# a method of estimating the EFFECT OF A CHANGE IN A SET of decremental rates 

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INTRODUCTION
Three of the writer's earlier notes in this fournal (see References) have discussed respectively the effect, upon pension fund contribution rates, of changes in the salary scale, the rate of interest and the entry age, and approximate methods have been supplied for estimating rapidly the effects of the two last-mentioned changes. The object of the present note is to propose a method for estimating the effect on the survivorship ratio $l_{y} / l_{x}$ according to a service table, of changing one of the sets of decremental rates in the same proportion at all ages, or of removing a set altogether, or of introducing a new set in addition to those already present. For example, given a service table based on certain rates of mortality and withdrawal, we shall be able to estimate, with a negligible amount of arithmetical work, the new values of such ratios as $l_{60} l_{25}$ resulting from removing the mortality rates altogether, from doubling or halving the withdrawal rates or from introducing a new set of decremental rates. The values of $l_{y}^{\prime} / l_{x}^{\prime}$, where accented functions refer to the amended table, would enable us to calculate the revised value of a pension, and the effect of the change in rates on the value of contributions could be estimated by splitting the ${ }^{8} \bar{D}_{x}$ column into age-groups and applying the ratios $l^{\prime} / l$ (at appropriate mean ages) to the sums of ${ }^{s} \bar{D}_{x}$ in the groups. Once we know these ratios, in fact, the possible applications are numerous and varied and will not concern us in this note; our purpose is to propose an approximate method whereby the ratios can be calculated. Experience has shown that the results are seldom in error by more than $1 \frac{1}{2} \%$ even when we are changing withdrawal rates, and that much greater accuracy than this is obtained when the decremental rates are relatively small, as
in the case of mortality rates in service; in general, the accuracy is great when the rates to be changed are small, and when the degree of change is small; e.g. we obtain better results if we are adding $10 \%$ to the rates than if we are adding $100 \%$. For changes of up to $50 \%$ or so in mortality rates, the results are so good that for most practical purposes it is no longer necessary to construct the revised service table even when 'exact' answers are required.

## THE METHOD

Let $x, y$ be any two different ages in the service table such that $y>x$. For any age $t$ let $\beta_{t}$ denote the (dependent) decremental rate at age $t$ which is to be changed, and let $\gamma_{t}$ denote the sum of all other rates at this age, so that

$$
l_{t+1}=l_{t}\left(\mathrm{I}-\beta_{t}-\gamma_{t}\right) .
$$

Now suppose that, for every age $t$ such that $x \leqslant t<y$, the rate $\beta_{l}$ is multiplied by a constant $k$ which may be greater or less than unity, and may be zero. Then

$$
l_{t+1}^{\prime}=l_{t}^{\prime}\left(\mathrm{x}-k \beta_{t}-\gamma_{t}\right) .
$$

Therefore

$$
\begin{equation*}
l_{y} / l_{x}=\left(\mathrm{I}-\beta_{x}-\gamma_{x}\right)\left(\mathrm{I}-\beta_{x+1}-\gamma_{x+1}\right) \ldots\left(\mathrm{I}-\beta_{y-1}-\gamma_{y-1}\right) \tag{1}
\end{equation*}
$$

and

$$
\begin{equation*}
l_{y}^{\prime} / l_{x}^{\prime}=\left(\mathrm{I}-k \beta_{x}-\gamma_{x}\right)\left(\mathrm{I}-k \beta_{x+1}-\gamma_{x+1}\right) \ldots\left(\mathrm{I}-k \beta_{y-1}-\gamma_{v-1}\right) . \tag{2}
\end{equation*}
$$

Since

$$
\frac{\mathrm{I}-k \beta_{t}-\gamma_{t}}{\mathrm{I}-\beta_{t}-\gamma_{t}}=\mathrm{I}+(\mathrm{I}-k) \frac{\beta_{t}}{\mathrm{I}-\beta_{t}-\gamma_{t}}=\mathrm{I}+(\mathrm{I}-k) \beta_{t} l_{t} / l_{t+1},
$$

we have, by dividing (2) by (1) and taking logarithms,

$$
\log R=\sum_{t=x}^{\nu-1} \log \left\{\mathrm{I}+(\mathrm{I}-k) \beta_{l} l_{t} / l_{t+1}\right\}
$$

where $R$ is the ratio which $l_{y}^{\prime} / l_{x}^{\prime}$ bears to $l_{y} / l_{x}$. Since $\beta_{t}$ is a decremental rate, we shall generally have $\left|(\mathrm{I}-k) \beta_{t} l_{t} / l_{t+1}\right|<\mathrm{I}$, and if this is true for all values of $t$ such that $x \leqslant t<y$,

$$
\log R=(1-k) \sum_{x}^{\nu-1} \beta_{t} l_{t} / l_{t+1}-\frac{1}{2}(\mathrm{I}-k)^{2} \sum_{x}^{y-1} \beta_{t}^{2} l_{t}^{2} / l_{t+1}^{2}+\ldots a d \text { inf. }
$$

This expression is exact, and we now proceed to approximate. The second term will generally be small: e.g. if $\beta$ is about 04 , $k=0$ or 2 and $y-x=40$, its value is only about $\cdot 0$; the third and subsequent terms will be still smaller. However, if we retain only the first term we overestimate the value of $\log R$, and a better approximation is obtained by replacing $\beta_{l} l_{t} / l_{t+1}$ in the first term by $\beta_{i}$; since $l_{t}>l_{t+1}$ this reduces the value of the first term and introduces an error in the opposite direction to that due to ignoring the second and subsequent terms of the sum. Thus it is proposed to take
or

$$
\left.\begin{array}{c}
\log R \doteqdot(\mathrm{I}-k) \sum_{x}^{y-1} \beta_{t}  \tag{3}\\
\log _{10} R \doteqdot \cdot 4343(\mathrm{I}-k) \sum_{x}^{y-1} \beta_{t}
\end{array}\right\}
$$

It will be seen that the only arithmetical work involved in finding $R$ consists of summing the rates $\beta$ (i.e. the rates to be changed), multiplying by 4343 ( $\mathrm{r}-k$ ) and looking up the antilogarithm of the product.

As (3) is only approximate, it is necessary to demonstrate that it is self-consistent. If we multiply the $\beta$ 's by a constant $k_{1}$, and then use the method again to multiply the new ( $k_{1} \beta$ )'s by $k_{2}$, we must show that the final result is the same as would be yielded by the method if the original $\beta$ 's had been multiplied by $k_{1} k_{2}$ in one step. If $R_{1}$ is the ratio of the survivorship factors on the first application, and $R_{2}$ is the ratio on the second application, we have

$$
\begin{align*}
\log R_{1} & =\left(\mathrm{I}-k_{1}\right) \sum_{x}^{y-1} \beta_{t}, \\
\log R_{2} & =\left(\mathrm{I}-k_{2}\right) \sum_{x}^{y-1} k_{1} \beta_{t} . \\
\text { Adding, } \quad \log R_{1} R_{2} & =\left\{\mathrm{I}-k_{1}+k_{1}\left(\mathrm{I}-k_{2}\right)\right\} \sum_{x}^{y-1} \beta_{t} \\
& =\left(\mathrm{I}-k_{1} k_{2}\right) \sum_{x}^{y-1} \beta_{t} .
\end{align*}
$$

However, $R_{1} R_{2}$ is the ratio of the final to the initial survivorship factor, and the right-hand side of (4) is the result which would have been obtained from (3) with $k=k_{1} k_{2}$. Thus the method is self-
consistent. In particular, if $k_{1} k_{2}=1$, we correctly reproduce the original survivorship factor if we multiply the $\beta$ 's by a constant $k_{1}$ and then use the method again to divide the new $\left(k_{1} \beta\right)$ 's by the same constant.

## ARITHMETICAL EXAMPLES

Examples $1-4$ are based on a service table involving mortality and withdrawals only. The dependent rates of mortality are the A ${ }_{\text {949-52 }}$ (ultimate) values of $q_{x}$, and the withdrawal rates commence at 042 at age 25 and reduce to zero at age 46 . The sum of the mortality rates from age 25 to age 59 inclusive is $\cdot 15434$ and the corresponding sum of the withdrawal rates is $\mathbf{3 3 3 0 0}$. For illustration we consider the ratio $l_{60} / l_{25}$, which on the service table has the value 6 II I20.

Example 1. To remove the mortality rates altogether.
With $k=0$ we have
whence

$$
\begin{aligned}
\log _{10} R & =\cdot 4343 \times \cdot 15434 \\
& =.06703
\end{aligned}
$$

$$
R=1 \cdot 167
$$

Hence

$$
l_{60}^{\prime} / l_{25}^{\prime}=1 \cdot 167 \times \cdot 61120=\cdot 713 .
$$

By actual construction of the revised table, the true value is also $\cdot 713$ (to three decimal places).

Example 2. To reduce the withdrawal rates by $33 \frac{1}{3} \%$.
With $k=\frac{2}{3}$ we have
therefore

$$
\begin{aligned}
\log _{10} R & =4343\left(\mathrm{I}-\frac{2}{3}\right)(\cdot 33300) \\
& =\cdot 0482 \mathrm{I}, \\
R & =1 \cdot 117 .
\end{aligned}
$$

Hence

$$
l_{60}^{\prime} / l_{25}^{\prime}=1 \cdot 117 \times \cdot 61120=\cdot 683 .
$$

By actual construction of the revised table, the true value is $\cdot 684$.
Example 3. To double the withdrawal rates.
With $k=2$ we have

$$
\begin{aligned}
\log _{10} R & =\cdot 4343(\mathrm{r}-2)(\cdot 33300) \\
& =-\cdot 1446 \text { or } 1 \cdot 8554, \\
R & =\cdot 7168 .
\end{aligned}
$$

therefore

Hence

$$
l_{60}^{\prime} / l_{25}^{\prime}=\cdot 7168 \times \cdot 61120=438
$$

By actual construction of the revised table, the true value is $\mathbf{4 3 3}$.
Example 4. To remove the withdrawal rates altogether.
With $k=0$ we have

$$
\begin{aligned}
\log _{10} R & =\cdot 4343 \times \cdot 33300 \\
& =\cdot 1446
\end{aligned}
$$

therefore

$$
R=1 \cdot 395
$$

Hence

$$
l_{60}^{\prime} / l_{25}^{\prime}=1.395 \times \cdot 61120=853
$$

From the A 1949-52 ultimate table, the true value is 856 .
Example 5. To introduce a set of rates.
For illustration, we consider the simplest case where there are no existing rates (so that $l_{60} / l_{25}=1$ ) and we introduce a set of rates whose sum from 25 to 59 inclusive is $\cdot 15434$. Let us suppose the table to have been constructed; then with $k=0$ we remove the rates and produce the known value of $l_{60} / l_{25}$. Thus

$$
\begin{aligned}
\log _{10} R & =\cdot 4343 \times \cdot 15434 \\
& =\cdot 06703
\end{aligned}
$$

therefore

$$
R=\mathrm{I} \cdot 167
$$

Hence
or

$$
\begin{aligned}
1 \cdot 167 \frac{l_{60}^{\prime}}{l_{25}^{\prime}} & =1 \\
\frac{l_{60}^{\prime}}{l_{25}^{\prime}} & =857 .
\end{aligned}
$$

We have thus estimated the ratio of $l_{60}$ to $l_{25}$ on the A 1949-52 ultimate table being given only $\sum_{25}^{59} q_{x}$. From the table, the true value is 856 .

## REFERENCES

Bizley, M. T. L. (i949). The comparison of salary scales, J.S.S. 9, 105.
Bizley, M. T. L. (1950). The effect on pension fund contributions of a change in the rate of interest, $\mathcal{F} . S . S$. no, 47.
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