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OPTION PRICING MODELS

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ABSTRACT

The paper discusses two important models of option pricing: the binomial model and the Black-Scholes model. It begins with a brief description of options.

1. OPTIONS

An *option* contract gives the right but not the obligation to buy or sell an underlying security at a fixed price (the *exercise price* or *strike price*) at or before a specific date (the *maturity date* or *expiry date*). A *call* option gives the right to *buy* the security while a *put* option gives the right to *sell* the security. In order to give effect to the right to buy or sell the option has to be *exercised.* A *European* option can only be exercised on the expiry date whereas an *American* option can be exercised at any time before the expiry date. In return for the insurance offered by the option, a price (i.e. *option premium*) has to be paid.

Table 1. *Premiums on ABC options (April 1)*

Exercise	Calls			Puts		
Price	June	Sept.	Dec.	June	Sept.	Dec.
105p	12	22	31	6	11	15
115p	9	17	22	12	16	20
125p	3	4	5	21	26	30

On April 1, ABC was trading at 115p per share; the ABC option trades in units of 1000 shares.

Table 1 gives the premiums payable on both call and put options on ABC shares trading on the International Stock Exchange in London. There are three expiry months and three exercise prices.

On April 1, ABC was trading at 115p per share. At the same time, the call option with an exercise price of 115p and an expiry month of June is trading at 9p. Clearly no one will purchase this option and exercise it immediately because the exercise price is equal to the current share price and so the option has no *intrinsic value*. Nevertheless, the buyer still has to pay 9p for such an option. This is because there is a chance that the share price will rise above the exercise price before the expiry date and the option will acquire intrinsic value; indeed if the share price rises above 124p, the buyer can exercise the option and make a profit. The option buyer is prepared to pay something for this possibility and this component of the option premium is called the *time value* of the option.

The option premium therefore has two components

$$\text{Option premium} = \text{Intrinsic value} + \text{Time value.} \tag{1}$$

For a call option, where the share price is trading either at or below the exercise price (i.e. an *at-the-money* or an *out-of-the-money* option), the intrinsic value is zero and the option has only time value. For a call option where the share price is trading above the exercise price (i.e. an *in-the-money* option), the intrinsic value equals

$$\text{Intrinsic value} = \text{Share price} - \text{Exercise price.} \tag{2}$$

Figure 1 shows the profit and loss (P/L) profile for the buyer of a June call option on ABC (i.e. a *long* call option) with an exercise price of 125p when ABC is currently trading at 115p and the premium on the option is 3p. The short-run P/L profile passes through the current share price of 115p. If the share price rose to 135p, the short-run profit on the option would be AC, of which AB constitutes time-value and BC constitutes intrinsic value. Over time, the time-value of the option declines and eventually, at expiration, vanishes. If the share price stood at 135p when the option expired in June, the time-value of the profit on the option would be zero (AB = 0), but the option could be exercised to give an intrinsic-value profit of BC. The premium on the option at expiration would therefore be

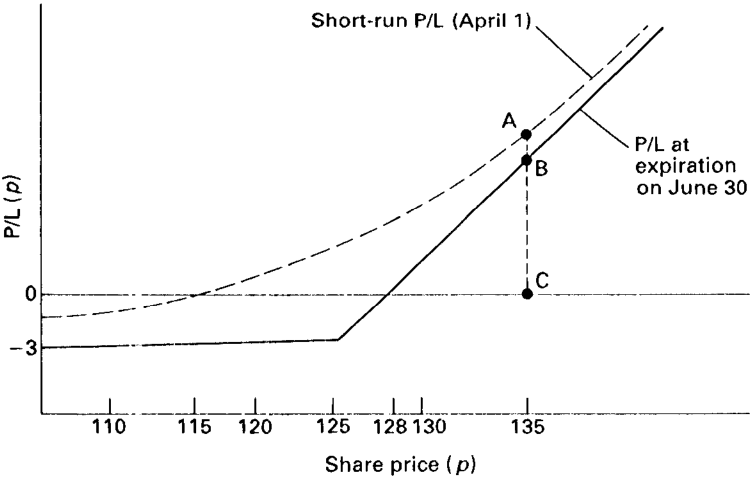

Figure 1. Profit and loss profile of the buyer of a call option.

BC+3p. If, at expiry, the share price was at or below 125p, the option would expire worthless (i.e. without either time value or intrinsic value). In this case, the loss would be 3p. The share price has to be above 128p before any profits are made at expiry.

When a call option is sold or *written*, the seller (or writer) of the option gives the right to purchase the shares to the buyer in return for receiving the premium. Figure 2 illustrates the profit and loss profile to the seller of the June call option on ABC (i.e. a *short* call option) discussed in Figure 1. It is a mirror image of the call P/L profile about the horizontal zero-profit axis. The maximum profit that the seller can make is the premium. He will make some profit if the price of the share at expiration is less than 128p. But he will make losses, possibly without limit, if the share price moves above 128p.

Figure 3 illustrates the profit and loss profile to the buyer of an in-the-money June put option (i.e. a *long* put option) with an exercise price of 125p when the share price is trading at 115p. The premium on the put option is 21p which comprises 10p in intrinsic value (the difference between the exercise price and the lower share price: it is worth at least 10p to have the right to sell for 125p an ABC share that can be purchased in the market place for 115p) and 11p in time value. The buyer will make a profit at expiration if the share price falls below 104p.

The profit and loss profile for the seller of the put option (i.e. a *short* put option) considered in Figure 3 is given in Figure 4. The maximum gain is the 21p premium, while the maximum potential loss is 104p, equal to the exercise price less the premium.

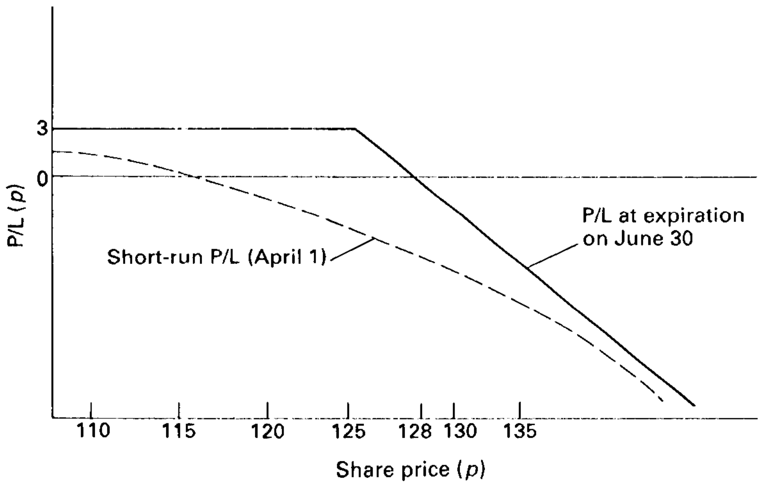

Figure 2. Profit and loss profile of the seller of a call option.

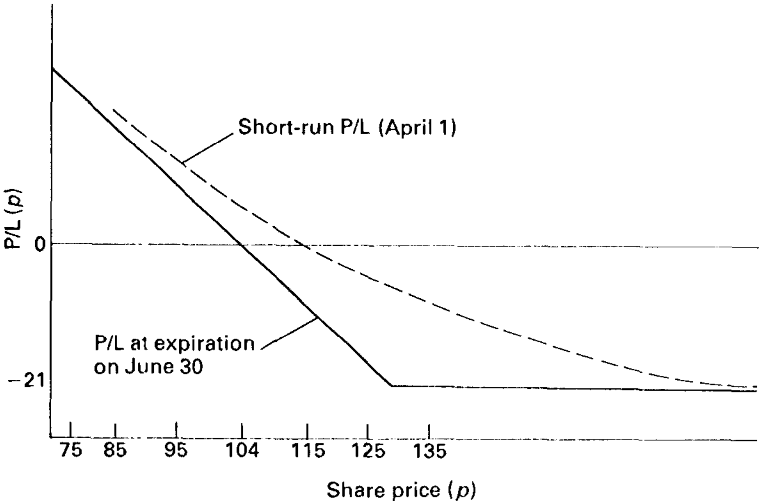

Figure 3. Profit and loss profile of the buyer of a put option.

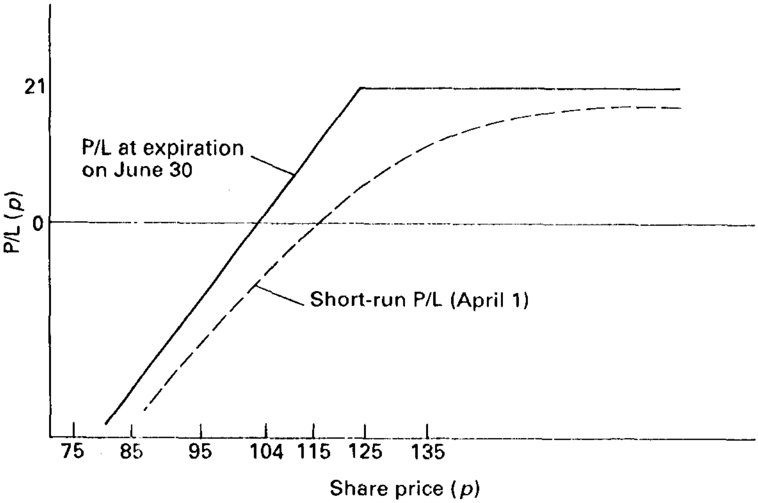

Figure 4. Profit and loss profile of the seller of a put option.

2. FACTORS INFLUENCING THE PRICE OF AN OPTION

There are five factors that influence the price or premium at which an option contract is traded. We will consider these factors for a European call option, which was the first type of option contract to be priced theoretically. The premium for a European call option depends on

$$\begin{array}{l} P^C = F(P^S, \quad X, \quad T, \quad r, \quad \sigma^2) \\ \qquad\quad (+)\ (-)\ (+)\ (+)\ (+) \end{array} \tag{3}$$

where
P^C = premium on a European call option
P^S = spot price of the underlying security
X = exercise price
T = time to expiry
r = riskless rate of interest
σ^2 = variance (or volatility) of the price of the underlying security.

It is clear that the higher the share price the more valuable the option, so that P^C in (3) will be positively related to P^S.

Similarly, the lower the exercise price, the more valuable the option, so that P^C will be negatively related to X.

The greater the time to maturity, the greater the time-value of the option, so that P^C will be positively related to T.

An increase in the riskless rate of interest increases the value of the option because the money saved by purchasing the option rather than the underlying security can be invested at the riskless rate of interest until the option expires. An increase in r increases the attractiveness of holding the option relative to the underlying security and hence raises the option premium.

An increase in the variance of the share price increases the value of the option. This is because an increase in variance increases the chance that the share price will lie in the tails of the distribution of the share price when the option expires. However, only the distribution of the share price above the exercise price is relevant for option pricing. This is because the maximum loss on the option is the premium paid whereas the potential gains on the option are unlimited. Therefore an increase in the chance of the share price lying in the tail of the distribution above the exercise price increases the value of the option while the same increase in the chance of the share price lying in the tail of the distribution below the exercise price does not decrease the value of the option symmetrically. Consequently P^C in (3) is positively related to σ^2. This is demonstrated in Figure 5 which shows that the proportion of the distribution with the high variance lying below the exercise price is greater than with the distribution with the low variance. Since the option buyer is paying a premium to lose the distribution below the exercise price, the greater the proportion of the distribution lost, the greater the premium. The proposition can also be demonstrated using a simple

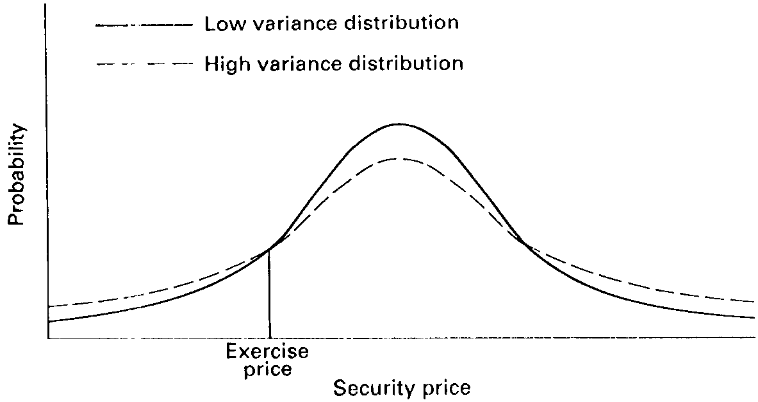

Figure 5. The effect of increasing variance on the option premium.

Table 2. *Option payoffs in different states*

State of the world	Probability	Low variance share: Share price at expiry	Low variance share: Option payoff if $X=45$p	High variance share: Share price at expiry	High variance share: Option payoff if $X=45$p
Slump	·2	40p	0p	20p	0p
Normal	·6	50p	5p	50p	5p
Boom	·2	60p	15p	80p	35p

arithmetic example.

Consider Table 2 which shows the payoffs to call options (with an exercise price of 45p) on a low variance share and a high variance share in three states of the world. The option payoff from the high variance share is at least as great in all states of the world as, and greater in at least one state of the world than, the option payoff from the low variance share. This is a definition of *stochastic dominance* and, if one option payoff pattern stochastically dominates that of another, the option will have a higher value.

3. BOUNDARY CONDITIONS FOR THE PRICE OF AN OPTION

Before deriving the exact formula for pricing options, it will be useful to derive some boundary conditions for the option premium. This time we will consider the boundary conditions for an American call option.

The option must have a positive value throughout its life because there is always some chance, however small, that it will expire in-the-money. Therefore it must be the case that

$$P^C \geqslant 0. \tag{4}$$

An American call option must have a value at least as high as its intrinsic value. If this was not the case, then someone could buy the option, exercise it immediately and earn a riskless arbitrage profit. Therefore it must be the case that

$$P^C \geqslant P^S - X. \tag{5}$$

A call option cannot be worth more than the underlying security, i.e. no one is going to pay more for the option on a security than the security itself. Therefore

$$P^C \leqslant P^S. \tag{6}$$

A call option with a low exercise price will be worth more than a call option with a high exercise price

$$P^C(X_1) > P^C(X_2) \qquad \text{if} \qquad X_2 > X_1. \tag{7}$$

This follows from (5).

A call option with a greater time to expiry will be worth more than a call option with a shorter time to expiry

$$P^C(T_1) > P^C(T_2) \qquad \text{if} \qquad T_1 > T_2. \tag{8}$$

This again follows from (5), where the right hand side of (5) is the (intrinsic) value of the shorter (T_2) option at expiry and the left hand side is the value of the longer (T_1) option.

The value of a call option will always be greater than the value of the security price minus the discounted value of the exercise price

$$P^C \geqslant P^S - Xe^{-rT}. \tag{9}$$

This is a stronger condition than (5) and is proved by stochastic dominance. Consider two portfolios A and B. Portfolio A consists of a share with current price P^S. Portfolio B consists of a European call option on the share with current premium P^C and exercise price X plus a riskless pure discount bond paying X on the expiry date (T) of the option. The current price of the bond is Xe^{-rT}. The payoffs to the two portfolios at the expiry date of the option depends on whether the share price at expiry (P^S_T) is less than the exercise price (X) in which case the option expires worthless, or exceeds the exercise price in which case the option is worth the intrinsic value $P^S_T - X$. The current value of the two portfolios are V^A and V^B and the expiry values are V^A_T and V^B_T.

Portfolio	Current value	Value at expiry $P < X$	Value at expiry $P^S_T \geqslant X$
A	$V^A = P^S$	$V^A_T = P^S_T$	$V^A_T = P^S_T$
B	$V^B = P^C + Xe^{-rT}$	$V^B_T = 0 + X$	$V^B_T = (P^S_T - X) + X$
		$V^B_T > V^A_T$	$V^B_T = V^A_T$

Because portfolio B stochastically dominates portfolio A at expiry, its current value must exceed that of A. Hence

$$P^{C} + Xe^{-rT} \geqslant P^{S}$$

or

$$P^{C} \geqslant P^{S} - Xe^{-rT}. \tag{10}$$

Combining (4) and (10) implies that

$$P^{C} \geqslant \mathrm{Max}(0,\ P^{S} - Xe^{-rT}) \tag{11}$$

i.e. that the value of a European call option cannot be less than the current security price minus the discounted value of the exercise price. In addition, an American call option must be worth more than a European call option because it can be exercised at any time and not just at maturity. Therefore

$$P^{CA} \geqslant P^{CE} \geqslant \mathrm{Max}(0,\ P^{S} - Xe^{-rT}) \tag{12}$$

where P^{CA} is the price of an American call and P^{CE} is the price of a European call. There are two implications of (12).

One of the implications of (12) is that an American call option (on a non-dividend paying share) will never be exercised before maturity. This is because

$$P^{C} \geqslant P^{S} - Xe^{-rT} > P^{S} - X \tag{13}$$

for $r, T > 0$. Since $P^{S} - X$ is all that would be received if the option was exercised, (13) suggests that it would be better to sell the option than to exercise it.

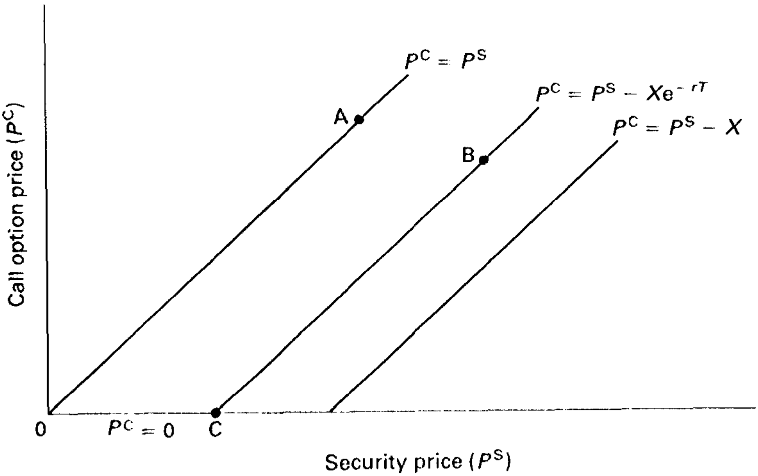

Figure 6. Boundary conditions for a call option.

The second implication of (12) concerns the risk-free rate. If the risk-free rate increases, the discounted value of the exercise price falls and this increases the value of the option.

Figure 6 shows the boundary conditions for a call option premium. The option premium will lie in the area OABC.

4. THE BINOMIAL MODEL OF THE PRICING OF A EUROPEAN CALL OPTION

The simplest model for determining the price of an option is the *binomial model* developed by Cox, Ross and Rubinstein (1979), and the simplest type of option to price is the European call option on a security that makes no cash payments (e.g. a non-dividend paying share).

The binomial model assumes a discrete-time one-period world in which the security price follows a stationary binomial stochastic process so that at the end of the period it can be higher or lower than at the start of the period, as shown in Figure 7. We assume the following notation

P^S = current security price (e.g. 50p)
q = probability that security price will rise (e.g. ·5)
$1-q$ = probability that security price will fall (e.g. ·5)
r = risk-free rate of interest (e.g. $r=\cdot 1$)
u = multiplicative upward movement in security price, $u>1+r>1$ (e.g. 1·3)
d = multiplicative downward movement in security price, $d<1$ (e.g. ·7).

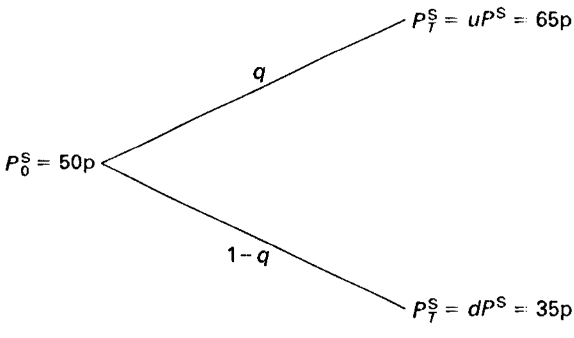

Figure 7. A one-period binomial stochastic process for a security price.

With these assumptions, the security price will increase to uP^S (i.e. 65p) with probability $q=\cdot 5$, or decrease to dP^S (i.e. 35p) with probability $1-q=\cdot 5$. It is necessary that $u > (1+r) > 1 > d$ otherwise there would be opportunities for riskless arbitrage.

Now we consider a call option on the security with an exercise price of 45p. It is clear from Figure 8 that the expiry value of the option has to be either $P_u^C = \text{Max}(0, uP^S - X) = 20\text{p}$ or $P_d^C = \text{Max}(0, dP^S - X) = 0$ and there is a 50% probability that it could be either. What is the value of the call at the beginning of the period?

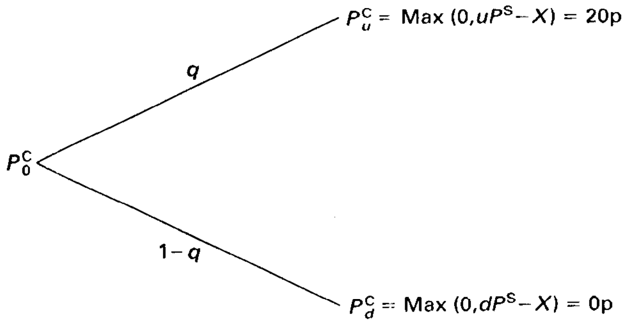

Figure 8. Expiry value of a one-period call option.

To answer this question, we need to examine the return on a *riskless hedge portfolio* constructed from a long position in the underlying security and a short position in h units of the call option (where h is the *hedge ratio*). The value of the riskless hedge is given by

$$V^H = P^S - hP^C. \tag{14}$$

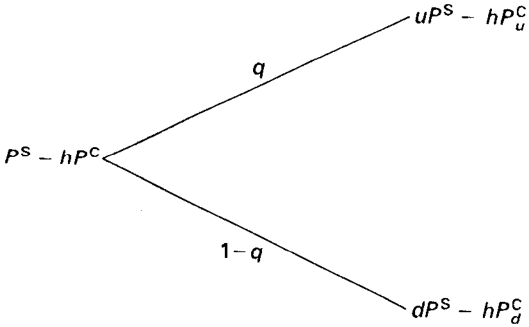

Figure 9. Expiry value of riskless hedge portfolio.

Figure 9 shows the expiry value of the riskless hedge portfolio. We can use the fact that a riskless hedge portfolio must have the same terminal value in all states

$$uP^{S} - hP^{C}_{u} = dP^{S} - hP^{C}_{d} \tag{15}$$

in order to determine the appropriate hedge ratio, h, i.e. the number of call options to be written against the underlying security

$$\begin{aligned} h &= \frac{P^{S}(u-d)}{P^{C}_{u} - P^{C}_{d}} \\ &= \frac{50(1{\cdot}3 - {\cdot}7)}{20 - 0} \\ &= 1{\cdot}5. \end{aligned} \tag{16}$$

A riskless hedge portfolio is constructed from one security and 1·5 short call options. Substituting $h = 1{\cdot}5$ into (15) implies that the riskless hedge portfolio has a terminal value of 35p in all states of the world, so that any increase in the value of the share component is always exactly offset by the fall in the value of the options component.

Because the hedge portfolio is riskless, it must be the case that the current value of the portfolio can be found by discounting the known terminal value by the riskless rate of interest

$$P^{S} - hP^{C} = \frac{uP^{S} - hP^{C}_{u}}{1+r}. \tag{17}$$

Substituting (16) into (17) and solving for p^{C} we get the fair price of the call option (P^{C}_{0})

$$P^{C}_{0} = \frac{fP^{C}_{u} + (1-f)P^{C}_{d}}{1+r} \tag{18}$$

where

$$f = \frac{1+r-d}{u-d}$$

is known as the *hedging probability*. From (18), it is clear that the fair option premium is simply the expected discounted value of the sum of the expiry values of the option.

Using the numerical values above, we get

$$\begin{aligned} P^{C}_{0} &= \frac{\left[\dfrac{1{\cdot}1 - {\cdot}7}{1{\cdot}3 - {\cdot}7}\right]20 + \left[1 - \left[\dfrac{1{\cdot}1 - {\cdot}7}{1{\cdot}3 - {\cdot}7}\right]\right]0}{1{\cdot}1} \\ &= \frac{({\cdot}667)20 + ({\cdot}333)0}{1{\cdot}1} \\ &= 12{\cdot}12\text{p}. \end{aligned}$$

From (17) and (18), it is clear that the fair option premium depends on all the factors given in (3): the current security price (P^S), the exercise price (X), the time to expiry ($T=1$ in the one-period model since the period is one year) and the variance of the security price as determined by u, d and q. The variance of the security price implied by u, d and q can be found as follows. The expected terminal value of the security price is

$$\begin{aligned} E(P_T^S) &= quP^S + (1-q)dP^S \\ &= \cdot 5(1\cdot 3)(50) + \cdot 5(\cdot 7)(50) \\ &= 50\text{p} \end{aligned}$$

and the variance is (assuming zero covariance since u and d are independent of each other)

$$\begin{aligned} \sigma^2(P_T^S) &= q(uP^S - E(P_T^S))^2 + (1-q)(dP^S - E(P_T^S))^2 \\ &= \cdot 5(65-50)^2 + \cdot 5(35-50)^2 \\ &= 225. \end{aligned}$$

5. THE BLACK–SCHOLES MODEL OF THE PRICING OF A EUROPEAN CALL OPTION

The binomial model assumes a discrete time stationary binomial stochastic process for security price movements. In the limit, as the discrete time period becomes infinitely small, this stochastic process becomes a *diffusion process* (also called a *continuous time random walk*, an *Ito process*, or *geometric Brownian motion*). This was the process assumed by Black–Scholes (1973) in their derivation of the option pricing formula. As with the binomial model, Black–Scholes begin by constructing a riskless hedge portfolio, long in the underlying security and short in call options. This portfolio generates the riskless rate of return, but the internal dynamics of the portfolio are driven by the diffusion process for the security price. The structure of the hedge portfolio can be put into a form that is identical to the heat equation in physics. Once this was recognized, the solution to the equation was easily derived.

The Black–Scholes formula for the fair price of the call option is

$$P_0^C = P^S N(d_1) - Xe^{-rT} N(d_2) \tag{19}$$

where

P_0^C = fair price of call option
P^S = current price of security
X = exercise price
r = riskless rate of interest
T = time to expiry in fractions of a year (e.g. one quarter, $T=\cdot 25$; one year, $T=1\cdot 00$)

σ = instantaneous standard deviation (or volatility)

$d_1 = \dfrac{\ln(P^S/X)+rT}{\sigma\sqrt{T}}+\frac{1}{2}\sigma\sqrt{T}$

$d_2 = d_1-\sigma\sqrt{T}$

$N(d_i)$ = cumulative probability distribution for standard normal variate from $-\infty$ to d_i.

Again the formula in (19) depends on all the factors given in (3): P^S, X, T, r and σ^2, the instantaneous variance of the security price.

All the factors in (19) are readily observable (i.e. P^S, X, T and r) except for the variance σ^2 which has to be estimated. There are two main ways of estimating σ^2.

The first method uses historical data on the security's price movements and calculates the variance based on log price relatives. We will illustrate this using Table 3 which lists the average weekly price for a share for the previous 6 weeks.

Table 3. *Calculating volatility.*

Week	Share price (P^S_t)	Price relative (P^S_t/P^S_{t-1})	Log price relative ($\ln(P^S_t/P^S_{t-1})$)
1	47·38p	—	—
2	48·17	1·0167	·016536
3	49·21	1·0216	·021360
4	50·79	1·0321	·031603
5	51·83	1·0205	·020270
6	52·62	1·0152	·015127
Arithmetic mean	50·00p		·020979
Variance	4·33		·003856
Standard deviation	2·08		·062119

Table 3 also lists the price relatives and the log price relatives.

The arithmetic mean is calculated as

$$\text{Arithmetic mean} = \bar{X} = \left(\sum_{t=1}^{N} X_t\right)/N \tag{20}$$

where

$X_t = P^S_t$ or $\ln(P^S_t/P^S_{t-1})$

N = number of observations ($N=6$ for P^S_t; $N=5$ for $\ln(P^S_t/P^S_{t-1})$).

The variance is calculated as

$$\text{Variance} = \left(\sum_{t=1}^{N} (X_t-\bar{X})^2\right)/(N-1) \tag{21}$$

and the standard deviation is the square root of the variance.

The variances and standard deviations given in Table 3 are weekly variances and standard deviations. But we need annual variances and standard deviations for the Black–Scholes formula. The annual variance is given by

$$\text{Annual variance} = \text{Weekly variance} \times 52 \tag{22}$$

implying

$$\underset{\text{(share price)}}{\text{Annual variance}} = 4{\cdot}33 \times 52 = 225$$

and

$$\underset{\text{(log price relative)}}{\text{Annual variance}} = {\cdot}003856 \times 52 = {\cdot}200658$$

The annual standard deviation is given by

$$\text{Annual standard deviation} = \text{weekly standard deviation} \times \sqrt{52}$$

implying

$$\underset{\text{(share price)}}{\text{Annual standard deviation}} = 2{\cdot}08 \times \sqrt{52} = 15$$

and

$$\underset{\text{(log price relative)}}{\text{Annual standard deviation}} = {\cdot}062119 \times \sqrt{52} = {\cdot}447948.$$

The appropriate variance estimator for the Black–Scholes model is the annual variance of the log price relatives. However, what is usually quoted in the options pricing literature is *volatility*, which is the annual standard deviation of the log price relatives expressed as a percentage. For the case in point,

$$\text{Volatility} = 44{\cdot}79\%$$

meaning that $\sigma = {\cdot}4479$. As we have seen this is equivalent to an annual standard deviation for the share price of 15.

The second method of estimating σ^2 uses (19) in reverse. It takes the current market option premium for call options with a particular exercise price and term to expiry, together with the current price of the security and riskless rate of interest, and solves for the standard deviation. In other words, this method calculates the volatility implied by the market option premium itself. The problem is that there can be different implied volatilities even for the same expiry date, so that the different volatility estimates have to be combined into a single composite volatility estimate. Nevertheless, this method provides a theoretically superior estimate of volatility because it is essentially forward-looking. The first method uses backward-looking historical data.

To illustrate the Black–Scholes model, we will use the same data as for the binomial model, i.e. $P^S = 50$p, $X = 45$p, $r = {\cdot}1$, $T = 1$, $\sigma = {\cdot}4479$ (implying an annual variance for the share price of 225, the same as for the binomial model). From (19), we have

$$d_1 = \frac{\ln(50/45) + \cdot 1(1)}{\cdot 4479\sqrt{1}} + \tfrac{1}{2}(\cdot 4479)\sqrt{1}$$

and

$$\begin{aligned} d_2 &= \cdot 68245 - \cdot 4479\sqrt{1} \\ &= \cdot 23455. \end{aligned}$$

Using the cumulative normal distribution tables, we get

$$N(d_1) = N(\cdot 68245) = \cdot 7525$$

and

$$N(d_2) = N(\cdot 23455) = \cdot 5927.$$

(Note: (a) linear interpolation is used if d_1 or d_2 lies between the relevant numbers in the tables, e.g. d_1 lies about a quarter of the way between ·68 and ·69 in the tables; (b) if normal distribution tables which present only the upper half of the standard normal distribution are used, then ·5 has to be added to the result, e.g. with d_1 the normal distribution table gives a value of ·2525 and ·5 has to be added to this to give ·7575.)

Substituting these values into (19) gives

$$\begin{aligned} P_0^C &= 50(\cdot 7525) - 45e^{-\cdot 1}(\cdot 5927) \\ &= 13\cdot 49\text{p} \end{aligned}$$

of which 5p is intrinsic value and 8·49p is time value. Comparing with the binomial model, we see that the Black–Scholes model predicts a slightly higher price for the option. There could be a number of reasons for the discrepancy. Firstly the variance estimate is approximate and it may not be the case that the true annual variance is 52 times the weekly average variance. Secondly, the Black–Scholes model is a continuous time model requiring an estimate of the instantaneous variance; however, what we have estimated is an annual variance.

6. PROPERTIES OF THE BLACK–SCHOLES MODEL

We can now examine some of the properties of the Black–Scholes model. We are particularly interested in the sensitivity of the option premium to changes in the determining factors: P^S, T and σ^2.

(a) *Option Delta*

The first sensitivity factor is called *delta* (δ). It measures the change in the option premium as the security price moves one point. From the Black–Scholes formula, the option delta is given by

$$\text{Delta} = \frac{\partial P^C}{\partial P^S} = N(d_1). \tag{23}$$

This is an exact result even though d_1 and d_2 are functions of P^S. From (23), it is clear that delta always lies between zero and unity. In the above example, delta = ·7525.

From (23), delta measures the slope of the option price profile with respect to the security price (at the current value of the security price). This is shown in Figure 10(a). As the share price rises, delta approaches unity, while as the share price falls, delta approaches zero. Delta is also equal to the slope of the short-run profit and loss profile as shown in Figure 1.

Delta is also related to the riskless hedge ratio. The value of the hedge portfolio is given by (14) and its rate of change over time is given by

$$\frac{dV^H}{dt}=\frac{dP^S}{dt}-h\frac{dP^C}{dt}. \tag{24}$$

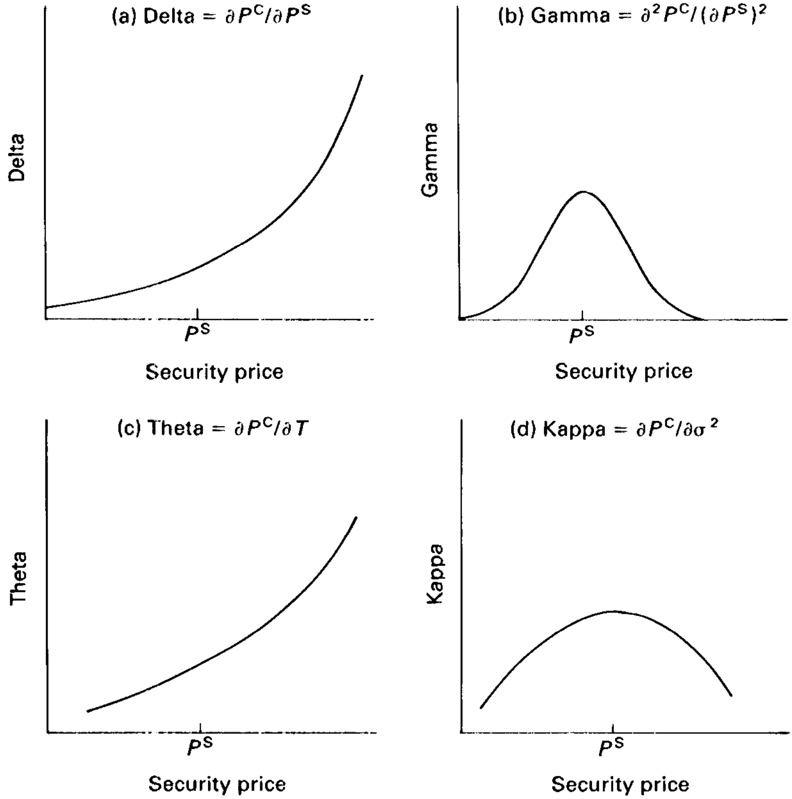

Figure 10. Sensitivity factors for a call option.

If the hedge portfolio is riskless, the value of (24) will be zero implying that (using (23))

$$h = \frac{dP^S}{dP^C} = \frac{1}{N(d_1)} = \frac{1}{\text{delta}}. \tag{25}$$

So delta is the inverse of the hedge ratio. To illustrate, if delta $= \cdot 7525$ then $h = 1 \cdot 33$. Given that a standard options contract in the United Kingdom is for 1000 shares, this implies that to hedge 1000 shares, it is necessary to write 1·33 call options (or to hedge 3000 shares, 4 calls have to be written). Again the discrepancies with the binomial model, where the hedge ratio was estimated to be 1·5, should be noted.

For long calls and short puts the delta is positive, while for short calls and long puts the delta is negative. One of the key characteristics of deltas is that they are linearly additive. This means that the delta for a portfolio of options is easy to calculate. A portfolio consisting of ten long 125p calls with delta of ·8, five short 135p calls with delta of $-\cdot 55$, fifteen short 125p puts with delta of ·15 and ten long 145p puts with delta of $-\cdot 75$, will have a portfolio delta of

$$\begin{aligned}\text{Portfolio delta} &= 10(\cdot 8) + 5(-\cdot 55) + 15(\cdot 15) + 10(-\cdot 75)\\ &= 0.\end{aligned}$$

Therefore the portfolio is totally riskless for small changes in the underlying security price. Delta can therefore be used as a measure of both option risk and option portfolio risk.

Having interpreted $N(d_1)$, it is interesting to give an interpretation to $N(d_2)$. While not immediately obvious, $N(d_2)$ can be interpreted as the probability that the option will finish in-the-money. In the above example, $N(d_2) = \cdot 59$ implying that there is a 59% probability that the option will expire in-the-money.

(b) *Option Gamma*

From Figure 10(a), it is clear that as the security price changes, so does the option delta. This is because the short-run price profile for the option is not linear. *Gamma* (γ), the second sensitivity factor, measures the change in delta as the share price moves one point (in other words it is the second derivative of the option premium with respect to the share price)

$$\text{Gamma} = \frac{\partial\ \text{delta}}{\partial\ P^S} = \frac{\partial^2\ P^C}{(\partial\ P^S)^2} \tag{26}$$

So, for example, if the share price moves from 50p to 51p and this leads the delta on the 45p call option to move from ·75 to ·77, this means that the gamma of the 45p call option is ·02. The average delta over the range of a price change is delta $\pm \frac{1}{2}$ gamma, depending on whether the security price rose or fell. The gamma function is symmetric about the current share price (see Figure 10(b)).

(c) *Option Theta*

The third sensitivity factor is *theta* (θ) which measures the change in the option premium as the time to expiry increases

$$\text{Theta} = \frac{\partial P^C}{\partial T}. \tag{27}$$

It is an increasing function of the security price (see Figure 10(c)). Time is always against the holder of an option so theta is always positive for a call option (see Figure 10(c)). An option is an asset whose time-value decays over time according to the square root of time (see Figure 11). For example: suppose that an option priced at 5p with 10 days to expiry has a theta of ·5. This means that the following day the option price will be 4·5p.

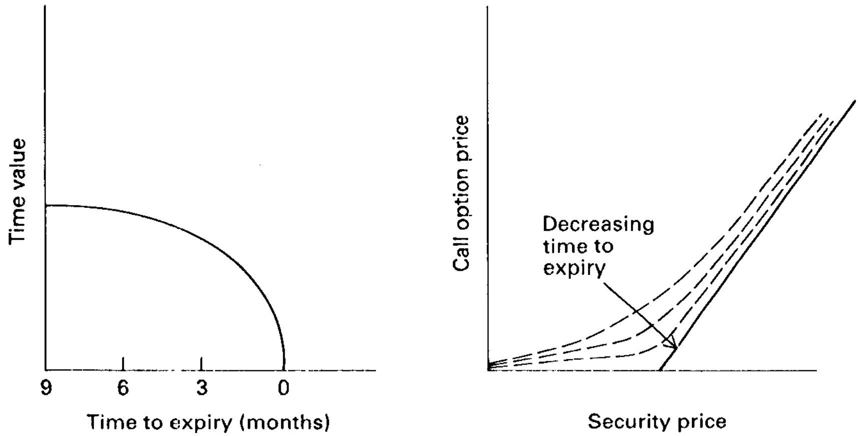

Figure 11. The decay of time value.

(d) *Option Kappa*

The fourth sensitivity factor is *kappa* (κ) (also known as *epsilon*, *eta* or *omega*) which measures the change in the option premium following a 1% change in security price volatility or variance

$$\text{Kappa} = \frac{\partial P^C}{\partial \sigma^2}. \tag{28}$$

The kappa function is symmetric about the current share price (see Figure 10(d)).

7. PRICING A EUROPEAN PUT OPTION

Once we know the value of a European call option, we can use it to calculate the value of a European put option written on the same underlying security with the same exercise price and expiration date. We can do this using the *put-call parity* theorem of Stoll (1969). Stoll argues that it is possible to create a riskless hedge portfolio by combining long positions in the security and the put option with a short position in the call option (with the same exercise price X and expiry period T). At expiration, if the price of the security equals or exceeds the exercise price ($P_T^S \geqslant X$) then the value of the portfolio is

$$\begin{array}{rl} \text{Value of security} & P_T^S \\ +\,\text{value of put option} & 0 \\ -\,\text{value of call option} & \underline{-(P_T^S - X)} \\ & X \end{array}$$

while if the price of the share is less than the exercise price ($P_T^S < X$), then the value of the portfolio is

$$\begin{array}{rl} \text{Value of security} & P_T^S \\ +\,\text{value of put option} & X - P_T^S \\ -\,\text{value of call option} & \underline{0} \\ & X \end{array}.$$

So in either case, the value of the portfolio at expiration is X. Such a portfolio is completely riskless and will therefore earn a riskless return r. Hence the value of the portfolio at the beginning of the period if it earns a riskless return of r is Xe^{-rT}. Therefore the following relationship (known as the *put-call parity relationship*) must hold for European options at the beginning of the period

$$P^S + P^P - P^C = Xe^{-rT}$$

or rearranging

$$P^P = P^C - P^S + Xe^{-rT} \tag{29}$$

where P^P is the premium on the put option. Therefore, once we know P^C (using the Black–Scholes model) we can easily determine P^P.

Example of European Put Option Pricing

Using the same data as for the call option, namely

$$\begin{aligned} P^S &= 50\text{p} \\ X &= 45\text{p} \\ T &= 1 \\ r &= \cdot 1 \\ P^C &= 13{\cdot}5 \end{aligned}$$

we have using (29)

$$P^{P} = 13\cdot5 - 50 + 45\,(e^{-(\cdot 1)(1)})$$
$$= 4\cdot22\text{p}.$$

8. MODIFICATIONS TO THE BLACK–SCHOLES MODEL

Finally, we will briefly consider a number of modifications to the Black-Scholes model considered before (see e.g. Fitzgerald (1987)).

(a) *The Payment of Dividends*

The basic Black–Scholes formula (19) is valid for a European call option on a share which does not pay dividends. It is also valid for an American call option on a non-dividend paying share since such an option will not be exercised early; it is more profitable to sell rather than to exercise. However, the payment of dividends requires the Black–Scholes model to be modified slightly.

When dividends are paid, there is the possibility of early exercise in order to receive the dividend payment on the delivered share. Nevertheless it can be shown that the only time that it becomes profitable to exercise an American call option (apart from on the expiry date) is just prior to the ex-dividend date. Suppose that the share price falls by exactly the amount of the dividend, D, when a dividend is paid, i.e. it falls to $P^S - D$. If the call option is exercised just prior to the ex-dividend date, the option holder receives $P^S - X$ (i.e. $D + (P^S - D) - X$). If the option is not exercised, the option holder's position is worth $P^C(P^S - D, X, T)$. It becomes profitable to exercise early whenever

$$P^S - X > P^C(P^S - D, X, T). \tag{30}$$

However, if the present value of the dividends is less than the present value of the interest earned on an investment equal to the exercise price, then the American call option will not be exercised before expiry.

The opposite result holds for an American put option. If the present value of the dividends is greater than the present value of the interest earned on an investment equal to the exercise price, then the American put will not be exercised before expiry.

To illustrate the effect of dividends payments, we will consider a European call option on a share paying dividends continuously with a constant dividend yield, d. The Black–Scholes model in this case becomes

$$P^C = e^{-dS}N(d_1) - e^{-rX}N(d_2) \tag{31}$$

where

$$d_1 = \frac{\ln(P^S/X) + (r-d)T}{\sigma\sqrt{T}} + \tfrac{1}{2}\sigma\sqrt{T}$$

$$d_2 = d_1 - \sigma\sqrt{T}.$$

The basic Black–Scholes formula (19) is also valid for a European call option on a cash stock index with no dividend payments, while (31) is also valid for a European call option on a cash stock index with a constant dividend yield, d.

(b) *Options on Futures Contracts*

The modification to the Black–Scholes formula when the call option is on a futures contract is straightforward.

$$P^C = e^{-rT}[F\,N(d_1) - X\,N(d_2)] \tag{32}$$

where

$$F = \text{current futures price}$$

$$d_1 = \frac{\ln(F/X)}{\sigma\sqrt{T}} + \tfrac{1}{2}\,\sigma\sqrt{T}$$

$$d_2 = d_1 - \sigma\sqrt{T}.$$

Similarly the modification to the put-call parity formula when the put option is on a futures contract is

$$\begin{aligned} P^P &= P^C - e^{-rT}[F - X] \\ &= e^{-rT}[F\,N(d_1) - X\,N(d_2)] - e^{-rT}[F - X] \\ &= e^{-rT}[F(N(d_1) - 1) - X(N(d_2) - 1)] \\ &= e^{-rT}[X\,N(-d_2) - F\,N(-d_1)]. \end{aligned} \tag{33}$$

(c) *Options on Currency Contracts*

The modification to the Black–Scholes formula when the call option is on a currency option is

$$P^C = e^{-r_\$ T}[P^S N(d_1) - e^{-r_£ T} X\,N(d_2)] \tag{34}$$

where

P^S = spot rate of currency to be delivered in foreign units of domestic currency (e.g. \$ per £).

$r_£$ = domestic riskless rate of interest

$r_\$$ = foreign riskless rate of interest

$$d_1 = \frac{\ln(P^S/X) + (r_£ - r_\$)}{\sigma\sqrt{T}} + \tfrac{1}{2}\sigma\sqrt{T}$$

$$d_2 = d_1 - \sigma\sqrt{T}.$$

The put option premium is determined by

$$P^P = P^C - P^S + e^{-r_£ T}X. \tag{35}$$

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