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A NOTE ON THE EQUATED TIME OF A SERIES OF CASH FLOWS

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ABSTRACT

A well-known result concerning the equated time of a series of cash payments is extended to more general cash flows, including continuous payment streams.

KEYWORDS

Cash Flows; Equated Time; Jensen's Inequality

1. INTRODUCTION

The definition and common rule for the equated time of a finite number of positive cash flows is well known: see, for example, McCutcheon & Scott[1], Example 3.2.3. It is shown in that example that, at any positive force of interest, δ, the (true) equated time, T, is not greater than the approximate equated time, t^*, calculated by the common rule. The purpose of this note is to extend this result to more general cash flows, including continuous payment streams. The proof is based on Jensen's inequality.

2. MORE GENERAL CASH FLOWS

Let us consider more general positive cash flows, and let:

$M(t)$ = the total cash to be received up to (and including) time t (where time is measured in suitable time-units).

$M(t)$ is a non-decreasing right-continuous function, and so generates a Borel measure on $(-\infty, \infty)$, which we shall denote by M. (See, for example, Rudin[2], Chapter 2, for a discussion of Borel measures, and Norberg[3] for a discussion of their applications to payment streams.)

We shall assume that:

$$M(t) \text{ is bounded, and}$$

$$\int_{-\infty}^{\infty} t\,dM < \infty .$$

(These conditions are satisfied if payments cease at some time, and for certain decreasing perpetuities.)

3. The Equated Time

Let δ be the force of interest per time-period; we assume that $\delta > 0$. We define the *true equated time* of the cash flows given above as the time T, such that a cash payment of:

$$\int_{-\infty}^{\infty} dM$$

(i.e., the total cash to be received) at time T has the same present value, at force of interest δ, as the given set of cash flows; that is, T is such that:

$$e^{-\delta T} = \frac{\int_{-\infty}^{\infty} e^{-\delta t} dM}{\int_{-\infty}^{\infty} dM}. \tag{3.1}$$

The *approximate equated time by the common rule* is defined as:

$$t^* = \frac{\int_{-\infty}^{\infty} t dM}{\int_{-\infty}^{\infty} dM}. \tag{3.2}$$

Theorem 3.1 $T \leqslant t^*$.

Proof
Let μ denote the measure:

$$\frac{M}{\int_{-\infty}^{\infty} dM}.$$

Since the function $e^{-\delta t}$ is convex, Jensen's inequality (see Rudin[(2)], Theorem 3.3) shows that:

$$\exp\left(-\delta \int_{-\infty}^{\infty} t d\mu\right) \leqslant \int_{-\infty}^{\infty} e^{-\delta t} d\mu.$$

That is:

$$e^{-\delta t^*} \leqslant e^{-\delta T}$$

from which the theorem follows.

Remark
Suppose that the force of interest may vary. Let it be $\delta(t)$ per unit time at time t from the present, and let us define T by the equation:

$$\exp\left(-\int_0^T \delta(t)dt\right) = \frac{\int_{-\infty}^{\infty} \exp\left(-\int_0^t \delta(r)\,dr\right)dM}{\int_{-\infty}^{\infty} dM}.$$

Theorem 3.1 may be extended to include this case if the function:

$$g(t) = \exp\left(-\int_0^t \delta(r)\,dr\right)$$

is convex, but this is not necessarily so.

4. Practical Applications

In practice we may ignore the 'singular' component of $M(t)$, and so write:

$$M(t) = \sum_{r \leqslant t} c_r + \int_0^t \rho(r)\,dr \tag{4.1}$$

where c_r is the discrete cash flow at time r ($r \geqslant 0$), and $\rho(r)$ is the rate of payment of cash at time r ($r \geqslant 0$); $\rho(t)$ is usually continuous or piecewise continuous. The result of Theorem 3.1 may be stated in the form:

$$T \leqslant t^* = \frac{\Sigma t c_t + \int_0^\infty t\rho(t)\,dt}{\Sigma c_t + \int_0^\infty \rho(t)\,dt} \tag{4.2}$$

where T is such that:

$$e^{-\delta T} = \frac{\Sigma e^{-\delta t} c_t + \int_0^\infty e^{-\delta t}\rho(t)\,dt}{\Sigma c_t + \int_0^\infty \rho(t)\,dt}. \tag{4.3}$$

If there is no continuous payment stream, we have:

$$T \leqslant t^* = \frac{\Sigma t c_t}{\Sigma c_t} \tag{4.4}$$

where:

$$e^{-\delta T} = \frac{\Sigma e^{-\delta t} c_t}{\Sigma c_t}. \tag{4.5}$$

Formula 4.4 includes the 'traditional' result, the case when the number of payments is finite.

If there are no discrete cash flows, we have:

$$T \leqslant t^* = \frac{\int_0^\infty t\rho(t)\,dt}{\int_0^\infty \rho(t)\,dt} \tag{4.6}$$

where:

$$e^{-\delta T} = \frac{\int_0^\infty e^{-\delta t}\rho(t)\,dt}{\int_0^\infty \rho(t)\,dt}. \tag{4.7}$$

5. An Example

Let us consider a continuous payment stream at the rate of £t p.a. at time t years ($0 \leqslant t \leqslant 10$). By formula 4.6:

$$t^* = \frac{\int_0^{10} t^2\,dt}{\int_0^{10} t\,dt} = 6{\cdot}6667$$

so the true equated time, T, is not greater than 6·6667 years.

When $\delta = 0{\cdot}05$, for example, we have:

$$\begin{aligned} e^{-\delta T} &= \frac{\int_0^{10} te^{-\delta t}\,dt}{\int_0^{10} t\,dt} \\ &= \frac{\frac{1}{\delta^2}[1 - e^{-10\delta}(1 + 10\delta)]}{50} \quad \text{(by integration by parts)} \\ &= 0{\cdot}72163. \end{aligned}$$

Hence $T = 6{\cdot}5248$, which is not very much less than t^*.

References

(1) McCutcheon, J. J., & Scott, W. F. (1986). *An introduction to the mathematics of finance*. Heinemann, London.
(2) Rudin, W. (1986). *Real and complex analysis*. 3rd edition. McGraw-Hill, New York.
(3) Norberg, R. (1990). Payment measures, interest and discounting. *Scand. Actuarial J.*, 14 33.