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A NOTE ON LIFE TABLE AND MULTIPLE-DECREMENT TABLE FUNCTIONS

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ABSTRACT

This note considers the mortality table functions, and shows that the usual formulae hold under less restrictive assumptions than those usually made. The foundations of the theory of multiple-decrement tables are also considered, in the context of probability theory.

KEYWORDS

Life Tables; Decrement Tables.

1. INTRODUCTION

The properties of the mortality table functions are usually derived under the assumption that l'_x exists and is continuous (see Neill, Section 1.6). We shall show that the usual formulae hold under less restrictive assumptions, viz. that l_x and μ_x exist and are continuous. The mathematical basis for this approach is similar to that used in connection with the force of interest in McCutcheon & Scott, Section 2.4. We also consider the foundations of the theory of multiple-decrement tables, which we attempt to place in the context of probability theory. In particular, we prove the 'identity of the forces' (see Hooker & Longley-Cook, Section 20.7).

2. SOME MATHEMATICAL RESULTS

Theorem 2.1

Let $I=[A,B)$ or $(-\infty, B)$, where B may be ∞. Suppose that $f(x)$ is continuous on I and that its right-hand derivative, $f'_+(x)$, is zero on I. Then $f(x)$ is constant.

Proof (Adapted from Hobson, p. 365)

Let us consider the case when $I=(-\infty, B)$; the case when $I=[A,B)$ is similarly dealt with. If the conclusion of the theorem be false, there must be a and b in I, with $a<b$, such that $f(a)<f(b)$ or $f(a)>f(b)$.

Let us suppose that $f(a)<f(b)$, the case when $f(a)>f(b)$ being treated by a similar argument. For each k, let:

$$\phi(x,k)=f(x)-f(a)-k(x-a)$$

which is continuous on I. Note that $\phi(a,k)=0$. Let k be positive but so small that $\phi(b,k)=2q>0$.

Let $M=\{x: a\leq x\leq b \text{ and } \phi(x,k)\leq q\}$ and $\xi=\sup M$, which is clearly such that

$a < \xi < b$. Let $x_n \to \xi -$. We have $\phi(x_n,k) \leq q$, which shows that $\phi(\xi,k) \leq q$. Now let $x_n \to \xi +$. Since $\phi(x_n,k) > q$ for all n, $\phi(\xi,k) \geq q$, and so $\phi(\xi,k) = q$. Also:

$$\phi'_+(\xi,k) = \lim_{n \to \infty} \frac{\phi(x_n,k) - \phi(\xi,k)}{x_n - \xi} \geq 0.$$

But this contradicts the fact that $\phi'_+(x,k) = -k < 0$ for all x in I. Q.E.D.

This theorem leads easily to the following result, which is used in McCutcheon & Scott, Appendix 1.

Theorem 2.2

Let I be as in Theorem 2.1. Suppose that $f(x)$ and $f'_+(x)$ are continuous on I. Then $f(x)$ is differentiable on I.

Proof

For each x in I, let:

$$g(x) = f(x) - \int_a^x f'_+(y)dy$$

where a is any number in I. An application of Theorem 2.1 shows that $g(x)$ is constant, and is therefore equal to $f(a)$. Hence:

$$f(x) = f(a) + \int_a^x f'_+(y)dy$$

for all x in I, so $f(x)$ is differentiable. (Its derivative is, of course, equal to $f'_+(x)$.) Q.E.D.

3. APPLICATION TO LIFE TABLES

For $\alpha \leq x < \omega$, let:

$$s(x) = \text{the probability that a life aged } \alpha \text{ survives to age } x \tag{3.1}$$

which is usually called the 'survivorship function'. For simplicity, let us suppose from now on that the limiting age, ω, is ∞; the case when $\omega < \infty$ requires minor modifications. We may also write:

$$l_x = l_\alpha\, s(x) \tag{3.2}$$

where l_α is the *radix* of the life table (generally a large number such as 100,000.) Let $s(x)$ be positive and continuous. (In some applications, generally not to life tables, but to tables of retirements, etc., $s(x)$ may be discontinuous at certain points. In such cases we may apply the techniques given here over each section of the table, making special allowances at the discontinuities.)

For $\alpha \leq x \leq x + t$, let:

$${}_tp_x = \Pr\{\text{a life aged } x \text{ survives to age } x+t\}. \tag{3.3}$$

We also define:

$$\begin{aligned} {}_tq_x &= 1 - {}_tp_x \\ &= \Pr\{\text{a life aged } x \text{ does not survive to age } x + t\}. \end{aligned} \tag{3.4}$$

(If $t = 1$, we may write ${}_tq_x = q_x$ and ${}_tp_x = p_x$.) We assume that, for $x \geq \alpha$:

$$\begin{aligned} \mu_x &= \lim_{h \to 0+} \frac{\Pr\{\text{a life aged } x \text{ dies within time } h\}}{h} \\ &= \lim_{h \to 0+} \frac{{}_hq_x}{h} \end{aligned} \tag{3.5}$$

exists, and is a continuous function on $[\alpha, \infty)$. We also assume that, for $\alpha \leq x_1 \leq x_2 \leq x_3$:

$${}_{x_3 - x_1}p_{x_1} = {}_{x_2 - x_1}p_{x_1} \cdot {}_{x_3 - x_2}p_{x_2}. \tag{3.6}$$

It follows on, setting $x_1 = \alpha$, $x_2 = x$ and $x_3 = x + t$, that, for each x, t such that $\alpha \leq x \leq x + t$:

$${}_tp_x = \frac{s(x + t)}{s(x)} = \frac{l_{x+t}}{l_x}. \tag{3.7}$$

Theorem 3.1

For each x and t such that $\alpha \leq x \leq x + t$:

$${}_tp_x = \exp\left[-\int_0^t \mu_{x+s}\, ds \right]. \tag{3.8}$$

Proof

By (3.5) and (3.7):

$$\begin{aligned} \mu_x &= -\lim_{h \to 0+} \left[\frac{s(x + h) - s(x)}{h\, s(x)} \right] \\ &= \frac{-s'_+(x)}{s(x)}. \end{aligned}$$

Consequently, $s(x)$ and $s'_+(x)$ are continuous on $[\alpha, \infty)$, and Theorem 2.2 shows that $s'(x)$ exists and equals $s'_+(x)$. It follows that:

$$s'(x) = -s(x)\mu_x \qquad (x \geq \alpha)$$

which may easily be solved. Using the initial condition, $s(\alpha) = 1$, we have:

$$s(x) = \exp\left[-\int_\alpha^x \mu_y\, dy \right]$$

Formula 3.8 now follows from this and formula 3.7. Q.E.D.

It now follows easily (on substituting $u = \int_0^t \mu_{x+s}\, ds$) that:

$$_hq_x = \int_0^h {}_tp_x \mu_{x+t}\, dt. \tag{3.9}$$

Now let x and t be fixed, with $\alpha \leq x < x+t$. As in Scott, lemma 1.1, we have:

$$_hq_y = \mu_y h + o(h) \tag{3.10}$$

uniformly for $x \leq y < y+h \leq x+t$.

Example 3.1 (Gompertz' law)

Let us assume that, for $x \geq 25$, say, $s(x)$ is continuous and that:

$$\Pr\{\text{a life aged } x \text{ dies within time } h\} = Bc^x\, h + o(h).$$

This implies that:

$$\mu_x = Bc^x$$

i.e. Gompertz' law holds for $x \geq 25$, and μ_x is clearly continuous. If we also suppose that formula 3.6 holds, Theorem 3.1 shows that, for $x \geq 25$ and $t \geq 0$:

$$\begin{aligned} {}_tp_x &= \exp\left[-\int_0^t Bc^{x+s} ds\right] \\ &= \exp\left[-Bc^x(c^t - 1)/\log c\right]. \end{aligned}$$

4. THE MULTIPLE-DECREMENT TABLE

Let us consider only two modes of decrement, β and γ, and suppose that we are mainly interested in mode β. (If there are more than two modes, all except mode β may be combined.) It is supposed that each mode of decrement has the 'life table' functions μ_x^β, ${}_tp_x^\beta$, etc., as in Section 3. Lives are considered to continue in existence with respect to a given mode after exit by the other mode of decrement. To avoid philosophical problems when one of the modes of decrement is death, one may, for example, take mode β as exit by marriage and mode γ as exit by withdrawal from service among the bachelor employees of a large organisation, mortality being ignored.

Let T_1, T_2 denote the times to exit by modes β and γ respectively for a life aged x. It follows by formula 3.9 that T_1, T_2 have probability density functions ${}_{t_1}p_x^\beta \mu_{x+t_1}^\beta$ $(t_1 > 0)$ and ${}_{t_2}p_x^\gamma \mu_{x+t_2}^\gamma$ $(t_2 > 0)$ respectively. We assume that T_1 and T_2 are independent, and define:

${}_{t}(ap)_x =$ the probability that a life aged x, subject to both modes of decrement, survives to age $x+t$

$$= {}_tp_x^{\beta} \cdot {}_tp_x^{\gamma} \tag{4.1}$$

by the independence of T_1 and T_2. (If $t=1$, we may omit it.) We also define:

${}_{t}(aq)_x^{\beta} =$ the probability that a life aged x, subject to both modes of decrement, will exit by mode β before age $x+t$, exit by mode γ not having previously occurred

$$= \Pr\{T_1 \leq t \text{ and } T_1 \leq T_2\}$$

$$= \iint\limits_{\substack{t_1 \leq t_2 \\ \text{and } t_1 \leq t}} {}_{t_1}p_x^{\beta}\mu_{x+t_1}^{\beta} \cdot {}_{t_2}p_x^{\gamma}\mu_{x+t_2}^{\gamma} dt_2 dt_1$$

$$= \int_0^t {}_{t_1}p_x^{\beta}\mu_{x+t_1}^{\beta} \cdot \int_{t_1}^{\infty} {}_{t_2}p_x^{\gamma}\mu_{x+t_2}^{\gamma} dt_2 dt_1$$

$$= \int_0^t {}_{t_1}p_x^{\beta}\mu_{x+t_1}^{\beta} \cdot {}_{t_1}p_x^{\gamma}\, dt_1. \tag{4.2}$$

Let x and t be fixed, with $\alpha \leq x \leq x+t$. It follows, as in Scott, lemma 1.1, that:

$$ {}_{h}(aq)_y^{\beta} = \mu_y^{\beta}h + o(h) \tag{4.3}$$

uniformly for $x \leq y \leq y+h \leq x+t$. Incidentally, this proves the result known as the 'identity of the forces', viz.:

$$(a\mu)_x^{\beta} = \lim_{h \to 0+} \frac{{}_{h}(aq)_x^{\beta}}{h} = \mu_x^{\beta}. \tag{4.4}$$

(See Hooker & Longley-Cook, Section 20.7.)

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