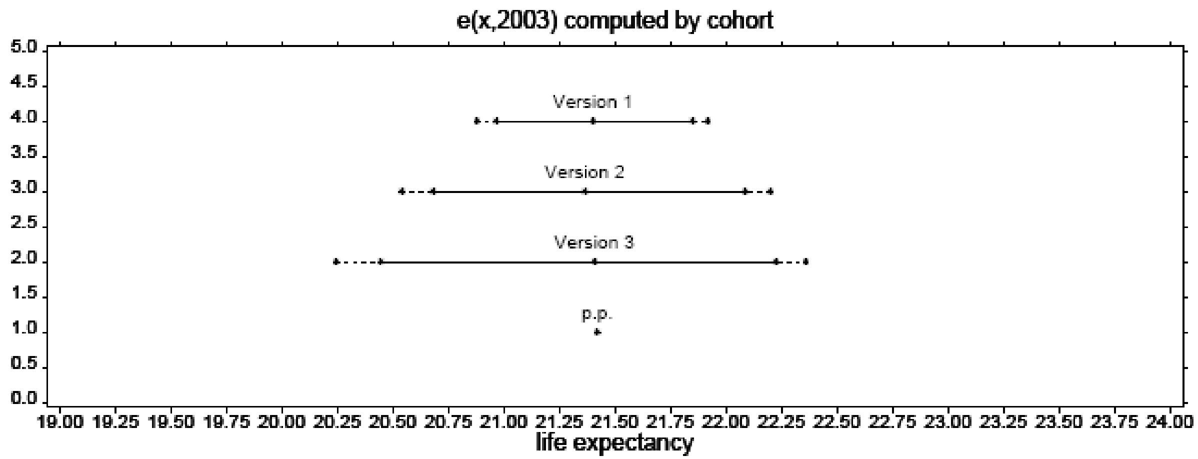
$$V(u) = u + \lambda_x u^2$$



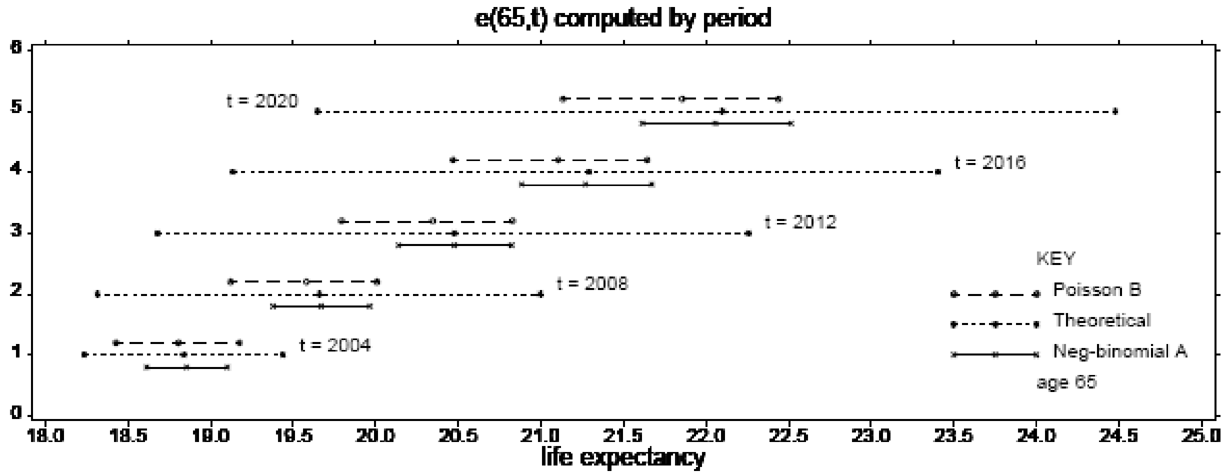


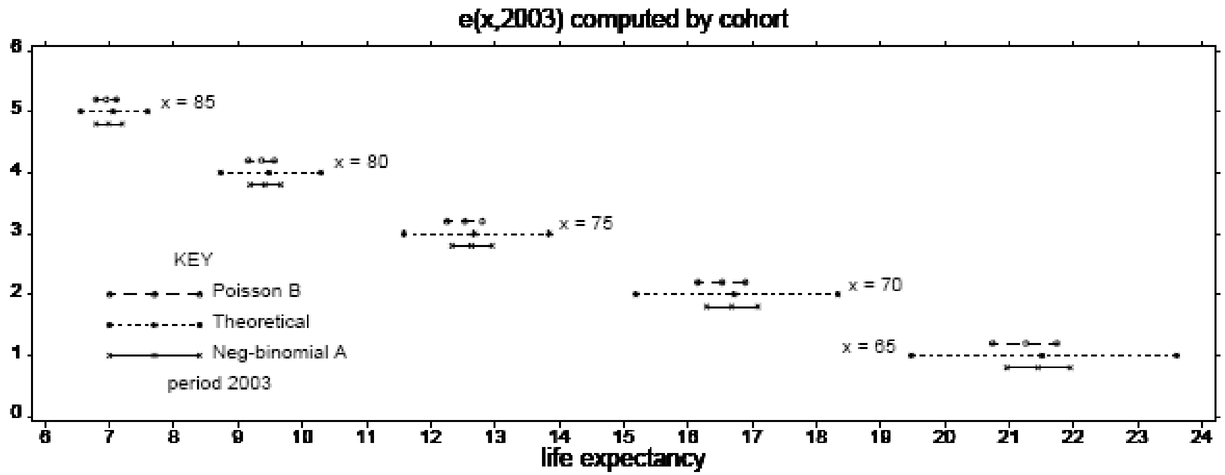
UK 1983-2003 male pensioner mortality experience (ages 51-104).
Poisson LC: age-period (log-bilinear) fitted parametric structure.
Life expectancy: comparison of 2.5, 50, 97.5 precentile based PIs
using three different simulation algorithms, with point predictor (p.p.).



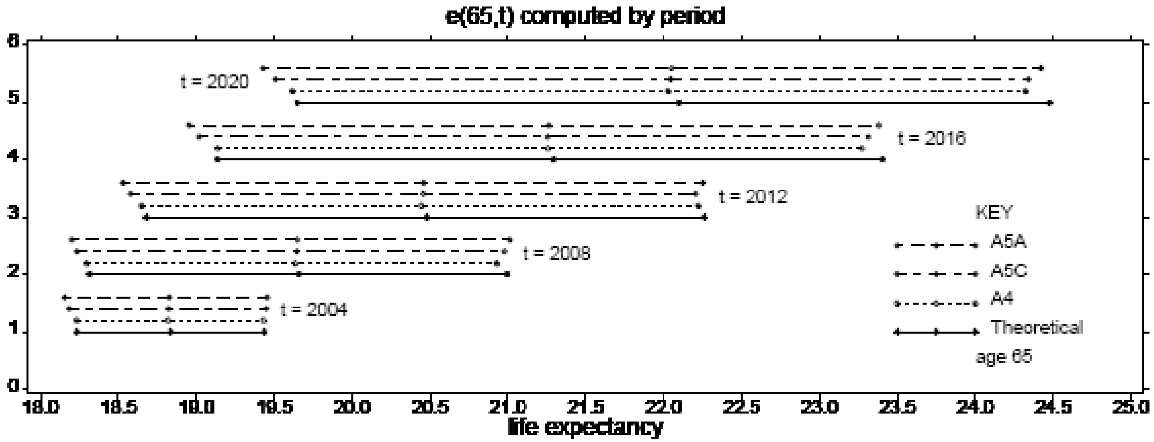


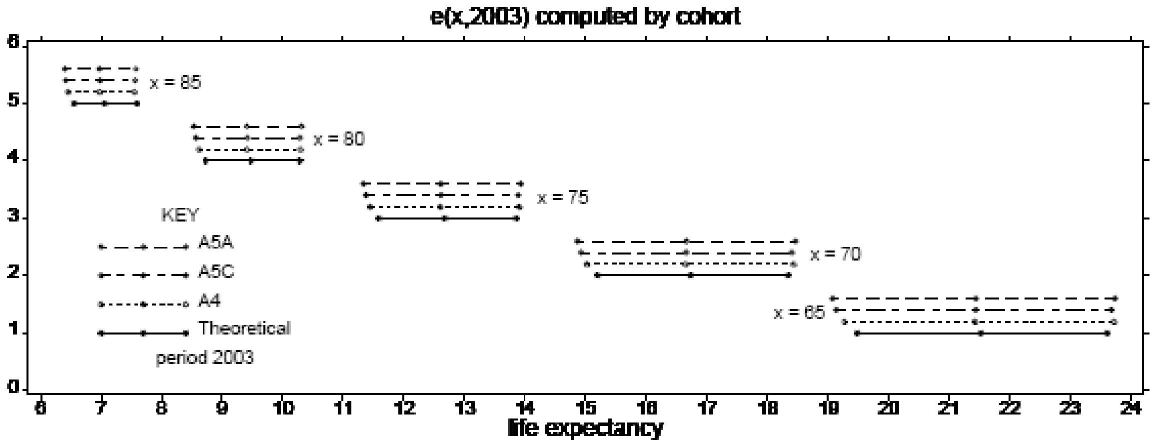
UK 1983-2003 male pensioner mortality experience (ages 51-104).
Poisson LC: age-period (log-bilinear) fitted parametric structure, with and without the inclusion of a free-standing (constant) scale parameter.
Life expectancy: comparison of 2.5, 50, 97.5 precentile based PIs using different versions of simulation Algorithm B, with point predictor (p.p.).



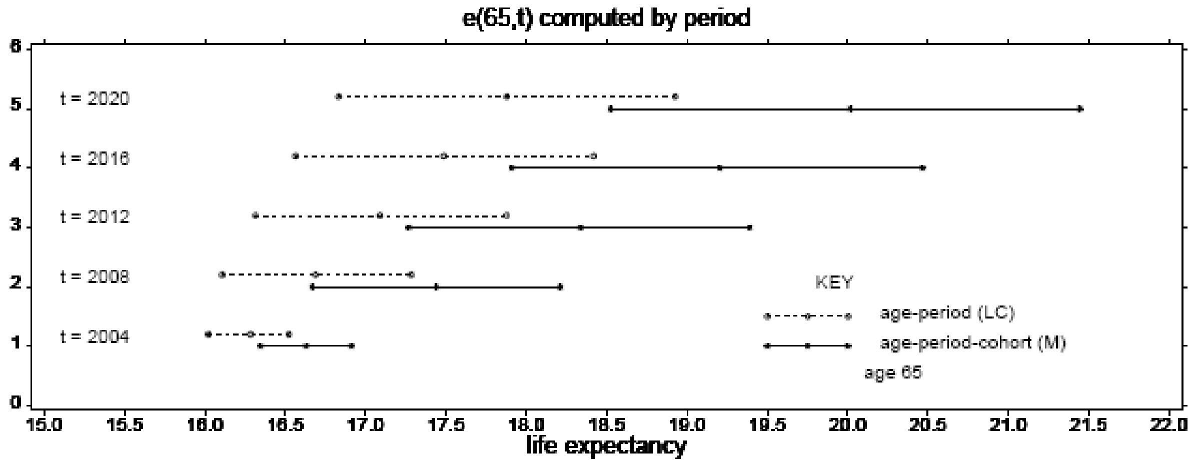


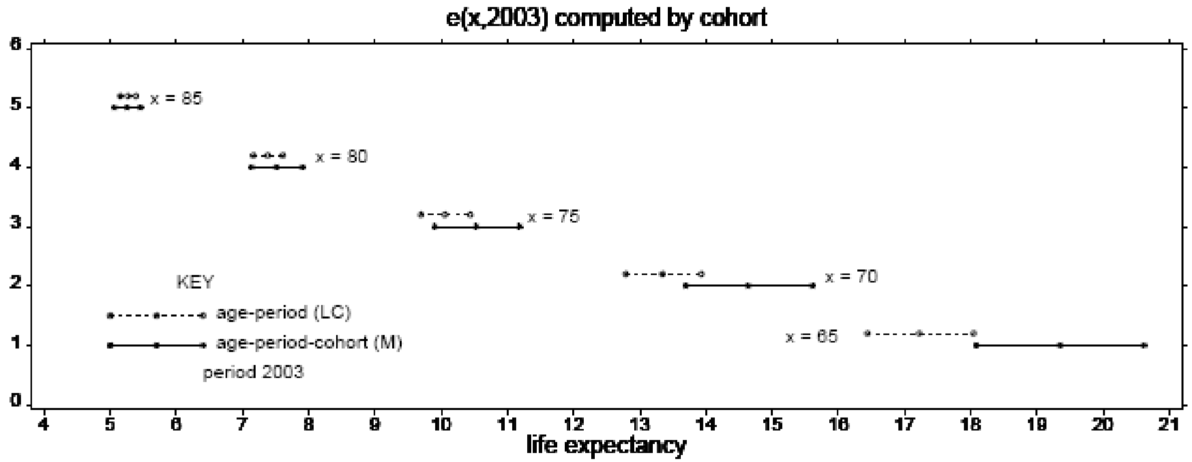
UK 1983-2003 male pensioner mortality experience (ages 51-104).
LC: age-period (bilinear) structure in combination with random walk.
Life expectancy: comparison 2.5, 50, 97.5 precentile based PIs using
(i) Algorithm B, Poisson (version 1). (ii) By theory (Denuit (2007))
(iii) Algorithm A, negative binomial.



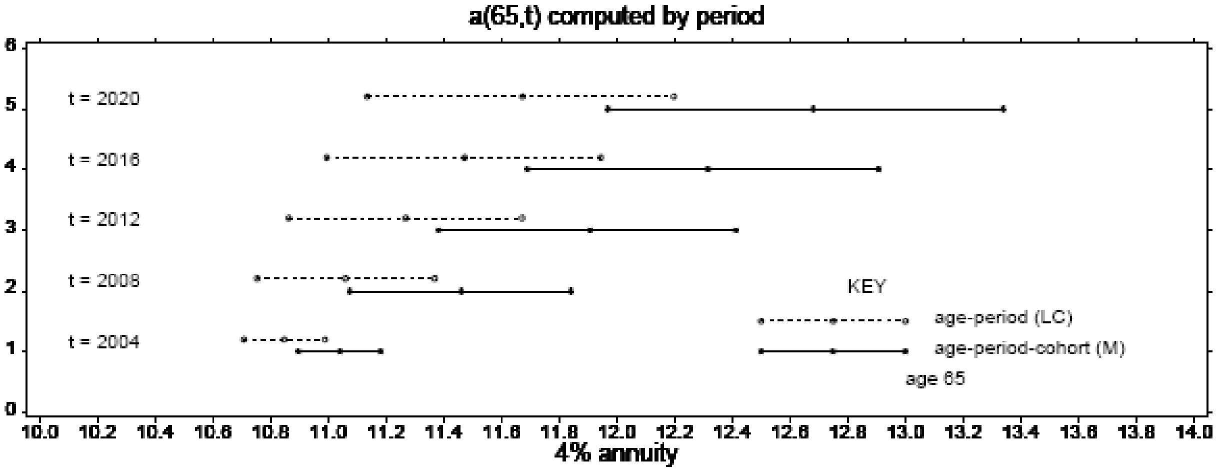


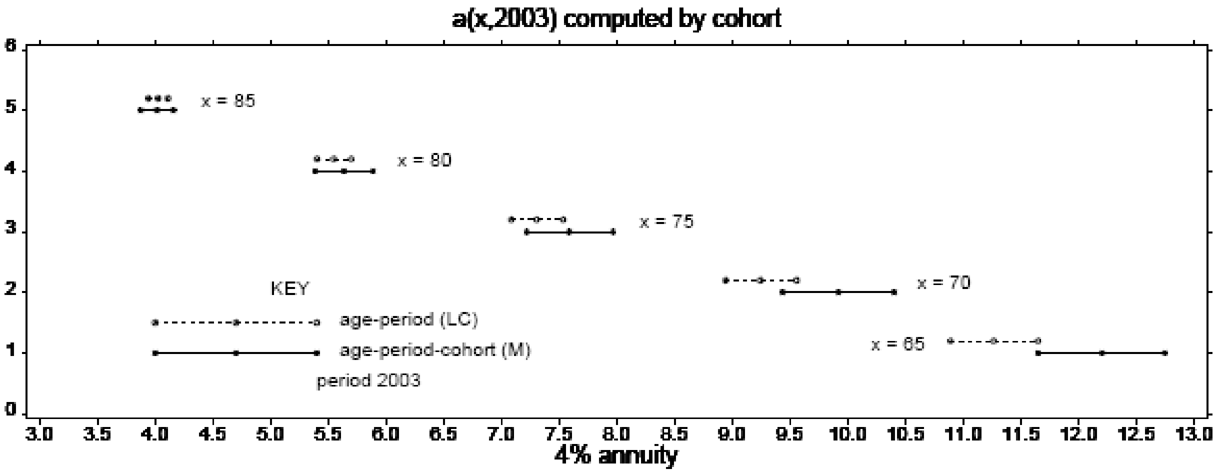
UK 1983-2003 male pensioner mortality experience (ages 51-104).
Poisson LC: age-period (log- bilinear) structure, with period random walk.
Comparison of life expectancy predictions (various age-period start points)
based on 2.5, 50, 97.5 percentiles: (i) By theory. (ii) By bootstrapping
the prediction error in the period component time series (Algorithm A4).
(iii) By bootstrapping the time series prediction error and including
model fitting simulated error (Algorithms A5A or A5C).





E+W male mortality: comparison life expectancy predictions using (i) age-period-cohort and (ii) age-period Poisson structures. Predictions with intervals by bootstrapping the time series prediction error in the period (and cohort) components, and selecting the resulting 2.5, 50, 97.5 percentiles.





E+W male mortality: comparison 4% fixed rate annuity predictions using (i) age-period-cohort and (ii) age-period Poisson structures. Predictions with intervals by bootstrapping the time series prediction error in the period (and cohort) components, and selecting the resulting 2.5, 50, 97.5 percentiles.

FINAL COMMENT

- Other approaches to prediction intervals for Lee-Carter models – Bayesian methods
- Adding simulation techniques to APC model
- Extreme ages – extrapolation methods needed where data are scarce
- Problems with forecasting structural changes
- Time series methods and their application to long forecasting periods

FINAL COMMENT (continued)

- Effect of β_x on smoothness of projected age profiles: need for smoothing of estimates
- Assessing performance of forecasting methods.
- Quality of data sources and appropriateness for particular applications: “basis risk”.
- Model error – essential to investigate more than one modelling framework.
- Sources of uncertainty – process, parameter, model, judgement. <u>Not</u> all sources of uncertainty can be quantified.

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