JIA 73 (1947) 0423-0426

A NOTE ON THE GOMPERTZ TABLE

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The object of the present note is to point out that in a Gompertz table the values of l and of d can be stated in a form which makes practical numerical integration and interpolation possible over an important range of the life table.

In this note D is used for d/dx, and λ for $\log_e$. By definition,

$$\mu_x = -\frac{1}{l_x}\mathrm{D}l_x$$

and $$\mathrm{D}l_x = -(\mu l)_x.$$

The negative sign is used so that μ may be a positive quantity, Dl being by its nature negative.

In a Gompertz curve the values of μ are in geometrical progression, and we can write

$$\mu_x = \mathrm{B}c^x = \mu_0 c^x,$$

so that $$\mathrm{D}\mu = \mu\lambda c$$

and
$$\begin{aligned}\mathrm{D}(\mu l) &= l\mathrm{D}\mu + \mu\mathrm{D}l\\ &= l\mu\lambda c - \mu^2 l\\ &= (\mu l)(\lambda c - \mu).\end{aligned}$$

The graph of (μl) may be called the curve of deaths. In tables for good-class lives, excluding deaths by accident and violence, the ordinates of the curve at early ages are very low. They increase with age, and in current tables there is a maximum value at about the age of 80. The position of the maximum has been steadily advancing during the present century, and in tables which represent the mortality of classes leading well-regulated lives there seems to be good reason to expect a further advance. We shall refer to the maximum value as the 'peak' of deaths. The value of μ at the peak is λc, and observation suggests that this value does not differ much from $\frac{1}{12}$. The Gompertz curve can only be employed after the ages of childhood and youth, but it is convenient to think of it as extending indefinitely in both directions.

Returning to the primary definition,

$$\mu_x = -\frac{1}{l_x}\mathrm{D}l_x = -\mathrm{D}\lambda l_x,$$

and therefore $$\lambda l_x = -\int \mu_x dx.$$

But $\mu_x = \mu_0 c^x$, and the integral of μ_x is $\mu_x/\lambda c$. Integrating between the limits $-\infty$ and x, μ_x vanishes when we take $x = -\infty$ and therefore

$$[\lambda l_x]_{-\infty}^{x} = -\frac{\mu_x}{\lambda c}.$$

Without any loss of generality we can take the limiting value of l when $x \to -\infty$ to be unity and we then have

$$\lambda l_x = -\frac{\mu_x}{\lambda c},$$

whence $$l_x = e^{-\mu_x/\lambda c}.$$

If τ represents the period in which μ is doubled,

$$l_{x+\tau} = e^{-2\mu_x/\lambda c} = l_x^2,$$

and therefore $$\frac{l_{x+\tau}}{l_x} = l_x,$$

so that l_x represents the probability of surviving a doubling period.

At the peak, $$\mu = \lambda c, \quad \text{and} \quad l = e^{-\mu/\lambda c} = 1/e = \cdot 368\ldots.$$

The value of l at one doubling period before the peak is

$$e^{-\frac{1}{2}} = \cdot 606\ldots,$$

and at one doubling period after the peak is

$$e^{-2} = \cdot 135\ldots.$$

The deaths in the two doubling periods, one on each side of the peak, are represented by the fraction ·471. They are nearly one-half of the deaths of the whole table, and are almost equally distributed between the two doubling periods, ·238... on the younger side and ·233... on the older side.

At the peak $el = 1$, whatever the value of c may be.

In modern normal tables the doubling period is nearly 8 years.

The peak is at the centre of the greatest concentration of deaths, and the suggestion evidently occurs that we should choose the position of the peak as the origin of co-ordinates. When we do this,

$$\mu_0 = \lambda c = e(\mu l)_0, \quad el_0 = 1.$$

By Maclaurin's theorem,

$$(\mu l)_x = (\mu l)_0 + x\mathrm{D}(\mu l)_0 + \frac{x^2}{2}\mathrm{D}^2(\mu l)_0 + \ldots.$$

Now,
$$\begin{aligned}
\mathrm{D}(\mu l) &= l\mathrm{D}\mu + \mu \mathrm{D}l \\
&= (\mu l)(\lambda c - \mu), \\
\mathrm{D}^2(\mu l) &= (\lambda c - \mu)\mathrm{D}(\mu l) - \mu^2 l\lambda c \\
&= (\mu l)(\lambda c - \mu)^2 - \mu^2 l\lambda c \\
&= (\mu l)\{(\lambda c - \mu)^2 - \mu\lambda c\}, \\
\mathrm{D}^3(\mu l) &= \{(\lambda c - \mu)^2 - \mu\lambda c\}\mathrm{D}(\mu l) + (\mu l)\{-2(\lambda c - \mu)\mathrm{D}\mu - \lambda c\,\mathrm{D}\mu\}.
\end{aligned}$$

Putting $\mu = \lambda c$, and $el = 1$, these expressions become, at the peak,

$$\begin{aligned}
\mathrm{D}(\mu l)_0 &= 0, \\
e\mathrm{D}^2(\mu l)_0 &= -(\lambda c)^3, \\
e\mathrm{D}^3(\mu l)_0 &= -(\lambda c)^4;
\end{aligned}$$

and therefore, including terms up to $\mathrm{D}^3(\mu l)$,

$$e(\mu l)_x = \lambda c\left\{1 - \frac{(x\lambda c)^2}{2} - \frac{(x\lambda c)^3}{6}\ldots\right\}.$$

Applying higher powers of D to (μl) we can find further terms in the expansion of (μl).

Since $Dl = -(\mu l)$, and at the peak $l = 1/e$, we can express l_x in the form

$$l_x = \frac{1}{e}\left\{A_0 + A_1(x\lambda c) + A_2\frac{(x\lambda c)^2}{2!} + A_3\frac{(x\lambda c)^3}{3!} + \ldots + A_r\frac{(x\lambda c)^r}{r!} + \ldots\right\},$$

where $A_0 = 1, \quad A_3 = 1, \quad A_6 = -9,$

$A_1 = -1 \quad A_4 = 1, \quad A_7 = -9,$

$A_2 = 0, \quad A_5 = -2, \quad A_8 = 50,$

....

The first few terms have been found with ease. Some additional terms can be found with care, but the labour becomes forbidding.

There is, however, another method of approach which is numerically much more convenient. I assume the reader to be familiar with the 'abacus' by means of which the powers of x can be expressed in terms of factorials. The abacus is given on p. 55 of Steffensen's *Interpolation*. Lidstone has pointed out to me that it was used by Stirling 200 years ago. I give below the first few rows of the abacus, in which I have inserted a term $=1$, in column (0), row (0), which does not appear in Steffensen's table.

	Col. (0) $x^{(0)}$	Col. (1) $x^{(1)}$	Col. (2) $x^{(2)}$	Col. (3) $x^{(3)}$	Col. (4) $x^{(4)}$	Col. (5) $x^{(5)}$	Col. (6) $x^{(6)}$	Col. (7) $x^{(7)}$	Col. (8) $x^{(8)}$
Row (0)	1	—	—	—	—	—	—	—	—
(1)	—	1	—	—	—	—	—	—	—
(2)	—	1	1	—	—	—	—	—	—
(3)	—	1	3	1	—	—	—	—	—
(4)	—	1	7	6	1	—	—	—	—
(5)	—	1	15	25	10	1	—	—	—
(6)	—	1	31	90	65	15	1	—	—
(7)	—	1	63	301	350	140	21	1	—
(8)	—	1	127	966	1701	1050	266	28	1

$$x^{(c)} = x(x-1)(x-2)\ldots(x-c+1).$$

The abacus is to be read as follows:

$$x = x^{(1)},$$

$$x^{(2)} = x^{(1)} + x^{(2)},$$

$$x^{(3)} = x^{(1)} + 3x^{(2)} + x^{(3)},$$

$$x^{(4)} = x^{(1)} + 7x^{(2)} + 6x^{(3)} + x^{(4)},$$

....

The law of formation of the abacus is indicated in the following scheme:

	Col. $(c-1)$	Col. (c)
Row (r)	L	A
Row $(r+1)$	—	B

$$B = cA + L.$$

In the abacus alter the signs of the alternate columns, as in the following table. Then the sums of the rows give the coefficients $A_0, A_1, A_2, \ldots$, which we require in order to state the expansions of l_x and of $(\mu l)_x$ in powers of x.

	$x^{(0)}$	$x^{(1)}$	$x^{(2)}$	$x^{(3)}$	$x^{(4)}$	$x^{(5)}$	$x^{(6)}$	$x^{(7)}$	$x^{(8)}$	Values of A
Row (0)	1	—	—	—	—	—	—	—	—	1
(1)	—	−1	—	—	—	—	—	—	—	−1
(2)	—	−1	1	—	—	—	—	—	—	0
(3)	—	−1	3	−1	—	—	—	—	—	1
(4)	—	−1	7	−6	1	—	—	—	—	1
(5)	—	−1	15	−25	10	−1	—	—	—	−2
(6)	—	−1	31	−90	65	−15	1	—	—	−9
(7)	—	−1	63	−301	350	−140	21	−1	—	−9
(8)	—	−1	127	−966	1701	−1050	266	−28	1	50

This result is so simple that I feel it ought to be possible to explain it briefly and simply; but any proof I can offer at present involves a lengthy disquisition on the properties of the abacus. The question was discussed in a long correspondence between Lidstone and myself several years ago. After the destruction of my own records by operations of war, he returned to me (in 1941) many of my notes with the expression of a hope that they might be arranged for publication. It is from the notes so returned that I have found material for the present contribution.