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Pension plan solvency and extreme market movements: a regime switching approach

A background paper for discussion

By Mark Freeman, Iain Clacher, David Hillier,
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Pension Plan Solvency and Extreme Market Movements: A Regime Switching Approach

Mark Freeman, Iain Clacher, David Hillier,
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Abstract

The global credit crunch has forcefully highlighted the impact of extreme market events on both the asset and liability side of a pension plan's balance sheet. Over the course of 2008 there were severe falls in global stock markets because of the financial crisis and, in particular, the failure of Lehman Brothers. However, the dramatic collapse in pension plan solvency that was observed in early 2009 was not caused solely by the significant falls in the value of pension plan assets that had occurred. In March 2009 quantitative easing pushed bond yields significantly lower, thereby increasing the present value of defined benefit pension liabilities. As a consequence, the net funding position of defined benefit pension plans in the UK swung dramatically from a surplus of 149.2bn in June 2007 to a deficit of 208.6bn in March 2009, back to a surplus of 35.5bn in February 2011 and then back to a deficit again of 206.2bn in March 2012.

These turbulent conditions highlighted the need to capture extreme market movements in the modelling of future pension fund solvency risk. Traditional models of asset returns assume that the statistical parameters that drive asset returns and interest rate changes remain constant over time. In these 'one state' models, stock markets might produce average returns of approximately 10% a year every year. It has recently been recognized that average returns, the variance of returns and the covariances between returns to different asset classes can be more accurately modelled using parameters that switch between more than one set of values. As a result, these multi-state' models are gaining in popularity as the benefits of applying such models to practical problems, such as asset allocation decisions, are becoming increasingly recognized in a number of fields. In particular, these models can capture rare, but extreme, market events, the time-variation in asset return volatility and the "fat-tailed" nature of stock returns.

We therefore model the future solvency of defined-benefit pension plans using regime switching models and compare the outcomes with those of traditional one-state models. Our

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results show that future projections of pension plan solvency are highly model-dependent. In particular, if discount rates are assumed to remain constant, then regardless of the choice of model (one-state or multi-state) the probability of future deficits is dramatically understated compared to forecasts where a stochastic discount rate process is used. Moreover, our results suggest that by allowing for leptokurtosis in asset returns the probability of a future deficit is much greater. However, it is also important to note that allowing for correlations between asset returns and the discount rate significantly reduces the assessed probability of underfunding in the future.

1 Background

This paper focuses on the estimation of future pension plan solvency in the presence of extreme market movements. The relevance of these issues was highlighted by the Actuarial Professions Benchmarking Stochastic Models (BSM) working party that considered modelling extreme market events. The BSM working party paper reviewed a number of techniques for both modelling and calibrating extreme market scenarios. One of the most important conclusions was that severe equity falls that are assigned a 0.5% probability by the normal distribution of stock returns occur much more frequently than this in practice. Consequently, such events need to be more carefully analysed in any modelling of future events. Moreover, in looking at the recent past, there are a number of scenarios that present extreme risks to pension plan solvency, and the full extent of these risks only emerges when the estimation of future asset and liabilities are considered jointly.

Our core empirical methodology involves the use of multivariate Gaussian regime switching models. This framework is well suited to modelling asset returns and extreme market events. Due to their ability to capture different states of the economy, Markov Switching models have been widely used broadly in economics and finance literature. For example, early economics studies used Markov switching models to capture structural breaks in the economy (see Hamilton, 1989; Lam, 1990; Raymond and Rich, 1997; Storer, 1995). In addition, these models have also been used in a wide range of other settings. For example, combining Markov Switching with other models such as GARCH (Hamilton and Susmel, 1994); error correction (Psaradakis, Sola and Spagnolo, 2004); and causality, (Ravn, Psaradakis and Sola, 2005).

In finance Markov Switching has also been used in option pricing, (Boyle and Draviam, 2007); bond pricing, (Elliott and Siu, 2009) and portfolio selection, (Zhou and Yin, 2003). Moreover, Regime Switching models have been shown to more accurately capture the increased correlations between asset returns that often occur in bear markets (Ang and Bekaert, 2002) and that such models estimate the 1% Value-at-Risk levels for portfolios better than many alternative approaches, suggesting that this is a useful way of modelling rare events (Kawata & Kijima, 2007).

As a result, this method has also been shown to be applicable in practice and it has been regularly applied to asset allocation problems (See for example, Guidolin and Timmerman, 2008). There has, however, been only limited use of this framework for assessing pension fund solvency risk. Although Chen and Yang (2010) do apply this approach, they present a highly stylized framework that would not be readily adapted for practical application as

they use closed form models and focus on dividend payout policy.

We base our analysis on $N = 5$ asset classes for the asset side of the balance sheet and one discount rate for the liabilities side.¹ On the asset side, we use the FTSE All Share total returns from Datastream. All the other data for the asset side of the balance sheet are taken from Global Financial Data. These are the United Kingdom 10 year Government Bond total return index, S&P500 total returns index, Japanese Topix total returns index and MSCI Europe total returns index. Data are taken at monthly frequency over the interval January 1970 to December 2010. All returns are calculated in sterling terms. In order to calculate the present value of the liabilities, we use the 10-year UK Treasury bond yield, rather than the AA corporate bond rate, as given by Datastream. Notice that this is an ex-ante yield rather than an ex-post total return.

In the table below, we present summary statistics for the total monthly nominal returns to each series that we consider on the asset side of the balance sheet. Throughout this study we use lognormal returns; $r_{nt} = \ln(I_{nt}/I_{nt-1})$, where I_{nt} is the total returns index of asset class n :

	Mean	Standard Deviation	Skewness	Excess Kurtosis
UK Equity Returns	0.809%	5.730%	-0.710	7.436
US Equity Returns	0.895%	5.264%	-0.637	2.448
European Equity Returns	0.936%	4.716%	-1.010	3.904
UK Treasury Returns	0.784%	1.742%	0.446	2.424
Japanese Equity Returns	0.833%	6.012%	-0.054	0.517

with correlations over the period (with the assets in the same order):

$$\begin{bmatrix} 1.00 & 0.56 & 0.83 & 0.25 & 0.33 \\ & 1.00 & 0.70 & 0.02 & 0.40 \\ & & 1.00 & 0.15 & 0.47 \\ & & & 1.00 & 0.06 \\ & & & & 1.00 \end{bmatrix}$$

Our purpose is to construct a statistical model that captures the broad characteristics of asset returns as reflected in these summary statistics. Under a frequentist approach, it is then assumed that these stochastic properties will not change in the future. While this is a common assumption in both industry and academia, it is not uncontroversial. For example, there is an extensive literature that argues that average historic returns to equity substantially overestimate the current ex-ante equity premium (Freeman, 2011, and the references therein). Based on the data given above, the simple annual expected returns to UK equities is $\exp(12(0.809\% + 0.5 \times 5.730\%^2)) - 1 = 12.4\%$. By contrast, recent surveys of the equity premium (by, for example, Welch for academics and Graham and Harvey for practitioners) suggest that most experts believe that the S&P500 will only give around 5% above the T-bill rate in the medium term future.

¹All work in this study is done using asset classes rather than individual assets. This is because the Markov Switching processes that we use estimates a large number of parameter values. Suppose there are N assets and S states. Taking the AR(0) model that is employed extensively below, this requires the estimate of $S(0.5N(N + 1) + S - 1)$ parameters. With 100 assets and four states, this is 20,212 parameter values. Even with the limited number of asset classes that we do consider, the software does not always converge.

For the purposes of this paper, we make no adjustments for potential limitations with the frequentist approach. However, we would note that it would be possible to do so. For example, if one were to believe that unconditional expected equity returns are too high, then the mean estimates could be manually adjusted downwards to reflect this belief. The regime-switching model would still add value by capturing some of the complex dynamics of the volatility process.

2 The asset & liability sides of the balance sheet

In order to understand the funding risks for pension funds, it is necessary to model the asset side of the balance sheet, the liabilities side of the balance sheet and the discount rate process. For the first of these, let w_i represent the proportion of wealth invested in asset i , with $w_1 + \dots + w_N = 1$. Let w be the N -vector with elements w_i . We choose weights that are broadly representative of a standard UK pension fund:

	Weightings
UK Equity	25%
US Equity	20%
European Equity	10%
UK Treasury Bond	40%
Japanese Equity	5%

It is assumed that the portfolio is rebalanced at the end of each month to keep these weightings fixed across time.

To keep our illustrative example simple, we assume a stylistic form for expected future liabilities of the defined benefits pension plan. At all times, the pension fund has future liabilities stretching over the next thirty years. The first expected liability, in one year's time, is C_1 . Following that, the liabilities are expected to grow at a fixed inflation rate i . This leads to the following schedule of expected future liabilities:

E_0 [Liabilities]	$T = 1$	$T = 2$	$T = 3$...	$T = 30$	$T = 31$	$T = 32$...
Year 0	C_1	$C_1 (i + 1)$	$C_1 (1 + i)^2$...	$C_1 (1 + i)^{29}$	0	0	...
Year 1	0	$C_1 (i + 1)$	$C_1 (1 + i)^2$...	$C_1 (1 + i)^{29}$	$C_1 (1 + i)^{30}$	0	...
Year 2	0	0	$C_1 (1 + i)^2$...	$C_1 (1 + i)^{29}$	$C_1 (1 + i)^{30}$	$C_1 (1 + i)^{31}$...
...

Given this schedule, the present value of liabilities that will be calculated at any time t is given by the growth annuity formula $PV(\text{Liabilities}_t) = C_1 (1 + i)^{t-1} G(r_{ft}, i, 30)$ where $G(r_{ft}, i, 30)$ is the thirty year growth annuity value based on the rate r_{ft} , that is used at time t to determine the present value of future liabilities.² The initial expected cash flow, C_1 , is set so that $C_1 G(r_{f0}, i, 30) = 1/(1 + \eta)$. η represents the initial funding level of the

²We ignore term structure issue here. Many financial economists would argue that each separate cash flow should be discounted at an individual rate that reflects the shape of the term structure that prevails at that horizon. While we agree with this point, for simplicity, and in keeping with many practitioners, at time t we discount cash flows of all horizons at the same discount rate r_{ft} .

fund expressed so that, for every £1 of current liabilities the fund has market assets valued at £1 + η . For our illustrative example, we set $\eta = 15\%$. We also assume that there is no net contribution to the asset side of the balance sheet after time zero. New contributions exactly offset fund payouts. It should be stressed that these assumptions are for simplicity of exposition. The processes that we describe could easily be extended to more complex cash flow dynamics and net inflows / outflows from the fund.

There are three issues to consider here in relation to the liabilities side of the balance sheet. First, we wish to capture the stochasticity in the discount rate process and the impact this has on solvency risk. Second, we need to capture the correlation structure between the asset and liabilities sides of the balance sheet. Finally, consider our expectation today of the present value of future liabilities that we will calculate at some future time t ; $E_0 [PV (\text{Liabilities}_t)] = E_0 [C_1 (1 + i)^t G (r_{ft}, i, 30)]$. If our cash flow forecast at time t is independent of the prevailing discount rate r_{ft} , this can be divided into $E_0 [C_1 (1 + i)^t] E_0 [G (r_{ft}, i, 30)]$. Take the second of these terms, $E_0 [G (r_{ft}, i, 30)]$. At time zero, we do not know the value of r_{ft} and we need to be particularly careful about how we deal with this uncertainty. In particular, the expectation that we currently make of the annuity value that will be prevail at that date is higher than the annuity value at the expected interest rate; $E_0 [G(r_{ft}, i, T)] < G(E_0 [r_{ft}], i, T)$. To see this, set $i = 4\%$ and $T = 30$ and suppose that the discount rate in five years, r_{f5} might equal 8% or 2% with equal probability. In this case $E_0 [G(r_{f5}, i, T)] = 0.5 [G(8\%, 4\%, 30) + G(2\%, 4\%, 30)] = 28.236$. By contrast, $G(E_0 [r_{f5}], i, T) = G(5\%, 4\%, 30) = 24.955$. This effect is caused by Jensen's inequality and has been very heavily documented in environmental economics, where it leads to a declining schedule of discount rates with time horizon (Weitzman, 2001) and, in turn, a higher social cost of carbon. In a pension fund context, ignoring our uncertainty about the discount rates that we will use at future time t will lead to an underestimate of our expected annuity value at time t , and thus an understatement of the expected present value of future liabilities that will prevail at time t .

Given the focus of this paper on pension fund solvency positions, we define z_t as the market value of assets minus the present value of future liabilities as a proportion of the present value of the future liabilities:

$$z_t = \frac{V_t - C_1(1+i)^{(t-1)}G(r_{ft}, i, 30)}{C_1(1+i)^{(t-1)}G(r_{ft}, i, 30)}$$

Our central interest in this paper is in calculating the probability that $z_t < 0$ for all t up to a horizon of thirty years. Our analysis is based on six models. The first three will take a simple one-state lognormal process for asset returns while the last three will incorporate Markov switching. Models 1 and 4 will have a fixed interest rate: $r_{ft} = r_{f0}$ for all t . All other models will incorporate an AR(1), or, equivalently, a discrete-time Ornstein-Uhlenbeck (O-U) model for the discount rate. In Models 2 and 5 the discount rate process will be independent of the asset returns process. In Models 3 and 6, however, the correlation between discount rates and asset returns will be included.

3 Modelling the risk of underfunding: 1-state

3.1 The models

Model 1. Let μ_t denote the N -vector with elements $E_t[r_{nt+1}]$ and Σ_t to denote the $N \times N$ matrix with elements $Cov_t(r_{nt}, r_{mt})$. In Model 1 it is assumed that $\mu_t = \mu$ and $\Sigma_t = \Sigma$ for all t . In addition, it is assumed that log returns are normally distributed.

In this case, the T -period return has expected value and variance $m_T = Tw'\mu$ and $\sigma_T^2 = Tw'\Sigma w$. This characterisation of the data accurately captures the observed mean and standard deviation of historical asset returns, and the contemporaneous correlation between any two asset classes. It does, though, have a number of weaknesses. First, it does not capture any higher moments of asset returns — in particular their “fat-tailed” (leptokurtotic) properties that are clearly present in the summary statistics of the data. This means that this model underestimates extreme market events, and therefore underestimates the true spread of possible future portfolio values. Second, it is well-known that the variance-covariance matrix of asset returns is time-varying, and this is not captured within this setting.

Model 1 also assumes that the risk-free rate is constant: $r_{ft} = r_{f0}$ for all t . In this case, the probability that $z_t < 0$ is given by

$$\text{Prob}(z_t < 0) = \Phi \left(\frac{\ln(C_1(1+i)^{(t-1)}G(r_{f0}, i, 30)) - m_t}{\sigma_t} \right)$$

where $\Phi(\cdot)$ is the cumulative distribution function for a standard normal distribution.

Model 2. In the second model we assume that asset returns remain lognormally distributed but now allow for the discount rate to follow an independent AR(1), or discrete-time Ornstein-Uhlenbeck, process:

$$r_{ft} - r_{ft-1} = \begin{cases} a + br_{ft-1} + e_t \\ \theta(r_{ft-1} - \bar{r}) + e_t \end{cases}$$

where $\theta = b - 1$ and $\bar{r} = -a/\theta$. This gives the AR(1) process an economic interpretation; \bar{r} represents the long-run interest rate value to which the discount rate mean-reverts and θ is the parameter value that determines the speed and strength of the mean reversion. In this case the probability density function (pdf) of r_{ft} conditional on the current discount rate r_{f0} is:

$$f(r_{ft}) = N \left(\frac{1-b^t}{1-b}a + b^t r_{f0}, \frac{1-b^{2t}}{1-b} \sigma_e^2 \right)$$

This pdf can be used to determine through numerical integration the probability that the pension scheme will become underfunded.

$$\text{Prob}(z_t < 0) = \int_{-\infty}^{\infty} \Phi \left(\frac{\ln(C_1(1+i)^{(t-1)}G(r_{ft}, i, 30)) - m_t}{\sigma_t} \right) f(r_{ft}) dr_{ft}$$

Model 3. In model 3, we continue to model asset returns as one-state lognormal and also discount rates by the AR(1) process, but now allow for the observed correlation between

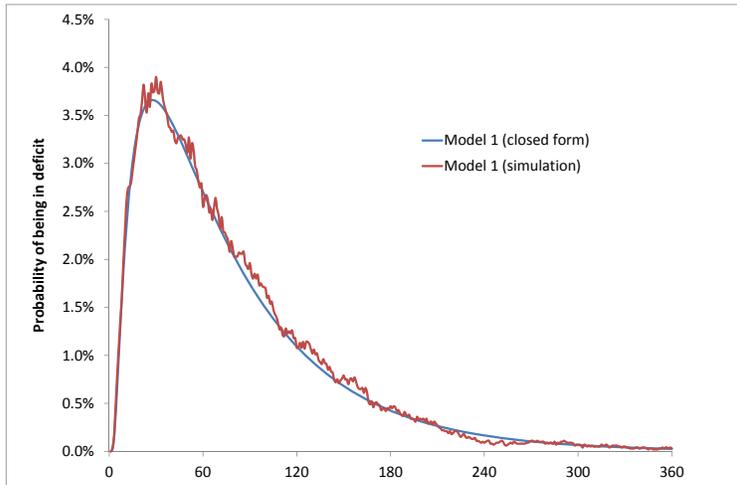


Figure 1: The probability of default at T months under Model 1, where the discount rate is non-stochastic and asset returns follow a one-state log-normal process.

e_t and the asset returns process. In this case, we need to run Monte Carlo simulations to calculate the probability that $z_t < 0$.

To run these simulations we construct the variable $x_t = r_{ft} - (1 + \hat{b})r_{ft-1} = \hat{a} + e_t$, where \hat{b} is estimated over the total sample. The variable x_t is not autocorrelated and is normally distributed and therefore shares the same characteristics as the asset returns process. Therefore it can be co-estimated in the variance-covariance matrix along with the five asset classes. By running 10,000 simulations of these six normally distributed, zero autocorrelation but non-zero cross-correlated variables, we can construct 10,000 values of V_t , r_{ft} and hence z_t . The proportion of these simulations that have a value of $z_t < 0$ reveals the probability of a pension scheme running into deficit.

3.2 The results

In figure 1, we present the results from Model 1 when $i = 4\%$. Results are presented both in closed form and as determined under Monte Carlo simulation. It can be seen that the probability of default rises quickly and then dissipates quickly. This is because, under a frequentist approach, the expected return to our portfolio is substantially above the assumed fixed inflation rate of 4%. Therefore the expected returns effect quickly dominates the stochasticity of the asset returns process. The maximal probability of default is under 4% given the initial solvency of 15%.

In figure 2, we allow for the independent O-U process for the interest rate. Basing our parameterisations on the deannualised monthly 10-year UK Treasury bond yield for the

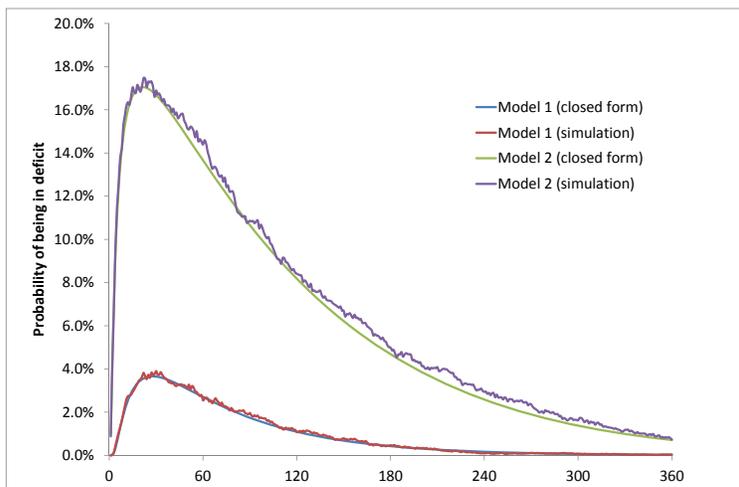


Figure 2: The probability of default at T months under Model 2, where the discount rate follows an Ornstein-Uhlenbeck process, asset returns follow a one-state log-normal process, and the discount rate process is independent of the asset returns process.

interval January 1970 to December 2010, we derive an estimate of $a = 1.651E - 05$ and $b = 0.9964$. As noted above, we do not advocate risk-free discounting of pension liabilities, the choice of this rate simply allows for a long-run interest rate series to enable us to estimate the models. When this is incorporated the results are dramatically different from Model 1 with the maximum probability of underfunding being close to 17%.

The intuition for this is clear. Suppose momentarily that the portfolio will deliver a non-stochastic return of r_p over the next twelve months. For the pension scheme now to be underfunded in one year

$$G(r_{f1}, i, 30) > G(r_{f0}, i, 30) (1 + \eta)(1 + r_p) / (1 + r_i)$$

Adding the assumption that $r_p = 8\%$ to previous ones ($\eta = 15\%$ and $i = 4\%$) and given the data value of $r_{f0} = 3.59\%$, this corresponds to $r_{f1} < 2.47\%$. Based on the calibrated 1-state Ornstein-Uhlenbeck model this movement is not unlikely. The 95% confidence interval for the interest rate in 12 months is 1.35% to 6.08%. Therefore even if there is no uncertainty over asset returns, the interest rate uncertainty effect is highly significant. This drives the difference between the results for Models 1 and 2.

In figure 3, we allow for the covariance between discount rates and asset returns. This somewhat reduces the perceived pension fund solvency risk, with a maximal probability of underfunding now being at about 13%. The reason for this is that, in general, asset returns are negatively correlated with the discount rate. This is most obviously true for the UK Treasury bonds, which make up 40% of the overall portfolio. As might be expected, the total return on this asset is highly negatively correlated with the discount rate (-86%). In addition, UK equity returns are also negatively correlated with bond yields (-32%). Therefore the

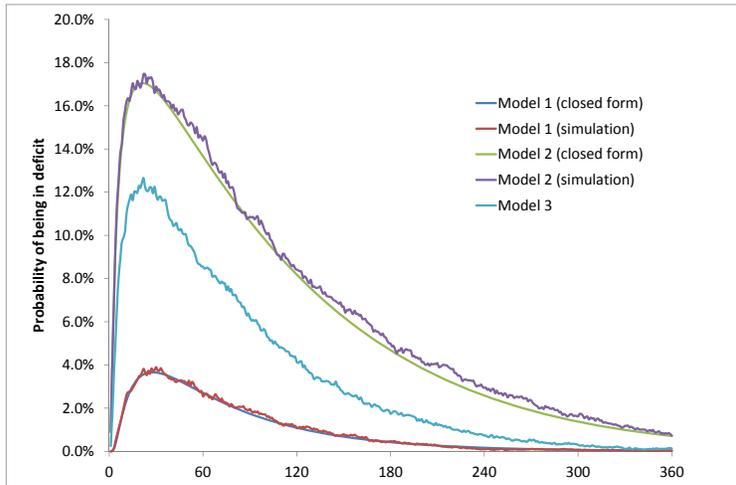


Figure 3: The probability of default at T months under Model 3, where the discount rate follows an Ornstein-Uhlenbeck process, asset returns follow a one-state log-normal process, and the discount rate process is correlated with the asset returns process.

market value of assets and the present value of liabilities tend to move up and down in tandem, providing a hedging effect for the pension fund trustees.

4 Modelling the risk of underfunding: 4-states

To overcome the weaknesses of the traditional model, we next extend our analysis to Markov Switching models. In contrast to the traditional model it is no longer assumed that μ_t and Σ_t are constant across time. Instead, we invoke the Markovian assumptions that, at any time t , the world lies in one of S states. We use the dummy variable $\delta_{st} \in \{0, 1\}$ for $s \in [1, S]$ to denote the state that occurs at time t . Market noise is again modelled within any state as being normally distributed.³

In this Markov world, the probability we assign at time t to the world being in state s at time $t + 1$, $\text{Prob}_t(\delta_{st+1} = 1)$, depends only on the state at time t . A simple, fixed, transition probability matrix, M , can then be used to fully describe the stochastic way in which the prevailing state changes over time. A more detailed description of Markov Switching processes, the alternatives, and their application in a pension fund context are given in Kemp (2011).

In order to estimate this multivariate regime switching environment, we invoke the MSVARlib package in GAUSS (<http://bellone.ensae.net/download.html>) written by Benoit Bellone. This code uses maximum likelihood methods to estimate regimes in a vector au-

³This contrasts with, for example, Elliott and Miao (2009, Quantitative Finance, 9, 747-755) who allow the error terms to be Student-t distributed within a Markov Switching evaluation of Value-at-Risk problems.

autoregressive framework. While Guidolin and Timmerman (2006) demonstrate how to best determine the specification of this model, we use a four-state process throughout, which is consistent with their choice of specification to jointly capture US stock and bond dynamics. This is sufficiently sophisticated to capture many of the broad statistical properties of the historical data, but is sufficiently limited so that the number of parameters for estimation does not get out of hand and so the model remains parsimonious.

We present results here only for AR(0) Markov Switching Models, which results in us not capturing any autocorrelation in the data. This is consistent with standard views of market efficiency that returns should not be predictable over time.⁴ This requires the software to estimate μ_s, Σ_s for each state, the transition probability matrix, M , and a time-series of smoothed probabilities that assign each period in the past to each state. The eigenvectors of M also give the ergodic probabilities, π , associated with each state — this is the proportion of time that the economy spends in each of the states over very long time-periods.⁵

4.1 The Markov environment

Based on the five asset classes on the asset side of the balance sheet, the empirical estimates of M and π under the AR(0) four state specification are:

$$M = \begin{bmatrix} 0.9791 & 0.0010 & 0.0353 & 0.0444 \\ 0.0010 & 0.9817 & 0.0010 & 0.0967 \\ 0.0120 & 0.0010 & 0.9641 & 0.0010 \\ 0.0080 & 0.0163 & 0.0000 & 0.8578 \end{bmatrix}, \quad \pi = \begin{bmatrix} 40.78\% \\ 37.84\% \\ 14.81\% \\ 6.58\% \end{bmatrix}$$

where element $M_{ij} = \text{Prob}(\delta_{it+1} = 1 | \delta_{jt} = 1)$. For example, M_{43} reveals that state 3 (almost) never exists into state 4.

The transition matrix allow us to estimate the expected period of time in any one state before transition into an alternate state. This is given by $\sum_{t=1}^{\infty} t M_{ii}^{t-1} (1 - M_{ii}) = (1 - M_{ii})^{-1}$. This has values of approximately 7 months for state 4 and over 2 years for all the other states.

State 1 For all four states we present three statistics; the average return to each asset, μ_s , the standard deviation of each asset's returns, σ_s , and also the correlation matrix, Σ^* . The variance-covariance matrix for that state is easily reconstructed by $\Sigma_{nm} = \sigma_{sn} \sigma_{sm} \Sigma_{snm}^*$

$$\Sigma_1^* = \begin{bmatrix} 1.00 & 0.74 & 0.91 & 0.02 & 0.47 \\ & 1.00 & 0.77 & -0.05 & 0.43 \\ & & 1.00 & 0.01 & 0.46 \\ & & & 1.00 & -0.04 \\ & & & & 1.00 \end{bmatrix}, \quad \mu_1 = \begin{bmatrix} 0.00583 \\ 0.00862 \\ 0.00720 \\ 0.00705 \\ -0.0022 \end{bmatrix}, \quad \sigma_1 = \begin{bmatrix} 0.047 \\ 0.055 \\ 0.052 \\ 0.013 \\ 0.066 \end{bmatrix}$$

⁴In unreported results, we have also run estimations with VAR(1) specifications for the data. This has no substantive impact on our results and comes at considerable computational cost.

⁵For each of the S states it is necessary to estimate N values of μ_s , $N(N-1)/2$ values of Σ_s . In addition, it is necessary to estimate $S(S-1)$ values of M . This is a total of $S(N + 0.5N(N-1)) + S(S-1) = S(0.5N(N+1) + S-1)$ estimates. Using $N = 5$ and $S = 4$, this is 72 parameter values.

This is the single “bear state” and is the most common state with a high probability of persistence. Average returns are somewhat below their unconditional averages, while standard deviations and correlations are close to their unconditional averages.

State 2 This is the second most common state, with the highest probability of persistence. This is the bull state with high expected returns, low volatilities and low covariances.

$$\Sigma_2^* = \begin{bmatrix} 1.00 & 0.27 & 0.70 & 0.45 & 0.15 \\ & 1.00 & 0.46 & -0.02 & 0.34 \\ & & 1.00 & 0.24 & 0.41 \\ & & & 1.00 & -0.02 \\ & & & & 1.00 \end{bmatrix}, \quad \mu_2 = \begin{bmatrix} 0.016 \\ 0.013 \\ 0.015 \\ 0.010 \\ 0.022 \end{bmatrix}, \quad \sigma_1 = \begin{bmatrix} 0.048 \\ 0.048 \\ 0.034 \\ 0.019 \\ 0.054 \end{bmatrix}$$

State 3 State 3 is the second rarest state and has a high probability of persistence. This is the low volatility state. While the means and correlations are close to the unconditional averages, the volatilities are low in this state.

$$\Sigma_3^* = \begin{bmatrix} 1.00 & 0.65 & 0.91 & -0.08 & 0.47 \\ & 1.00 & 0.76 & 0.01 & 0.37 \\ & & 1.00 & -0.10 & 0.42 \\ & & & 1.00 & 0.01 \\ & & & & 1.00 \end{bmatrix}, \quad \mu_3 = \begin{bmatrix} 0.013 \\ 0.010 \\ 0.015 \\ 0.004 \\ 0.008 \end{bmatrix}, \quad \sigma_3 = \begin{bmatrix} 0.021 \\ 0.028 \\ 0.024 \\ 0.012 \\ 0.046 \end{bmatrix}$$

State 4 State 4 is the rarest state and has the lowest persistence rate. Nevertheless, for the modelling purposes of this paper it is arguably the most interesting as it represents the “crash” state. It has very low expected returns, high correlations between asset classes and high volatility. The joint effect of low expected returns and high correlations is also reported by Ang and Bekaert (2002) on US data.

$$\Sigma_4^* = \begin{bmatrix} 1.00 & 0.68 & 0.88 & 0.32 & 0.43 \\ & 1.00 & 0.82 & 0.22 & 0.52 \\ & & 1.00 & 0.35 & 0.66 \\ & & & 1.00 & 0.51 \\ & & & & 1.00 \end{bmatrix}, \quad \mu_4 = \begin{bmatrix} -0.038 \\ -0.018 \\ -0.022 \\ 0.010 \\ -0.009 \end{bmatrix}, \quad \sigma_4 = \begin{bmatrix} 0.139 \\ 0.088 \\ 0.091 \\ 0.032 \\ 0.064 \end{bmatrix}$$

Simulating asset price dynamics In order to understand how well this AR(0) Markov Switching Process captures the unconditional statistical properties of the data as presented above we have simulated asset returns for 100,000 months. The summary statistics are

presented below:

	Mean	Standard Deviation	Skewness	Excess Kurtosis
UK Equity	0.781%	5.807%	-0.910	6.659
US Equity	0.863%	5.308%	-0.217	1.143
European Equity	0.920%	4.794%	-0.481	2.421
UK Bonds	0.786%	1.723%	0.206	1.631
Japanese Equity	0.743%	6.077%	-0.098	0.171

with correlations over the period

$$\begin{bmatrix} 1.00 & 0.58 & 0.83 & 0.24 & 0.34 \\ & 1.00 & 0.71 & 0.02 & 0.41 \\ & & 1.00 & 0.14 & 0.47 \\ & & & 1.00 & 0.06 \\ & & & & 1.00 \end{bmatrix}$$

Notice that, this process now broadly captures the excess kurtosis of asset returns. This is because of the time-varying nature of the variance-covariance matrix. As is well-known, condition heteroskedasticity leads to unconditional fat-tailed distributions when the process has a constant mean.⁶

Simulated portfolio values A priori, we would expect that the presence of fat-tails from this AR(0) Markov Switching environment would lead to wider 95% confidence intervals for V_T than in the traditional model. This would also be consistent with the findings of Kawata & Kijima (2007). To test this, we simulate forward the value of the portfolio, V_T . We do this 10,000 times for 360 months. We can then compare these value for $T = 5, 10$ and 30 years with those from the 1-state model.

V_T	4-state			1-state		
	$T = 5$	$T = 10$	$T = 30$	$T = 5$	$T = 10$	$T = 30$
Mean	1.71	2.93	26.98	1.69	2.84	23.16
Lower 2.5%	0.87	1.13	4.21	1.04	1.44	6.53
Upper 97.5%	2.80	6.05	98.28	2.58	5.13	60.17

As can be seen, there are significant differences. At a maturity of five years, the traditional model predicts with 97.5% confidence that the value of the portfolio will be at least 4% more than the initial value. By contrast, the AR(0) Markov Switching model gives a 2.5% chance of a fall of 13%. The differences between the models become ever greater with increased time horizons.

4.2 Results

In figure 4, we show the results from Model 4, which incorporates Markov switching in the asset returns process but also assumes that the discount rate $r_{ft} = r_{f0}$ for all t . Now that

⁶See, for example, <http://www.nematrion.com/MixturesOfNormalDistributions.aspx>.

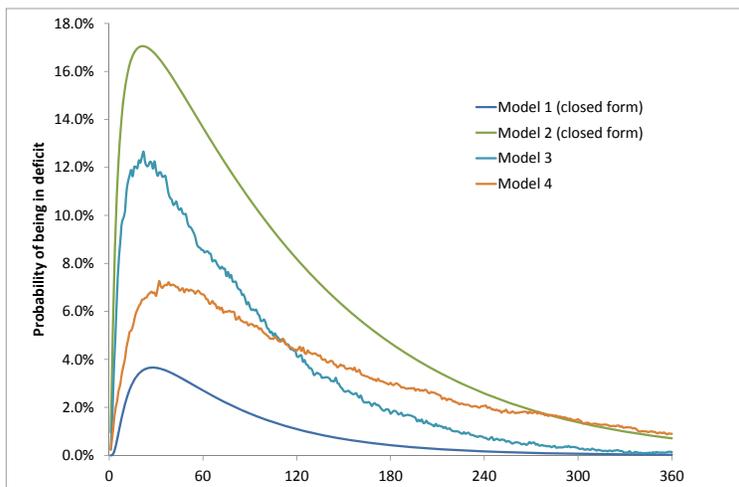


Figure 4: The probability of default at T months under Model 4, where the discount rate is non-stochastic and asset returns follow a four-state Gaussian Markov switching process.

asset returns are not unconditionally normally distributed, we again use 10,000 simulations in order to determine the statistical properties of z_t . There are two key elements to notice from this graph. First, by comparing Model 4 against Models 2 & 3, it is clearly that more accurately modelling the asset side of the balance sheet has less effect on our estimates of z_t than introducing stochasticity into the discount rate process. Second, by comparing Models 1 and 4, it can be seen that allowing for leptokurtosis in asset returns doubles the maximal perceived risk of default. In addition, the risk at longer horizons is much greater in Model 4 than Model 1.

Figure 5 presents the results from Model 5, where the discount rate is stochastic but independent of the Markov switching process. This has the highest default probabilities of all — nearly 20% at its peak. However, the differences between Model 5 and 4 are similar to the differences between Models 2 and 1.

Model 6 is our most sophisticated model. In this case x_t is included as a sixth variable within the Markov Switching calibration exercise. Within an Ornstein-Uhlenbeck interpretation of the interest rate process, this is equivalent to having the speed of mean reversion, θ , constant across states but allowing both the long-run interest rate value \bar{r} and the volatility of the discount rate process, σ_e^2 , to be state dependent. There are now 96 variables that need estimating.

There is one notable problem with this calibration. While we continue to capture leptokurtosis in the asset returns process, this is less pronounced than reported above. For example, the excess kurtosis of UK equities drops from 6.659 to 2.785, compared to 7.436 in the data. While this is a clear improvement on the one-state model there remains a danger that this calibration still somewhat underestimates the true funding risk. Nevertheless, the comparison between Models 6 and 5 are similar to those between Models 3 and 2. This is partly because the correlation between UK equities and discount rates remains negative

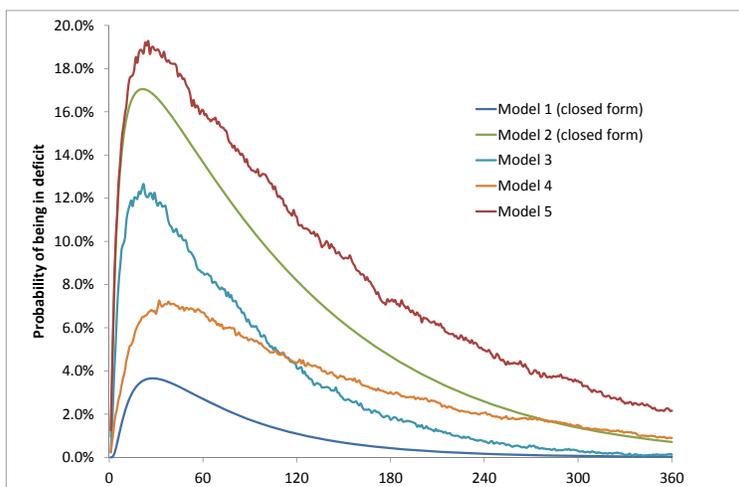


Figure 5: The probability of default at T months under Model 5, where the discount rate follows an Ornstein-Uhlenbeck process, asset returns follow a four-state Gaussian Markov switching process, and the discount rate process is independent of the asset returns process.

even in the crash state (-0.18), somewhat in contrast to recent experience in the credit crisis.

Our results suggest that including both interest rate dynamics and leptokurtosis in asset returns accurately has significant implications for evaluating pension fund solvency risk. In particular, the risks are greater and much longer-lasting than more naive models might suggest. A key warning, though, is that if interest rate dynamics are modelled separately from asset returns dynamics than this may well result in an overestimation of risk.

4.3 Modelling the future funding position of the fund

In addition to revealing the probability that $z_t < 0$, the simulations also present broader statistical information about this variable. The following table presents some summary statistics for z_t at horizons of 1 and 5 years for the four models with stochastic interest rates:

z_t	Mean	Median	Lower 2.5%	Upper 2.5%
	One year horizon			
Model 2	0.259	0.237	-0.188	0.834
Model 3	0.250	0.238	-0.127	0.699
Model 5	0.252	0.228	-0.197	0.843
Model 6	0.231	0.217	-0.141	0.680
	Five year horizon			
Model 2	0.762	0.636	-0.343	2.638
Model 3	0.723	0.645	-0.227	2.094
Model 5	0.741	0.595	-0.386	2.771
Model 6	0.610	0.500	-0.285	2.119

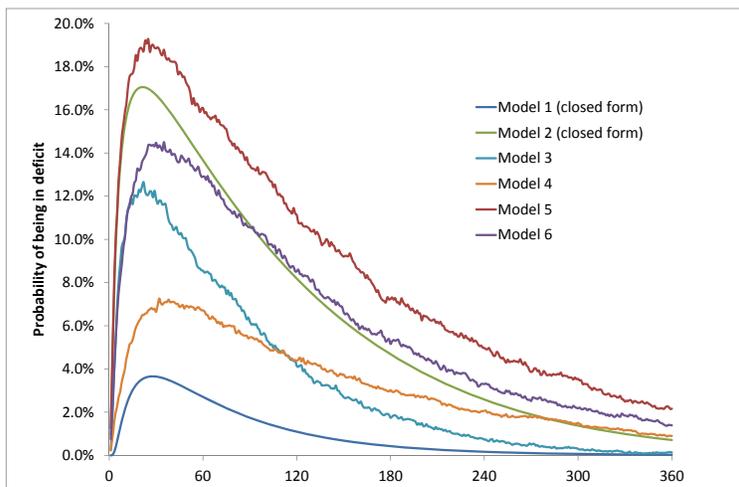


Figure 6: The probability of default at T months under Model 6, where the discount rate follows a four-state Markov switching Ornstein-Uhlenbeck process, asset returns follow a four-state Markov switching process process, and the discount rate process is correlated with the asset returns process.

Again, the effects of accurately modelling the underlying dynamics are dramatic. Despite starting with an initial funding surplus of $\eta = 15\%$, the most sophisticated model predicts that the lower 2.5% of funds will be more than 14% in deficit after 12 months and double that after five years. This again emphasises not only the probability of falling into a funding shortfall but also, in a Value-at-Risk sense, how severe such a fall might be.

5 Conclusion

This paper has examined the impact of extreme market movements on pension fund solvency. This issue was forcefully brought home by the credit crunch in 2008 which resulted in significant falls in asset values, and the subsequent policy response of QE, which dramatically reduced AA bond yields. For pension funds this presented the perfect storm. Asset values were depressed, while the discount rate for the present value of the liabilities were pushed lower, resulting in inflated liabilities. Jointly, the effect was to leave huge deficits on the balance sheet of pension funds. As a result, to better estimate future solvency scenarios it is critical that both asset returns and discount rates are modelled jointly.

To undertake this analysis we use Markov Regime Switching Models. The modelling choice seems appropriate given its widespread use in both academic and practitioner work. Moreover, this type of modelling addresses some of the key issues raised by the Actuarial Profession BSM Working Party in capturing extreme market movements. Markov Regime Switching Models can capture, rare but extreme market events, time-varying asset return volatility and the ‘fat-tailed’ nature of stock returns.

Our results show that future projections of fund solvency are extremely sensitive to modelling choices. In particular, our analysis shows the importance of estimating a stochastic interest process that is allowed to vary with asset returns. If interest rates are allowed to remain constant through time, then both standard ‘one-state’ model and the ‘multi-state’ Markov Regime Switching model significantly underestimate the likelihood of future pension plan underfunding. Where interest rates are estimated as a stochastic process that varies with asset returns, then our results suggest that both the traditional ‘one-state’ model and the ‘multi-state’ model predict a much higher proportion of underfunded pension plans in the future. Although the difference between the most sophisticated multi-state model and the traditional one-state model graphically do not appear to be large, the multi-state model predicts between approximately 2.5% and 4.5% more schemes underfunded and the increased number of underfunded schemes is persistent through time.

These results have a number of pension management and policy considerations. From a micro pension management perspective, the advice that is given to pension managers must be carefully explained. The presentation of one result or one type of result (i.e. a fixed discount rate), does not present an accurate picture for decision making. In particular a much broader and clearer discussion about potential outcomes would allow for more effective decision making around issues such as short-term and long term funding plans, potential risk management strategies and asset allocation decisions.

From a macro-prudential perspective understanding the potential impact of these different outcomes in both the short and long-run has huge implications for macro-prudential pension regulation. For example, if our models were re-calibrated to incorporate a sustained period of low asset returns then the percentage of funds that are likely to have deficits would increase at the further out projections. Consequently, a much richer data set of future pension outcomes could be estimated and better decisions could be made in terms of pension funding at a macro level. Moreover, from the perspective of the TPR scheme specific sensitivities to potential future outcomes could be considered and so the identification of ‘at risk’ schemes may become better, while for the PPF a richer set of future scenarios could be projected that may help with the identification of potential funding pressures.

6 References

Ang, A. and G. Bekaert, International asset allocation with regime shifts, *Review of Financial Studies*, 2002, 15, 1137–1187.

Billio, M. and L. Pelizzon., Value-at-Risk: a multivariate switching regime approach. *Journal of Empirical Finance*, 2000, 7, 531–554.

Boyle, P.P., and T. Draviam., Pricing exotic options under regime switching. *Insurance: Mathematics and Economics* 2007, 40, 267–282.

Chen, P., and H. Yang, Pension funding problem with regime-switching geometric Brownian motion assets and liabilities, *Applied Stochastic Models in Business and Industry*, 2010, 26, 125–141.

Elliott, R.J., and T. K., Siu. On Markov-modulated exponential-affine bond price formulae. *Applied Mathematical Finance*, 2009, 16, 1–15.

Freeman, M.C., The time-varying equity premium and the S&P500 in the twentieth

century, *Journal of Financial Research*, 2011, 34, 179–215.

Guidolin, M., and A. Timmerman, An econometric model of nonlinear dynamics in the joint distribution of stock and bond returns, *Journal of Applied Econometrics*, 2006, 21, 1–22.

Guidolin, M., and A. Timmerman, International asset allocation under regime switching, skew, and kurtosis preferences, *Review of Financial Studies*, 2008, 21, 889–935.

Hamilton, J.D., A new approach to the economic analysis of non-stationary time-series and the business cycle. *Econometrica*, 1989, 57, 357–3840.

Hamilton, J.D., and R. Susmel., Autoregressive conditional heteroskedasticity and changes in regime. *Journal of Econometrics*, 1994, 64, 307–333.

Kawata, R., and M. Kijima, Value-at-risk in a market subject to regime switching, *Quantitative Finance*, 2007, 7, 609–619.

Kemp, M., *Extreme events: Robust portfolio construction in the presence of fat tails*, John Wiley, 2011.

Lam, P. S., The Hamilton model with a general autoregressive component. *Journal of Monetary Economics*, 1990, 26,409–432.

Psaradakis, Z., M. Sola, and F. Spagnolo., On markov error-correction models, with an application to stock prices and dividends, *Journal of Applied Econometrics*, 2004, 19, 69–88.

Psaradakis, Z., M. O. Ravn, and M. Sola., Markov switching causality and the money-output relationship. *Journal of Applied Econometrics*, 2005, 20, 665–683.

Raymond, J.E., and R. W. Rich., Oil and the macroeconomy: A Markov state-switching approach. *Journal of Money, Credit and Banking*, 1997, 29,193–213.

Schwert, G. W., *Business cycles, financial crises, and stock volatility*. Carnegie- Rochester Conference Series on Public Policy, 1989, 31, 83–126.

Storer, P.A., and M. Van Audenrode., Unemployment insurance take-up rates in Canada: Facts, determinants and implications. *Canadian Journal of Economics*, 1995, 28, 822–835.

Turner, C. M., R. Startz, and C. R. Nelson., A Markov model of heteroscedasticity, risk and learning in the stock market. *Journal of Financial Economics*, 1989, 25, 3–22.

Weitzman, M., Gamma discounting, *American Economic Review*, 2001, 91, 260–271.

Zhou, X.Y., and G. Yin., Markowitz’s mean-variance portfolio selection with regime switching: A continuous-time model. *SIAM Journal on Control and Optimization*, 2003, 42, 1466–1482.

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