JIA 81 (1955) 0053-0057

OSCULATION OF HIGH ORDER

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INTRODUCTION

THE principle of ordinary finite-difference interpolation, when values of u_x are given at n points, is to ascertain a polynomial of the $(n-1)$th order which will reproduce the given values. Since there are n coefficients in the polynomial, to be determined from n values, there is only one solution. In subdividing intervals by the ordinary central formulae, each segment of interpolated values is smooth; and if all the given values actually do lie on a polynomial curve of the assumed order, the interpolated terms will of course all lie on the same polynomial curve from beginning to end. However, the method is generally employed in cases where the given series is of polynomial form only approximately and only when a limited range of values is considered; and in such a case the successive interpolated segments will not meet smoothly. When the formula is based on an odd number of points, in fact, the segments do not meet at all except by accident. In a central second-difference formula, for example, the segment from $u_{-.5}$ to $u_{.5}$ is calculated from u_{-1}, u_0 and u_1 and the segment $u_{.5}$ to $u_{1.5}$ from u_0, u_1 and u_2. If the third difference of the given u's does not actually vanish, we have two values for $u_{.5}$ according to whether we calculate it from the former data or the latter. When the formula is based on an even number of points the successive segments do meet, since they pass through the given points at each end; but at the point of junction the slopes will differ and a differential coefficient will have alternative values according as it is taken to the right or left.

Sprague devised the osculatory principle to mitigate this difficulty. The particular formula associated with his name is equivalent to accepting correctness to one order of difference lower than normal, thus leaving at disposal the coefficient of the highest order of difference in the formula. This coefficient is then determined so that each segment passes through two of the given points and has the required number of derivatives (two in the case of Sprague's own formula) in common with the adjoining segments. The order of the polynomial in Sprague's formula was specified as five, and this enabled the six necessary conditions (three at each end of a segment) to be met.

Lidstone remarked (*J.I.A.* **42**, 397) that

> we can, by increasing the degree of the coefficient of the final term, introduce other arbitrary constants, which enable us to satisfy further conditions, without introducing any more terms in the result.

In particular, it is obvious that we can by this means arrange for osculation of any desired order. (The term 'osculation of the rth order' is used to indicate the existence of r continuous derivatives, so that Sprague's formula has osculation of the second order and King's has osculation of the first order.)

At first sight there seems to be an anomaly in this conclusion. Since we can arrange for any number of continuous derivatives and each segment lies on a polynomial arc and is presumably smooth, it might seem that by imposing osculation of sufficiently high order we could produce a series that was perfectly smooth from beginning to end. This, however, is too good to be true. Since each segment is calculated from only a small number of adjacent values, the process could not lead to an interpolated series of one mathematical form from end to end.

The object of this Note is to derive a general formula for osculation of any order, and to examine what happens as the order of osculation becomes very high and finally approaches infinity.

For simplicity in explanation the proof is given for a four-term formula, so that when the order of osculation is unity the result is King's formula. In other words, we proceed to deduce a generalization of that formula to provide osculation of the rth order.

PROBLEM

To deduce a four-term central formula of polynomial interpolation, correct to second differences, with osculation of rth order.

As in a previous Note (*J.I.A.* **80**, 63) we commence by writing down the ordinary formula correct to third differences

$$u_x = u_{-1} + (x+1)\Delta u_{-1} + \tfrac{1}{2}(x+1)x\Delta^2 u_{-1} + \tfrac{1}{6}(x+1)x(x-1)\Delta^3 u_{-1}.$$

As pointed out in the previous Note, the only change we can make to this is in the coefficient of Δ^3. We will use $j(x)$ to indicate the coefficient that will satisfy our conditions.

Now, since we have $(r+1)$ conditions to satisfy at the ends of each interpolated segment, there must be $(2r+2)$ coefficients at disposal in $j(x)$, which must therefore be a polynomial of order $(2r+1)$ at least. The investigation is confined to order $(2r+1)$.

On the argument of the previous Note, we can write

$$j(x) = h(x) + \tfrac{1}{6}(x+1)x(x-1), \tag{1}$$

where $h(x)$ is such that

$$h(x) = -h(1-x).$$

Hence

$$\begin{aligned} j(x) + j(1-x) &= \tfrac{1}{6}(x+1)x(x-1) + \tfrac{1}{6}(2-x)(1-x)(-x) \\ &= -\tfrac{1}{2}x(1-x). \end{aligned} \tag{2}$$

Now from the argument of our previous Note, $j(x)$ and its first r derivatives must vanish when $x=0$. Hence $j(x)$ contains the factor x^{r+1}. It follows from (2) that $j(1-x)$ contains the factor x, and therefore $j(x)$ the factor $(1-x)$. Since then both $j(x)$ and $j(1-x)$ contain the factor $x(1-x)$, we may write $l(x) = j(x)/x(1-x)$, where $l(x)$ is a polynomial of degree $(2r-1)$, so that

$$l(x) + l(1-x) = -\tfrac{1}{2}. \tag{3}$$

Differentiating according to x, we find that

$$l'(x) = l'(1-x). \tag{4}$$

$l'(x)$ is a polynomial of degree $(2r-2)$ containing the factor x^{r-1}. From (4) $l'(1-x)$ contains the factor x^{r-1} and therefore $l'(x)$ the factor $(1-x)^{r-1}$.

Hence

$$l'(x) = kx^{r-1}(1-x)^{r-1},$$

$$l(x) = k\int_0^x y^{r-1}(1-y)^{r-1}dy, \tag{5}$$

no constant of integration being necessary since $l(0)=0$.

The integral can be readily ascertained as a series but, though easy for small values of r, for high values it would be laborious. We note, however, that the form is that of an Incomplete Beta-function; so that after ascertaining the value of k we can find the numerical values for $l(x)$ by the use of Karl Pearson's *Tables of the Incomplete Beta-function*.

Formula (5), in the Beta notation, becomes

$$l(x) = kB_x(r, r).$$

From (3) $\qquad l(1) = -\tfrac{1}{2}.$

Therefore $\qquad kB(r, r) = -\tfrac{1}{2},$

$$k = -1/2B(r, r) = -(2r-1)!/2(r-1)!(r-1)!,$$

$$j(x) = [-(2r-1)!/2(r-1)!(r-1)!]\,x(1-x)\,B_x(r, r). \tag{6}$$

This completes the solution; the formula for rth-order osculation is

$$u_x = u_{-1} + (x+1)\Delta u_{-1} + \tfrac{1}{2}(x+1)x\Delta^2 u_{-1} + j(x)\Delta^3 u_{-1}.$$

NUMERICAL VALUES OF $j(x)$ IN PARTICULAR CASES

To illustrate the effect of osculation the numerical values of $j(x)$ have been calculated for certain cases at the points ·2, ·4, ·5, ·6 and ·8, and are shown in the table below in comparison with corresponding values for ordinary third-difference interpolation and for ordinary central second-difference interpolation.

For second-difference interpolation the values at ·2 and ·4 are, of course, zero. At the points ·6 and ·8, however, the interpolated terms would be calculated from u_0, u_1 and u_2 in lieu of u_{-1}, u_0 and u_1, making the formula

$$\begin{aligned} u_x &= u_0 + x\Delta u_0 + \tfrac{1}{2}x(x-1)\Delta^2 u_0 \\ &= u_{-1} + (x+1)\Delta u_{-1} + \tfrac{1}{2}(x+1)x\Delta^2 u_{-1} + \tfrac{1}{2}x(x-1)\Delta^3 u_{-1}. \end{aligned}$$

To compare with the four-term formulae, on the basis we are using, it is therefore necessary to treat the coefficient of Δ^3 as zero for the first half of the segment and as $\tfrac{1}{2}x(x-1)$ for the second half. At the centre it can, as remarked in our introduction, be taken either way; and the value there is 0 or $-$·125.

The highest order of osculation for which the calculation could be made from Pearson's *Tables* was the fiftieth.

Table of $j(x)$

$x=$	·0	·2	·4	·5	·6	·8	1·0
Ordinary third difference	0	$-$·0320	$-$·0560	$-$·0625	$-$·0640	$-$·0480	0
1st order osculation	0	$-$·0160	$-$·0480	$-$·0625	$-$·0720	$-$·0640	0
2nd order osculation	0	$-$·0083	$-$·0422	$-$·0625	$-$·0778	$-$·0717	0
7th order osculation	0	$-$·0006	$-$·0275	$-$·0625	$-$·0925	$-$·0794	0
50th order osculation	0	$-$·0000	$-$·0026	$-$·0625	$-$·1174	$-$·0800	0
Central second difference	0	0	0	$-$·1250 or 0	$-$·1200	$-$·0800	0

Examination of the table, especially if each line of the table is graphed and the points joined to make a curve, will illustrate the known fact that osculation is purchased at the price of a certain waviness within the segment. As the order of osculation increases the wave becomes more pronounced.

Comparison with the last line of the table also suggests an answer to our question as to what occurs as r approaches infinity. The indication is that at infinity we simply return to ordinary central second-difference interpolation.

The position then is that as the order of osculation increases we certainly secure a more and more complete correspondence of the segments at the points of junction; but the fact that the polynomial is of higher order permits the curve to take a more pronounced wave, until (for an infinite order of osculation) the curve in effect shakes apart, that is the value of $j(x)$ in the centre alters from 0 to $-\cdot 1250$ in an infinitely small space. The junction of a segment of interpolated terms with the segments to the left and right is perfect when the order of osculation is infinite; but these adjoining segments are not fully reconcilable, and an adjusting wave is necessary in order to pass from the conditions at one end to those at the other. At infinity this wave becomes in effect a discontinuity. This, then, is the resolution of the paradox mentioned in our introductory remarks. It is, however, necessary to establish the above limit mathematically, and this we now proceed to do.

LIMIT OF $j(x)$ WHEN r IS INFINITE

For convenience we write $z(r)$ for $[(2r-1)!/(r-1)!(r-1)!]x^{r-1}(1-x)^{r-1}$.

$$z(r+1)/z(r) = 2\left(2+\frac{1}{r}\right)x(1-x)$$

$$= \left(1+\frac{1}{2r}\right)[1-4(\tfrac{1}{2}-x)^2].$$

No matter how close x may be to $\frac{1}{2}$ (provided that $x \neq \frac{1}{2}$) we can always find a positive value of r (say t) so large that the above expression is less than 1, and though t may be a large number it will not be infinite. As r increases the value of the expression decreases for finite values of r and comes to a minimum at infinity. We can therefore say that $z(t+m) \ngtr s^m z(t)$, where s is less than unity. $z(t)$ is finite and the limit of s^m is zero; moreover, $z(r)$ cannot be negative; hence the limit of $z(r)$ is zero (unless $x = \cdot 5$).

We can now pass to $l(x)$ which, as we have shown in formula (5), can be expressed as a definite integral between the limits 0 and x. Since $l'(x) = -\frac{1}{2}z(r)$, and since $y^r(1-y)^r$ increases with y in the range $(0, \frac{1}{2})$, we have for fixed r $0 \geqslant l'(y) \geqslant l'(x)$ for $0 \leqslant y \leqslant x < \frac{1}{2}$. Therefore $0 \geqslant l(x) \geqslant xl'(x)$. When $x < \frac{1}{2}$ and $r \to \infty$, the limit of $xl'(x)$ and hence of $l(x)$ and therefore also of $j(x)$ is zero.

In interpolating values from u_0 to $u_{.5}$ by a formula with osculation of infinite order, the coefficient of Δ^3 therefore disappears and the result is the same as an ordinary central second-difference interpolation calculated from u_{-1}, u_0 and u_1.

Now $j(x)+j(1-x) = \frac{1}{2}x(x-1)$ for all values of r. If x lies between $\cdot 5$ and 1, $(1-x) < \cdot 5$, and $j(1-x)$ vanishes as $r \to \infty$. The limit of $j(x)$ is therefore $\frac{1}{2}x(x-1)$. As shown above, the interpolation formula then becomes the ordinary second-difference formula based on u_0, u_1 and u_2.

Osculation of infinite order, in a four-term formula correct to second differences, is therefore equivalent to ordinary central second-difference interpolation.

(When $x = \cdot 5$ exactly, the interpolated term by ordinary second differences can be either of two values. The value by the osculatory formula, whatever the value of r, is half-way between the two.)

THE USEFUL DEGREE OF OSCULATION

It is clear that a very high order of osculation is not of general utility. In a previous investigation it was found that, on a test based on minimum squares of differences, Shovelton's formula (first-order osculation) appeared rather better than Sprague's. It appears desirable in actuarial work not to go beyond osculation of the second order, and perhaps not beyond the first order.