PORTFOLIO OPTIMIZATION

RENE SCHNIEPER

1998 GENERAL INSURANCE CONVENTION
AND
ASTIN COLLOQUIUM

GLASGOW, SCOTLAND: 7-10 OCTOBER 1998

Portfolio Optimization

René Schnieper

Summary

Based on the profit and loss account of an insurance company we derive a probabilistic model for the financial result of the company, thereby both assets and liabilities are marked to market. We thus focus on the economic value of the company.

We first analyse the underwriting risk of the company. The maximization of the risk return ratio of the company is derived as optimality criterion. It is shown how the risk return ratio of heterogeneous portfolios or of catastrophe exposed portfolios can be dramatically improved through reinsurance. The improvement of the risk return ratio through portfolio diversification is also analysed.

In section 3 of the paper we analyse the loss reserve risk of the company. It is shown that this risk consists of a loss reserve development risk and of a yield curve risk which stems from the discounting of the loss reserves. This latter risk can be fully hedged through asset liability matching.

In section 4 we derive our general model. The portfolio of the company consists of a portfolio of insurance risks and of a portfolio of asset risks. The efficient border of the company is a straight line with a slope equal to the risk return ratio. It makes therefore sense to maximize this ratio which leads to a generalisation of Markowitz's Model to insurance risks and asset risks. Our model allows for a simultaneous optimization of both portfolios of risks. A theorem is derived which gives the optimal retention policy of the company together with its optimal asset allocation.

1 Introduction

The **profit and loss account** of an insurance company typically details the following income items

earned premiums (net of premiums for outwards reinsurance)

investment income

realized capital gains

and the following expenditure positions

incurred claims (net of reinsurance recoveries)

expenses

dividends to policyholders

dividends to shareholders

We assume that the accounts of the company are on an accident year basis. Any other commonly used basis (e.g. underwriting year) can be dealt with after some minimal changes. We shall some times refer to the financial year which is the period covered by the company's accounts.

We split the premium into its different components

pure risk premium

loading for expenses

loading for profit

We split incurred claims into the following two components:

incurred claims pertaining to the current accident year

changes in claim amounts in respect of claims pertaining to previous accident years

We also take unrealized capital gains into account as an income item.

We make the following simplifying assumptions

- expenses and loading for expenses are identical and therefore cancel out
- dividends to policyholders are accounted for as claims
- we are interested in the change in value of the surplus of the company before dividend to shareholders. We therefore ignore this item.
- the period under consideration is the financial year of the company. This is an arbitrary assumption. We could take any other period e.g. a quarter or a multi year period corresponding to the planing horizon of the company
- Payments pertaining to a given period are made at the end of the period
- The premium written in a given period is earned in that period, i.e. the company has no unearned premium reserves. (This assumption can be dropped at the cost of a slight increase in the model complexity. The interest rate risk pertaining to the unearned premium reserves would be treated in a similar way as the interest rate risk pertaining to the loss reserves. Since the former is much less material than the latter, we have chosen to ignore it.)

We make the following model **assumptions**

1. All random variables appearing in the model have finite second order moments
2. The pure risk premium is the present value of the expected loss payments

3. The loss reserves are equal to the present values of expected future loss payments
4. The discount factors used to assess the pure risk premium and the loss reserves are based on the yield curve as defined by the bond market
5. The assets of the company are valued at market value

We introduce the following notation, where random variables are denoted by a tilde:

$\tilde{S}$ total claims amount pertaining to the current accident year.

$E(\tilde{S})$ the mathematical expectation of the above random variable; this is the pure risk premium.

ℓ the profit loading for assuming the underwriting risk $\tilde{S}$

$\tilde{\Delta}L$ increase in claim amounts in respect of claims pertaining to previous accident years.

$\tilde{\Delta}A$ investment income plus realized capital gains plus unrealized capital gains

u capital (economic value) of the company at the beginning of the financial year

$\tilde{\Delta}u$ increase in capital (economic value) during the financial year

The following relation holds true

$$\tilde{\Delta}u = E(\tilde{S}) + \ell - \tilde{S} - \tilde{\Delta}L + \tilde{\Delta}A$$

$\tilde{S} - E(\tilde{S})$ is referred to as the underwriting risk, $\tilde{\Delta}L - E(\tilde{\Delta}L)$ as the loss reserve risk, $\tilde{\Delta}A - E(\tilde{\Delta}A)$ as the asset risk and $\tilde{\Delta}u - E(\tilde{\Delta}u)$ as to the total risk of the company.

2 Underwriting Risk

2.1 Simplified Model

We split the assets of the company between a liability fund and a capital fund $A = A_L + A_U$. This means that some of the assets (A_L) are earmarked to cover the liabilities of the company and the rest of the assets (A_U) match the equity of the company. Since in this section we focus on the underwriting risk, we assume that there is no loss reserve risk and no asset risk. To be more specific, we make the following

Assumptions

- There is no loss reserve risk, i.e. amount and time of payment in respect of outstanding losses are perfectly known to the company.
- The liability fund, i.e. those assets which cover the liabilities, perfectly match the amounts and maturities of the liabilities. The liabilities are discounted with the discount factors corresponding to the liability fund. As a consequence any change in the yield curve will have a perfectly offsetting effect on $\tilde{\Delta}L$ and $-\Delta\tilde{A}_L$.
- The capital fund is invested at the risk free rate of return: $\tilde{\Delta}A_U = \rho_0 u$.

The increase in capital (the profit) now is

$$\tilde{\Delta}u = E(\tilde{S}) + \ell - \tilde{S} - \tilde{\Delta}L + \tilde{\Delta}A_L + \tilde{\Delta}A_U = E(\tilde{S}) + \ell - \tilde{S} + \rho_0 u$$

And we obtain

$$\rho = \frac{E(\tilde{\Delta}u)}{u} = \frac{\ell}{u} + \rho_0 \qquad \sigma = \frac{\sigma(\tilde{\Delta}u)}{u} = \frac{\sigma(\tilde{S})}{u}$$

From which it follows that

$$\rho - \rho_0 = r \cdot \sigma \quad \text{with} \quad r = \frac{\ell}{\sigma(\tilde{S})}$$

the trade off between risk (σ) and excess return $(\rho - \rho_0)$ is thus linear and the slope of the line is equal to the ratio of underwriting return (ℓ) and underwriting risk ($\sigma(\tilde{S})$). Our objective is therefore to maximize this underwriting risk return ratio r.

The above defined straight line is the efficient border of the set of all risk return pairs (σ,ρ) which can be achieved by the company. If (σ,ρ) is on the straight line, an increase in return can only be achieved by the company through an increase in risk.

The choice of a specific point (σ^*,ρ^*) on the efficient border is equivalent to the choice of the capital level of the company. Indeed if (σ^*,ρ^*) is given, then by the very definition of σ and ρ we have $u = \frac{\sigma(\tilde{\Delta}u)}{\sigma^*} = \frac{\ell}{\rho^*-\rho_0}$. On the other hand if u is given, we have $\sigma^* = \frac{\sigma(\tilde{\Delta}u)}{u}$ and $\rho^* = \rho_0 + \frac{\ell}{u}$ and it is easily verified that (σ^*,ρ^*) is on the efficient border.

The choice of a specific point on the efficient border is arbitrary. It depends on the balance between the investors' hunger for profit and their aversion for risk. It is usually formalized by sets of indifference curves, where it is assumed that investors are indifferent between all risk return pairs (σ,ρ) which are on a given curve $\rho = f(\sigma)$. The curves are upward sloping and it is usually assumed that they become steeper as σ increases. (For a discussion of indifference curves see for instance W.F. Sharpe (1970).) Given such a set of curves, there is usually exactly one optimal point on the above defined straight line. This is the optimal risk return pair (σ^*,ρ^*) which can be achieved by the company. The capital level of the company is derived from this optimum $u = \frac{\sigma(\tilde{\Delta}u)}{\sigma^*} = \frac{\ell}{\rho^*-\rho_0}$.

For illustrative purposes, we assume that the owners of the company, or the managers acting on behalf of the owners, have a **quadratic utility function**

$$V(\tilde{\rho}) = a + b\tilde{\rho} - c\tilde{\rho}^2 \qquad b,c \geq 0$$

This utility function is only meaningful for $\tilde{\rho} \leq \rho_{max} = \frac{b}{2c}$, since above ρ_{max} the function is decreasing.

If $\mathrm{Prob}(\tilde{\rho} > \rho_{max}) \simeq 0$, we obtain

$$v = E(V(\tilde{\rho})) \simeq a + b\rho - c\rho^2 - c\sigma^2.$$

This defines a set of indifference curves. All points (ρ,σ) which yield the same values of $v = E(V(\tilde{\rho}))$ are on the same indifference curve.

Assuming that the efficient border of the company is a straight line $\rho - \rho_0 = r\sigma$ it is easily seen that the risk return pair which maximizes the utility of the company is $\sigma^* = \dfrac{(\rho_{max} - \rho_0)r}{1 + r^2}$ $\quad\quad \rho^* = \rho_0 + r\sigma^*$

The corresponding amount of capital is $u = \dfrac{\sigma(\tilde{\Delta}u)}{\sigma^*}$.

For illustrative purposes we shall occasionally assume

$$E[V(\tilde{\rho})] = \rho - 2\rho^2 - 2\sigma^2, \quad \text{i.e. } \rho_{max} = 25\ \%.$$

We now turn to the problem of allocating capital to individual risks.

Let $\tilde{\Delta}u = \sum_{i=1}^{n} \tilde{X}_i$ be any split of the total risk of the company into individual risks. The capital is proportional to

$$\sigma(\tilde{\Delta}u) = \left(\sum_{i=1}^{n} \mathrm{Cov}(\tilde{X}_i, \tilde{\Delta}u)\right)^{1/2}$$

It is thus fair to allocate to each risk $\tilde{X}_i$ an amount of capital u_i which is proportional to the contribution of that risk to the overall volatility of the result of the company: $u_i = k \cdot \mathrm{Cov}(\tilde{X}_i, \tilde{\Delta}u)$. Since $u = \sum_{i=1}^{n} u_i$, we obtain $u_i = u \cdot \dfrac{\mathrm{Cov}(\tilde{X}_i, \tilde{\Delta}u)}{\mathrm{Var}(\tilde{\Delta}u)}$.

The excess return which the company expects to achieve for assuming the risk $\sigma(\tilde{\Delta}u)$ is equal to $(\rho - \rho_0)u$, where ρ_0 denotes the risk free rate of return. It is fair to split the excess return proportionally to the capital.

Definition

The **fair loading** of risk $\tilde{X}_i$ is $(\rho-\rho_0)u_i = (\rho-\rho_0)\cdot u\cdot\frac{Cov(\tilde{X}_i, \tilde{\Delta}u)}{Var(\tilde{\Delta}u)}$

It is equal to the cost of the capital needed for assuming risk $\tilde{X}_i$.

We assume that the company is a price taker, the fair loading is thus not a way to compute prices but a way to define benchmarks. In general there will be cross–subsidies. Certain risks will have a higher expected profit than the fair loading, others will have a lower expected profit. Later we show that if the portfolio of risks is optimized in an unconstrained way, the actual loading of each risk is equal to the fair loading. This is a further justification for our way to allocate capital to individual risks.

We now turn to the problem of maximizing the underwriting risk return ratio. Assuming that the loadings of individual risks are given there are two main possibilities to increase the above ratio: combining risks in a portfolio and buying reinsurance. We now illustrate the impact of reinsurance and the portfolio effect on the risk return ratio.

2.2 Portfolio Heterogeneity

Let $\tilde{X}_1,\tilde{X}_2,\ldots,\tilde{X}_n$ be the uncorrelated risks of a portfolio $\tilde{S}=\sum_{i=1}^{n}\tilde{X}_i$.

Let ℓ_i denote the loading of risk i and σ_i^2 its variance. We have thus

$$\ell = \Sigma\ell_i \text{ and } \sigma(\tilde{S})=(\Sigma\sigma_i^2)^{1/2}.$$

Let us assume that for each individual risk i the company keeps a share α_i for its own account and cedes a share $(1-\alpha_i)$ to its reinsurers.

Theorem

Under the above assumptions, the choice of $\alpha_1,\dots,\alpha_n$ which maximizes the net underwriting risk return ratio

$$r_{net} = \frac{\Sigma\alpha_i \ell_i}{(\Sigma\alpha_i^2\sigma_i^2)^{1/2}}$$

is

$$\alpha_i = c\,\frac{\ell_i}{\sigma_i^2}$$

where c is some norming constant which must be chosen in such a way that

$0 \le \alpha_i \le 1$

for all i. With the so defined set of retentions we have

$$r_{net} = \Big(\sum_i \frac{\ell_i^2}{\sigma_i^2}\Big)^{1/2}$$

Proof

Deriving r_{net} with respect to α_j and setting the derivative equal to 0 we obtain

$$\frac{\ell_j(\Sigma\alpha_i^2\sigma_i^2)^{1/2} - (\Sigma\alpha_i\ell_i)(\Sigma\alpha_i^2\sigma_i^2)^{-1/2}\alpha_j\sigma_j^2}{\Sigma\alpha_i^2\sigma_i^2} = 0$$

$$\ell_j(\Sigma\alpha_i^2\sigma_i^2) = (\Sigma\alpha_i\ell_i)\alpha_j\sigma_j^2$$

$$\alpha_j = \frac{\ell_j}{\sigma_j^2}\,\frac{\Sigma\alpha_i^2\sigma_i^2}{\Sigma\alpha_i\ell_i} = c\,\frac{\ell_j}{\sigma_j^2}$$

and the value of the optimal r_{net} is obtained by plugging the above value of α_j into the expression defining r_{net}.

$\square$

Special Case

Let

$$\tilde{X}_i = \begin{cases} L_i & \text{with probability } p \\ 0 & \text{with probability } 1-p \end{cases}$$

and

$$\ell_i = E(\tilde{X}_i)\lambda = p\,L_i\,\lambda$$

we now have

$$\mathrm{Var}(X_i) = p(1-p)L_i^2 \simeq pL_i^2 \qquad\qquad \text{for } p << 1$$

and the optimal retention becomes

$$\alpha_i = c\,\frac{\ell_i}{\sigma_i^2} \simeq c\,\frac{p\ L_i\ \lambda}{p\ \ L_i^2} = \frac{1}{L_i}\,c\,\lambda$$

$$\Longrightarrow \alpha_i L_i = c\lambda$$

and the retention of each risk is such that the net monetary amount retained is the same for all risks i.e. the reinsurance arrangement which maximizes the underwriting risk return ratio is a surplus treaty, where the retention is equal to the smallest sum insured.

On a gross basis the risk return ratio is

$$r = \frac{\sum_{i=1}^{n} L_i p \lambda}{\left(\sum_{i=1}^{n} L_i^2 p\right)^{1/2}} \simeq \lambda\sqrt{p}\,\frac{\sum_{i=1}^{n} L_i}{\left(\sum_{i=1}^{n} L_i^2\right)^{1/2}}$$

and on a net basis

$$r_{net} = \left(\sum_i \frac{\ell_i^2}{\sigma_i^2}\right)^{1/2} = \lambda\sqrt{pn}$$

It is seen that $r_{net} \geq r$. The inequality is strict unless all L_i's are equal.

Numerical Example

Let us assume that there are two types of risks

$$X_1 = \begin{cases} 1 & \text{with probability } 10^{-3} \\ 0 & \text{with probability } 0.999 \end{cases}$$

and

$$X_2 = \begin{cases} 100 & \text{with probability } 10^{-3} \\ 0 & \text{with probability } 0.999 \end{cases}$$

There are $n = 10^5$ risks of the first type, and $n=10^3$ risks of the second type. The profit loading is $\lambda = 3$ % of the pure risk premium. We have

$$\sigma(\tilde{S}) \simeq \sqrt{10^{-3}(10^5+10^7)} = 100.5, \qquad \ell = 6.0, \qquad r = 0.060$$

According to the above theorem, the reinsurance arrangement which maximizes the underwriting risk return ratio is a surplus treaty with a retention of 1. On a net basis we have

$$\sigma(\tilde{S}_{net})=\sqrt{10^{-3}\cdot(10^5+10^3)} = 10.05, \qquad \ell = 3.03, \qquad r = 0.301$$

The net underwriting risk return ratio is much higher than the gross.

2.3 Catastrophe Exposure

Let $\tilde{S} = \sum_{i=1}^{n} \tilde{X}_i$ be a portfolio of individual risks where each risk is the sum of an ordinary risk and of a catastrophe risk:

$$\tilde{X}_i = {}_0\tilde{X}_i + {}_c\tilde{X}_i.$$

We have thus

$$\tilde{S} = \sum_{i=1}^{n} {}_0\tilde{X}_i + \sum_{i=1}^{n} {}_c\tilde{X}_i.$$

It is further assumed that

$$\mathrm{Cov}({}_0\tilde{X}_i, {}_0\tilde{X}_j) = \delta_{ij}\, \sigma_0^2 \qquad \text{for all i,j,} \qquad \text{where } \delta_{ij} = \begin{cases} 1 & i=j \\ 0 & i \neq j \end{cases}$$

and that

$$\mathrm{Cov}({}_c\tilde{X}_i, {}_c\tilde{X}_j) = \sigma_c^2 \qquad \text{for all i,j}$$

i.e. ordinary risks are uncorrelated and catastrophe risks are perfectly correlated. It is further assumed that

$$\mathrm{Cov}({}_0\tilde{X}_i, {}_c\tilde{X}_j) = 0 \qquad \text{for all i,j.}$$

It follows that

$$\mathrm{Cov}(\tilde{X}_i,\tilde{X}_j) = \mathrm{Cov}({}_0\tilde{X}_i+{}_c\tilde{X}_i,{}_0\tilde{X}_j+{}_c\tilde{X}_j) = \delta_{ij}\sigma_0^2 + \sigma_c^2$$

and

$$\mathrm{Var}(\tilde{S}) = n\sigma_0^2 + n^2\sigma_c^2.$$

Let us now assume that the catastrophe exposure is reinsured through an excess of loss reinsurance with retention x

$$S_{net}=\sum_{i=1}^{n} {}_0\tilde{X}_i + \left(\sum_{i=1}^{n} {}_c\tilde{X}_i\right) \wedge x$$

where $x \wedge y$ denotes the minimum of x and y.

To compute the value of

$$\left(\sum_{i=1}^{n} {}_c\tilde{X}_i\right) \wedge x$$

as a function of x we would need to make distributional assumptions on the catastrophe risk. We make the extreme assumption that the catastrophe risk is fully reinsured, i.e. x=0.

As a consequence we have

$$\mathrm{Var}(\tilde{S}_{net}) = n\sigma_0^2.$$

Let μ_0 and μ_c denote the pure risk premium of an ordinary risk and of a catastrophe risk respectively. Let λ_0 and λ_c denote the premium loading of an ordinary risk and of a catastrophe risk respectively. We have

$$r = \frac{\ell}{\sigma(\tilde{S})} = \frac{n(\mu_0\lambda_0+\mu_c\lambda_c)}{(n\sigma_0^2+n^2\sigma_c^2)^{1/2}} = \frac{\mu_0\lambda_0+\mu_c\lambda_c}{(\frac{\sigma_0^2}{n}+\sigma_c^2)^{1/2}}$$

Assuming that the loading of the reinsurance premium for the catastrophe risk is the same loading as for the original catastrophe risk, we obtain

$$r_{net}=\sqrt{n}\,\frac{\mu_0\lambda_0}{\sigma_0}$$

which is usually much larger than r.

Numerical Example

$$ {}_0\tilde{X}_i = \begin{cases} 100 & \text{with probability } 10^{-3} \\ 0 & \text{with probability } 0.999 \end{cases} $$

$$ {}_c\tilde{X}_i = \begin{cases} 5 & \text{with probability } 10^{-2} \\ 0 & \text{with probability } 0.99 \end{cases} $$

${}_0\tilde{X}_i$ could be a fire claim and ${}_c\tilde{X}_i$ an earthquake claim from a given fire policy.

We have

$$ \mu_0 = 0.1,\ \mu_c = 0.05,\ \sigma_0 \simeq 10^{-3/2} \cdot 100 = 3.16,\ \ \sigma_c \simeq 10^{-1} \cdot 5 = 0.5 $$

Let us assume that

$$ \lambda_0 = 5\ \%,\ \ \lambda_c = 20\ \%\ \text{ and } n = 10^5. $$

We obtain

$$ \sigma(\tilde{S}) = 50'010 \qquad \ell = 1'500 \qquad r = 0.030 $$

$$ \sigma(\tilde{S}_{net}) = 1'000 \qquad \ell_{net} = 500 \qquad r_{net} = 0.500 $$

The net underwriting risk return ratio is much higher than the gross. Assuming $\rho_0 = 5\ \%$ and optimizing according to the quadratic utility function of section 2.1 we obtain the following optimal risk and excess return for the net portfolio

$$ \sigma^* = \frac{(\rho_{max} - \rho_0) \cdot r}{1 + r^2} = 8\ \% \qquad \rho^* - \rho_0 = r \cdot \sigma^* = 4\ \% $$

and the capital is

$$ u = \frac{\sigma(\tilde{S}_{net})}{\sigma^*} = \frac{\ell_{net}}{\rho^* - \rho_0} = 12'500. $$

The corresponding quantities for the gross portfolio are

$$ \sigma^* = 0.0060 \qquad \rho^* - \rho_0 = 0.0002 \qquad u = 8'342'502 $$

From this example it is seen that it would be totally uninteresting to insure the gross portfolio without being able to reinsure a sizeable part of the catastrophe exposure.

2.4 Portfolio Diversification

Let $\tilde{X}_1, \tilde{X}_2, \ldots \tilde{X}_n$ denote the different insurance portfolios of our company (e.g., homeowners, private automobile, commercial multiperil, commercial automobile, assumed reinsurance business, etc...).

Let

$$\pi(\tilde{X}_i) = E(\tilde{X}_i) + \ell_i$$

denote the premium of portfolio $\tilde{X}_i$, ℓ_i is thus the corresponding loading.

We use the following notation

$$\sigma_{ij} = \mathrm{Cov}(\tilde{X}_i, \tilde{X}_j) \qquad\qquad \Sigma = (\sigma_{ij})$$

We assume that the company keeps a share α_i of portfolio $\tilde{X}_i$ for own account and cedes a share $(1-\alpha_i)$ to its reinsurers.

The combined net portfolio of the company is thus

$$\tilde{S}_{net} = \alpha_1 \tilde{X}_1 + \alpha_2 \tilde{X}_2 + \ldots + \alpha_n \tilde{X}_n$$

and its combined net profit loading is

$$\ell_{net} = \alpha_1 \ell_1 + \alpha_2 \ell_2 + \ldots + \alpha_n \ell_n$$

Theorem

We assume that Σ^{-1} exists.

1) The vector $\underline{\alpha}' = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ which maximizes the net underwriting risk return ratio

$$r_{net} = \frac{\ell_{net}}{\sigma(\tilde{S}_{net})}$$

is given by

$$\underline{\alpha} = c \cdot \Sigma^{-1} \cdot \underline{\ell}$$

where $\underline{\ell}=(\ell_1,\ell_2,\ldots,\ell_n)$ and c is a scalar which is chosen in such a way that $\max_{i=1,\ldots,n} \alpha_i=1$.

The optimal risk return ratio is equal to

$$r_{net}=(\underline{\ell}\Sigma^{-1}\underline{\ell})^{1/2}$$

2) $\underline{\alpha}$ maximizes the risk return ratio if and only if the net loadings $(\alpha_i\ell_i \ i=1,..,n)$ are equal to the fair loadings.

Remark

The solution $\underline{\alpha}$ provided by the theorem is only meaningful if $\alpha_i \geq 0$ for all i. It is indeed unrealistic to assume that the company can take a short position in any of the insurance portfolios $\tilde{X}_i$. To find a solution $\underline{\alpha}$ which always satisfies the condition $\underline{\alpha} \geq 0$ is a convex optimization problem with restrictions. It is a standard problem in finance theory, see for instance W.F. Sharpe (1970).

Proof

1) We have to maximize the following expression

$$r = \frac{\alpha_1\ell_1+\alpha_2\ell_2+\ldots+\alpha_n\ell_n}{(\sum_{i,j} \alpha_i\alpha_j\sigma_{ij})^{1/2}}$$

deriving with respect to $\alpha_1,\alpha_2,\ldots,\alpha_n$ and equating the expression to 0, we obtain

$$\frac{\delta r}{\delta\alpha_1} = \frac{\ell_1\sigma(\tilde{S}_{net})-\ell_{net}\ \frac{1}{2}\ \sigma(\tilde{S}_{net})^{-1}(2\sum_{j=1}^{n}\alpha_j\sigma_{1j})}{\sigma^2(\tilde{S}_{net})} = 0$$

$$\vdots$$

$$\frac{\delta r}{\delta\alpha_n} = \frac{\ell_n\sigma(\tilde{S}_{net})-\ell_{net}\ \frac{1}{2}\ \sigma(\tilde{S}_{net})^{-1}(2\sum_{j=1}^{n}\alpha_j\sigma_{nj})}{\sigma^2(\tilde{S}_{net})} = 0$$

and after some straightforward rearrangement of terms

$$\ell_1 \sigma^2(\tilde{S}_{net}) = \ell_{net} \sum_{j=1}^{n} \alpha_j \sigma_{1j}$$

$$\ell_n \sigma^2(\tilde{S}_{net}) = \ell_{net} \sum_{j=1}^{n} \alpha_j \sigma_{nj}$$

or in matrix notation

$$\underline{\ell} \frac{\sigma^2(\tilde{S}_{net})}{\ell_{net}} = \Sigma \, \underline{\alpha}$$

$$\underline{\alpha} = c \, \Sigma^{-1} \underline{\ell}$$

This proves the first part of the theorem. (Note that by definition $\underline{\alpha}$ is only defined up to a norming constant c.)

We now prove the statement about r_{net}.

$$\mathrm{Var}(\tilde{S}) = \underline{\alpha}'\Sigma \, \underline{\alpha} = c^2 \, \underline{\ell}' \, \Sigma^{-1} \, \Sigma \, \Sigma^{-1} \, \underline{\ell} = (c \, \underline{\ell}')(c \, \Sigma^{-1} \, \underline{\ell}) = c \, \underline{\ell}' \underline{\alpha}$$

$$r_{net} = \frac{\underline{\alpha}' \ \underline{\ell}}{\sqrt{c}(\underline{\alpha}' \ \underline{\ell})^{1/2}} = \frac{1}{\sqrt{c}} (\underline{\alpha}' \, \underline{\ell})^{1/2} = \frac{\sqrt{c}}{\sqrt{c}} (\underline{\ell}' \, \Sigma^{-1} \, \underline{\ell})^{1/2}$$

$$r_{net} = (\underline{\ell}' \, \Sigma^{-1} \, \underline{\ell})^{1/2}$$

2) $\alpha_i \ell_i$ i=1,..,n are the fair loadings if and only if

$$\alpha_i \ell_i = c \cdot \mathrm{Cov}(\alpha_i \tilde{X}_i, \tilde{S}_{net}) \qquad i=1,\ldots n$$

for some constant c. This in turn is equivalent with the following system of equations

$$\alpha_i \ell_i = c \cdot \sum_{j=1}^{n} \alpha_i \alpha_j \sigma_{ij} \qquad i=1,2,\ldots,n$$

$$\ell_i = c \cdot \sum_{j=1}^{n} \sigma_{ij} \alpha_j \qquad i=1,2,\ldots,n$$

$$\underline{\ell} = c \, \Sigma \, \underline{\alpha}$$

$$\underline{\alpha} = c^{-1} \Sigma^{-1} \underline{\ell}$$

which is equivalent with $\underline{\alpha}$ maximizing the risk return ratio.

q.e.d.

Numerical Example

There are three portfolios with

$$\sigma_{11} = 1 \qquad \ell_1 = 0.2 \qquad \Longrightarrow \frac{\ell_1}{\sqrt{\sigma_{11}}} = 20\ \%$$

$$\sigma_{22} = 4 \qquad \ell_2 = 0.6 \qquad \Longrightarrow \frac{\ell_2}{\sqrt{\sigma_{22}}} = 30\ \%$$

We think of $\tilde{X}_1$ and $\tilde{X}_2$ as of a motor portfolio and a homeowners portfolio respectively. We assume that both portfolios are exposed to the same natural peril (e.g. storm), which is only reinsured in excess of a substantial retention. The correlation between the two portfolios is therefore positive. Let us assume that it is equal to 0.20.

The third class of business consists of industrial risks with

$$\sigma_{33} = 9\cdot(1.5)^2 = 20.25 \qquad \ell_3 = 1.8 \qquad \Longrightarrow \frac{\ell_3}{\sqrt{\sigma_{33}}} = 40\ \%.$$

The interpretation is that for the same premium income as the homeowners portfolio, the industrial portfolio has a standard deviation of 3, instead of 2 for the homeowners portfolio. The industrial portfolio has 50 % more volume than the homeowners portfolio. It is assumed that the industrial portfolio and each of the personal lines portfolio are uncorrelated. We have thus

$$\Sigma = \begin{pmatrix} 1 & 0.4 & 0 \\ 0.4 & 4 & 0 \\ 0 & 0 & 20.25 \end{pmatrix}, \qquad \underline{\ell} = \begin{pmatrix} 0.2 \\ 0.6 \\ 1.8 \end{pmatrix}$$

From our theorem we obtain that the optimal retentions are

$$\underline{\alpha}' = (1,\ 0.93,\ 0.61)$$

yielding

$$\sigma(\tilde{S}_{\text{net}}) = 3.57 \qquad \ell_{\text{net}} = 1.85 \qquad r_{\text{net}} = 0.518$$

Thus the optimal risk return ratio is much higher than each of the risk return ratios of the individual classes.

Let $\tilde{S}$ be the gross combined portfolio $\tilde{S} = \tilde{X}_1 + \tilde{X}_2 + \tilde{X}_3$ we have

$$\sigma(\tilde{S}) = 5.10 \qquad \ell = 2.60 \qquad r = \frac{\ell_1+\ell_2+\ell_3}{(\sum_{i,j} \sigma_{ij})^{1/2}} = \frac{2.6}{(26.05)^{1/2}} = 0.509$$

which is nearly as high as the optimal risk return ratio.To achieve the optimal ratio the company must cede 7 % of its homeowners business and 39 % of its industrial business. It must thus forgo an expected profit of 0.75 out of a total expected profit of 2.6. It is questionable whether in this case the slight improvement in the risk return ratio is worth this sacrifice.

Let $\rho_0 = 5$ %. We assume that each portfolio is insured separately and that insurance companies optimize their capital allocation according to the set of indifference curves given in section 2.1. We obtain the following results for the three individual portfolios, the combined portfolio and the portfolio with the optimal risk return ratio.

Portfolio number	r	σ^*	$\rho^* - \rho_0$	u
1	0.200	3.85 %	0.77 %	26.0
2	0.300	5.50 %	1.65 %	36.3
3	0.400	6.90 %	2.76 %	65.2
4	0.509	8.09 %	4.12 %	63.2
5	0.518	8.17 %	4.23 %	43.7

where portfolio number 4 is the combined portfolio and portfolio number 5 is the optimal portfolio.

This example illustrates that combining portfolios results in substantial improvements of the risk return ratio. This example also illustrates the fact that, when we combine portfolios in a non optimal way, there is a cross subsidization between portfolios: Let $\tilde{S}$ denote the gross combined portfolio. The fair loadings are

$$\ell_i = (\rho-\rho_0)\cdot u \cdot \frac{Cov(\tilde{X}_i,\tilde{S})}{Var(\tilde{S})}$$

thus

$$\ell_1 = 4.12\,\%\cdot 63.2\cdot\frac{1.4}{26.05} = 0.14 \qquad \ell_2 = 0.44 \qquad \ell_3 = 2.02$$

whereas the actual loadings are

$$\ell_1 = 0.20 \qquad \ell_2 = 0.60 \qquad \ell_3 = 1.80$$

There is a subsidization of $\tilde{X}_3$ from $\tilde{X}_1$ and $\tilde{X}_2$.

3 Loss Reserve Risk

3.1 Individual Accident Year

Since we only consider one accident year, we can assume that the development year t of risk $\tilde{X}$ is also the financial year t of the company. This amounts to a renumbering of the financial years.We first analyze the problem on an undiscounted basis. Later we introduce discounting.

Let $\tilde{X}$ denote a risk, or a portfolio of risks pertaining to a given accident year. Let $\pi(\tilde{X})$ and ℓ denote respectively the premium and the loading of risk $\tilde{X}$. We have

$$\pi(\tilde{X}) = E(\tilde{X}) + \ell.$$

As with all other random variables we assume that $E(X^2)$ is finite. Let us assume that $\tilde{X}$ is paid out over ω development years.

$$\tilde{X} = \sum_{t=1}^{\omega} \tilde{P}_t.$$

$\tilde{P}_t$ denotes the payment made in development year t in respect of risk $\tilde{X}$. Let $\mathcal{X}_t$ denote the information of the company on risk $\tilde{X}$ in development year t. $\mathcal{X}_0$ is the information on the risk prior to underwriting it and we have thus $E(\tilde{X}) = E(\tilde{X} | \mathcal{X}_0)$.

We further introduce the following notation

$$\tilde{X}_t = E(\tilde{X} | \mathcal{X}_t).$$

$\tilde{X}_t$ is the company's estimate of risk $\tilde{X}$ in development year t.

We assume that $\mathcal{X}_0, \mathcal{X}_1, \ldots, \mathcal{X}_t, \ldots$ is an increasing sequence of σ–algebras. It is easily seen that $\tilde{X}_t$ is a martingale.

Let

$$L_t = E(\tilde{P}_{t+1} + \tilde{P}_{t+2} + \ldots \mid \mathcal{X}_t)$$

be the loss reserve of the company at the end of development year t in respect of risk $\tilde{X}$.

Based on the pure risk premium $E(\tilde{X})$, the contribution to results produced by risk $\tilde{X}$ in the successive development years are as follows

$$\tilde{R}_t = \tilde{L}_{t-1} - \tilde{P}_t - \tilde{L}_t \qquad t=1,2,\ldots$$

and the following relation holds true

$$\tilde{R}_t = E(\tilde{X} | \mathcal{X}_{t-1}) - E(\tilde{X} | \mathcal{X}_t) \qquad t=1,2,\ldots$$

$\tilde{R}_t$ is the difference process of a martingale (i.e. of $E(-\tilde{X} | \mathcal{X}_t)$).

Note that according to our terminology, $\tilde{R}_1$ is the underwriting risk and $\tilde{R}_2 + \ldots + \tilde{R}_\omega$ is the loss reserve risk.

Theorem

$$E(\tilde{R}_t) = 0 \quad t=1,2,\ldots, \qquad Cov(\tilde{R}_t, \tilde{R}_s) = 0 \quad t \neq s, \qquad Var(\tilde{X}) = \sum_{t=1}^{\omega} Var(\tilde{R}_t)$$

Proof

– $\tilde{X}_t = -E(\tilde{X}|\mathcal{X}_t)$ is a martingale and $\tilde{R}_t$ is the corresponding difference process.

q.e.d

Let ℓ denote the loading for profit pertaining to risk $\tilde{X}$. We make the assumption that ℓ is earned over the whole development period of risk $\tilde{X}$. The amount earned during development year t is

$$\ell_t = \ell \cdot \frac{\mathrm{Var}(\tilde{R}_t)}{\mathrm{Var}(\tilde{X})}.$$

The above theorem ensures that $\sum_{t=1}^{\omega} \ell_t = \ell$.

We now introduce discounting. Let $\tilde{\delta}(u)$, a random variable, denote the interest rate intensity at time u. The present value at time s of one monetary unit paid at time t is then

$$\tilde{v}(s,t) = e^{-\int_s^t \tilde{\delta}(u)du}$$

Let $\mathcal{G}_t$ denote the cumulative information on the interest rate intensity up to the end of financial year t (which is also development year t of risk $\tilde{X}$). It is assumed that $\mathcal{G}_0, \mathcal{G}_1, \ldots \mathcal{G}_t, \ldots$ is an increasing sequence of σ–algebras.

We have now

$$\tilde{X} = \tilde{v}(0,1) \cdot \tilde{P}_1 + \tilde{v}(0,2) \cdot \tilde{P}_2 + \ldots + \tilde{v}(0,\omega) \cdot \tilde{P}_\omega$$

Let

$$L_t = E\left(\sum_{s=1}^{\omega-t} \tilde{v}(t,t+s)\, \tilde{P}_{t+s} \,|\, \mathcal{X}_t, \mathcal{G}_t\right)$$

be the loss reserve of the company in respect of risk $\tilde{X}$ at the end of development year t. As a special case we have $L_0 = E(\tilde{X})$.

The loss development risk in development year t is

$$\tilde{R}_t = L_{t-1} - P_t - L_t$$

$$\tilde{R}_t = E\Big(\sum_{s=1}^{\omega-t+1} \tilde{v}(t-1,t-1+s)\,\tilde{P} \qquad |\mathcal{X}_{t-1},\mathcal{G}_{t-1}\Big) - \tilde{P}_t$$
$$- E\Big(\sum^{\omega-t} \tilde{v}(t,t+s)\,\tilde{P} \qquad |\mathcal{X}_t,\mathcal{G}_t\Big)$$
$$= \Big[\, E\Big(\sum_{s=0}^{\omega-t} \tilde{v}(t-1,t+s)\,\tilde{P}_{t+s}|\mathcal{X}_{t-1},\mathcal{G}_{t-1}\Big) - E\Big(\sum_{s=0}^{\omega-t} \tilde{v}(t-1,t+s)\,\tilde{P}_{t+s}|\mathcal{X}_t,\mathcal{G}_{t-1}\Big)\,\Big]$$
$$+ \Big[\, E\Big(\sum_{s=0}^{\omega-t} \tilde{v}(t-1,t+s)\,\tilde{P}_{t+s}|\mathcal{X}_t,\mathcal{G}_{t-1}\Big) - E\Big(\sum_{s=0}^{\omega-t} \tilde{v}(t,t+s)\,\tilde{P}_{t+s}|\mathcal{X}_t,\mathcal{G}_t\Big)\,\Big]$$

$$\tilde{R}_t = {}_1\tilde{R}_t + {}_2\tilde{R}_t$$

Assumption 6

The interest rate process and the claims process are stochastically independent.

Under the above assumption we obtain

$${}_1\tilde{R}_t = \sum_{s=0}^{\omega-t} E(\tilde{v}(t-1,t+s)|\mathcal{G}_{t-1})\cdot(E(\tilde{P}_{t+s}|\mathcal{X}_{t-1}) - E(\tilde{P}_{t+s}|\mathcal{X}_t))$$

${}_1\tilde{R}_t$ is the loss reserve development risk. It is seen at once that $E({}_1\tilde{R}_t) = 0$. In addition the company will earn a profit loading ℓ_t, as defined above, for assuming the risk ${}_1\tilde{R}_t$.

We also have

$${}_2\tilde{R}_t = \sum_{s=0}^{\omega-t} E(\tilde{P}_{t+s}|\mathcal{X}_t)\cdot(\,E(\tilde{v}(t-1,t+s)\,|\mathcal{G}_{t-1}) - E(\,\tilde{v}(t,t+s)|\mathcal{G}_t)\,)$$

$${}_2\tilde{R}_t = \sum_{s=0}^{\omega-t} E(\tilde{P}_{t+s}|\mathcal{X}_t)\cdot(\,E(\tilde{v}(t-1,t+s)\,|\mathcal{G}_{t-1}) - E(\,\tilde{v}(t-1,t+s)|\mathcal{G}_t)$$
$$+ E(\,\tilde{v}(t-1,t+s)|\mathcal{G}_t) - E(\tilde{v}(t,t+s)|\mathcal{G}_t)\,)$$

$${}_2\tilde{R}_t = \sum_{s=0}^{\omega-t} E(\tilde{P}_{t+s}|\mathcal{X}_t)\cdot(\,E(\tilde{v}(t-1,t+s)\,|\mathcal{G}_{t-1}) - E(\,\tilde{v}(t-1,t+s)|\mathcal{G}_t)\,)$$
$$+ \sum_{s=0}^{\omega-t} E(\tilde{P}_{t+s}|\mathcal{X}_t)\cdot E(\,\tilde{v}(t-1,t+s)|\mathcal{G}_t)\cdot(\,1 - \tilde{v}^{-1}(t-1,t)\,)$$

and it is seen that the first term is the yield curve risk stemming from the discounting of the loss reserves and the second term is the unwinding of the discount.

$-{}_2\tilde{R}_t$ can be viewed as the yield in financial year t of a bond portfolio with the amounts $E(\tilde{P}_t|\mathcal{X}_t)$, $E(\tilde{P}_{t+1}|\mathcal{X}_t)$, ... , $E(\tilde{P}_\omega|\mathcal{X}_t)$ maturing at time t, t+1, ..., ω respectively. The risk ${}_2\tilde{R}_t$ can therefore be perfectly hedged through asset liability matching.

3.2 Different Accident Years

Let $\tilde{X}_1$, $\tilde{X}_2$, ...,$\tilde{X}_\omega$ denote a risk or a portfolio of risks pertaining to accident years 1,2,...,ω. Let $\tilde{P}_{t,s}$ denote the claims payment made in respect of accident year t, in development year s. It is assumed that each $\tilde{X}_t$ is paid over ω development years. We have

$$\tilde{X}_t = \sum_{s=1}^{\omega-t+1} \tilde{v}(t-1,t-1+s)\, P_{t,s}$$

where $\tilde{v}(s,t)$ is defined as in the preceding subsection.

$\mathcal{X}_{t,s}$ (s=1,2,...,ω) is the σ–algebra generated by $\{\tilde{P}_{t,1}, \tilde{P}_{t,2}, \ldots ,\tilde{P}_{t,s}\}$.

$\mathcal{G}_t$ is the σ–algebra generated by $\{\ \tilde{\delta}(u)\ |\ u \leq t\ \}$.

The loss reserve held by the company in respect of accident year t at the beginning of financial year ω is

$$L_{t,\omega-t} = E\left(\sum_{s=\omega-t+1}^{\omega} \tilde{v}(\omega-t,s)\, \tilde{P}_{t,s}\ |\ \mathcal{X}_{t,\omega-t}, \mathcal{G}_{\omega-1}\right).$$

At the end of financial year ω it pays $P_{t,\omega-t+1}$ and puts up a reserve

$$L_{t,\omega-t+1} = E\left(\sum_{s=\omega-t+2}^{\omega} \tilde{v}(\omega-t+1,s)\, \tilde{P}_{t,s}\ |\ \mathcal{X}_{t,\omega-t+1}, \mathcal{G}_{\omega}\right).$$

The risk materializing during financial year ω in respect of accident year t is

$$\tilde{R}_{t,\omega-t+1} = L_{t,\omega-t} - P_{t,\omega-t+1} - L_{t,\omega-t+1}$$

And the overall loss reserve risk is thus

$$\tilde{\Delta}L = -\sum_{t=1}^{\omega-1} R_{t,\omega-t+1}.$$

Note that $\tilde{R}_{\omega,1}$ is the underwriting risk in respect of accident year ω and is therefore not part of the loss reserve risk.

Upon rearranging terms, we obtain

$$\begin{aligned}
\tilde{R}_{t,\omega-t+1} &= E\Big(\sum_{s=\omega-t+1}^{\omega}\tilde{v}(\omega-t,s)\,\tilde{P}_{t,s}\mid \mathcal{X}_{t,\omega-t},\mathcal{G}_{\omega-1}\Big)\\
&\quad - E\Big(\sum_{s=\omega-t+1}^{\omega}\tilde{v}(\omega-t+1,s)\,\tilde{P}_{t,s}\mid \mathcal{X}_{t,\omega-t+1},\mathcal{G}_{\omega}\Big)\\
&= \Big[E\Big(\sum_{s=\omega-t+1}^{\omega}\tilde{v}(\omega-t,s)\,\tilde{P}_{t,s}\mid \mathcal{X}_{t,\omega-t},\mathcal{G}_{\omega-1}\Big)\\
&\quad - E\Big(\sum_{s=\omega-t+1}^{\omega}\tilde{v}(\omega-t,s)\,\tilde{P}_{t,s}\mid \mathcal{X}_{t,\omega-t+1},\mathcal{G}_{\omega-1}\Big)\Big]\\
&\quad + \Big[E\Big(\sum_{s=\omega-t+1}^{\omega}\tilde{v}(\omega-t,s)\,\tilde{P}_{t,s}\mid \mathcal{X}_{t,\omega-t+1},\mathcal{G}_{\omega-1}\Big)\\
&\quad - E\Big(\sum_{s=\omega-t+1}^{\omega}\tilde{v}(\omega-t+1,s)\,\tilde{P}_{t,s}\mid \mathcal{X}_{t,\omega-t+1},\mathcal{G}_{\omega}\Big)\Big]
\end{aligned}$$

$$R_{t,\omega-t+1} = {}_1R_{t,\omega-t+1} + {}_2R_{t,\omega-t+1}$$

Using assumption 6 we obtain

$$\begin{aligned}
{}_1R_{t,\omega-t+1} &= \sum_{s=\omega-t+1}^{\omega} E(\tilde{v}(\omega-t,s)\mid \mathcal{G}_{\omega-1})\cdot\big(E(\tilde{P}_{t,s}\mid \mathcal{X}_{t,\omega-t})\\
&\quad - E(\tilde{P}_{t,s}\mid \mathcal{X}_{t,\omega-t+1})\big)\\
{}_2R_{t,\omega-t+1} &= \sum_{s=\omega-t+1}^{\omega} E(\tilde{P}_{t,s}\mid \mathcal{X}_{t,\omega-t+1})\cdot\big(E(\tilde{v}(\omega-t,s)\mid \mathcal{G}_{\omega-1})\\
&\quad - E(\tilde{v}(\omega-t+1,s)\mid \mathcal{G}_{\omega})\big)
\end{aligned}$$

Let $\tilde{\Delta}L = \tilde{\Delta}L_1 + \tilde{\Delta}L_2$ with $\tilde{\Delta}L_i = -\sum_{t=1}^{\omega-1} {}_iR_{t,\omega-t+1}$ $i=1,2$.

$\tilde{\Delta}L_1$ is the loss reserve development risk and $\tilde{\Delta}L_2$ is the yield curve risk combined with the unwinding of the discount.

It is easily seen that $E(\tilde{\Delta}L_1) = 0$. In return for the assumption of the risk $\tilde{\Delta}L_1$ the company earns a profit loading

$$\ell_1 = \sum_{t=1}^{\omega-1} \ell_{t,\omega-t+1}$$

where $\ell_{t,\omega-t+1}$ is the profit loading pertaining to accident year t in development year $\omega-t+1$ (see section 3.1).

Upon rearranging terms we obtain

$$\tilde{\Delta}L_2 = -\sum_{t=1}^{\omega-1} {}_2R_{t,\omega-t+1}$$

$$\tilde{\Delta}L_2 = \sum_{t=1}^{\omega-1} \sum_{s=\omega-t+1}^{\omega} E(\tilde{P}_{t,s} \mid \mathcal{H}_{t,\omega-t+1}) \cdot (E(\tilde{v}(\omega-t+1,s) \mid \mathcal{G}_{\omega}) - E(\tilde{v}(\omega-t,s) \mid \mathcal{G}_{\omega-1}))$$

$$\tilde{\Delta}L_2 = \sum_{s=0}^{\omega-1} k_s (E(\tilde{v}(\omega,\omega+s) \mid \mathcal{G}_{\omega}) - E(\tilde{v}(\omega-1,\omega+s) \mid \mathcal{G}_{\omega-1}))$$

with

$$k_s = \sum_{t=s+1}^{\omega} E(\tilde{P}_{t,\omega+s+1-t} \mid \mathcal{H}_{t,\omega-t+1}).$$

Thus

$$\tilde{\Delta}L_2 = \sum_{s=0}^{\omega-1} k_s (E(\tilde{v}(\omega,\omega+s) \mid \mathcal{G}_{\omega}) - E(\tilde{v}(\omega-1,\omega+s) \mid \mathcal{G}_{\omega}) + E(\tilde{v}(\omega-1,\omega+s) \mid \mathcal{G}_{\omega}) - E(\tilde{v}(\omega-1,\omega+s) \mid \mathcal{G}_{\omega-1}))$$

$$\tilde{\Delta}L_2 = \sum_{s=0}^{\omega-1} k_s (E(\tilde{v}(\omega-1,\omega+s) \mid \mathcal{G}_{\omega}) \cdot (\tilde{v}^{-1}(\omega-1,\omega) - 1)) + \sum_{s=0}^{\omega-1} k_s (E(\tilde{v}(\omega-1,\omega+s) \mid \mathcal{G}_{\omega}) - E(\tilde{v}(\omega-1,\omega+s) \mid \mathcal{G}_{\omega-1}))$$

where the first term is the unwinding of the discount and the second term is the yield curve risk stemming from the discounting of the loss reserves. We have thus

$$\tilde{\Delta}L_2 = \tilde{R}_L \cdot L$$

where $L = \sum_{s=0}^{\omega-1} k_s \, E(\tilde{v}(\omega-1,\omega+s) \mid \mathcal{G}_{\omega-1})$ is the total discounted loss reserves at the beginning of financial year ω and $\tilde{R}_L$ is the yield for financial year ω of a bond portfolio with the amounts k_s maturing at the end of financial year $\omega+s$ ($s=0,1,\ldots,\omega-1$). $\tilde{R}_L$ is the rate of return of a bond portfolio with the same maturities as the liabilities of the company. $\tilde{\Delta}L_2$ can thus be perfectly hedged through asset liability matching.

In conclusion the loss reserve risk consists of two parts

$$\tilde{\Delta}L = (\tilde{\Delta}L_1 - \ell_1) + \tilde{R}_L \cdot L$$

a loss reserve development risk ($\tilde{\Delta}L_1$) and a yield curve risk ($\tilde{R}_L \cdot L$).

4 General Model Including Asset Risk

4.1 Optimality Criterion

We have obtained the following representation for the capital increase (profit) of the company during the financial year

$$\tilde{\Delta}u = (E(\tilde{S})+\ell-\tilde{S}) + (\ell_1-\tilde{\Delta}L_1) - \tilde{R}_L \cdot L + \tilde{\Delta}A$$

The first two terms are **insurance risks** (underwriting and loss reserve development risk), the last two terms are **financial risks** (yield curve risk and asset risk).

It is assumed that there are n different categories of assets. $\tilde{R}_j$, a random variable, denotes the return of asset category j. A_j denotes the amount invested by the company in asset category j. We have

$$\tilde{\Delta}A = \sum_{j=1}^{n} \tilde{R}_j \cdot A_j.$$

Let ρ_0 denote the return of the risk free asset. We obtain the following representation for the excess profit of the company

$$\tilde{\Delta}u - \rho_0 u = (E(\tilde{S})+\ell-\tilde{S}) + (\ell_1-\tilde{\Delta}L_1) - (\tilde{R}_L-\rho_0)\cdot L + \sum_{j=1}^{n}(\tilde{R}_j-\rho_0)\cdot A_j$$

where we have used the fact that the sum of the liabilities of the company is equal to the sum of its assets

$$L+u = \sum_{j=1}^{n} A_j$$

Let

$$E(\tilde{S}) + \ell - \tilde{S} = \sum_{i=1}^{m} E(\tilde{X}_i) + \ell_i - \tilde{X}_i$$

and

$$\ell_1 - \tilde{\Delta}L_1 = \sum_{i=1}^{m'} \ell_i' - \tilde{X}_i' \qquad (E(\tilde{X}_i') = 0)$$

be a split of the underwriting risk and of the loss reserve development risk respectively into individual risks (e.g. lines of business, market segments, etc....). We assume that company keeps a share α_i ($\alpha_i \in [0,1]$) of each individual underwriting risk and cedes $1 - \alpha_i$ via quota share reinsurance. Similarly the company retains a share β_j of loss reserve development risk j. The excess profit of the company now reads

$$\tilde{\Delta}u - \rho_0 u = \sum_{i=1}^{m} \alpha_i \cdot (E(\tilde{X}_i) + \ell_i - \tilde{X}_i) + \sum_{j=1}^{m'} \beta_j \cdot (\ell_j' - \tilde{X}_j') - (\tilde{R}_L - \rho_0) \cdot L$$
$$+ \sum_{i=1}^{n} (\tilde{R}_j - \rho_0) \cdot A_j$$

And it is seen that portfolio optimization amounts to an 'optimal' choice of the α's, β's and A's. We now derive the optimality criterion.

The company is interested in its return

$$\tilde{R} = \frac{\tilde{\Delta}u}{u}$$

and in particular in the first two moments of its return

$$\rho(u) = E(\tilde{R}) = \frac{E(\tilde{\Delta}u)}{u}$$

$$\sigma(u) = \sigma(\tilde{R}) = \frac{\sigma(\tilde{\Delta}u)}{u}$$

[illegible]

[illegible]

[illegible]

$$[illegible] \frac{\rho(u) - \rho_0}{\sigma(u)}$$

Since we have

$$r(u) = \frac{E(\tilde{\Delta}u) - \rho_0 \cdot u}{\sigma(\tilde{\Delta}u)}$$

it is easily seen from the above representation of the excess profit of the company, that r(u) is independent of u. Thus the **efficient frontier** of the company (i.e. those achievable pairs $(\rho(u),\sigma(u))$ which have the property that an increase in expected return $(\rho(u))$ can only be achieved through an increase in risk $(\sigma(u))$) is the straight line defined by the equation $\rho(u) - \rho_0 = r \cdot \sigma(u)$. Hence the following

Definition

A portfolio is **optimal** if and only if the corresponding risk return ratio $r(\underline{\alpha},\underline{\beta},\underline{A})$ is maximal.

Usually $r(\underline{\alpha},\underline{\beta},\underline{A})$ is maximized under certain constraints such as $\alpha_i \epsilon [0,1]$ and $\beta_j \epsilon [0,1]$.

Once the company portfolio has been determined, the risk return ratio and the efficient border of the company are given. The company still has to choose a specific point (ρ^*,σ^*) on the efficient frontier.

This choice is equivalent to the choice of the amount of capital of the company

$$u = \frac{\sigma(\tilde{\Delta}u)}{\sigma^*}.$$

Let $\tilde{\Delta}u = \sum_{i=1}^{n} \tilde{Z}_i$ be any split of the total risk of the company into individual risks. Since the amount of capital required to assume the total risk $\tilde{\Delta}u$ is proportional to

$$\sigma(\tilde{\Delta}u) = \left(\sum_{i=1}^{n} \mathrm{Cov}(\tilde{Z}_i,\tilde{\Delta}u)\right)^{1/2}$$

We allocate to each individual risk $\tilde{Z}_i$ an amount of capital u_i which is proportional to the contribution of that risk to the overall volatility of the result of the company

$$u_i = k \cdot \mathrm{Cov}(\tilde{Z}_i,\tilde{\Delta}u).$$

Since $\sum_{i=1}^{n} u_i = u$, we obtain

$$u_i = u \cdot \frac{Cov(\tilde{Z}_i, \tilde{\Delta}u)}{Var(\tilde{\Delta}u)}.$$

The excess profit which the company expects to achieve for assuming the risk $\sigma(\tilde{\Delta}u)$ is $(\rho-\rho_0)\cdot u$. It is fair to split the excess profit proportionally to the allocated capital. Thus

Definition

The **fair loading** of risk $\tilde{Z}_i$ is

$$(\rho-\rho_0)\cdot u_i = (\rho-\rho_0)\cdot u \cdot \frac{Cov(\tilde{Z}_i, \tilde{\Delta}u)}{Var(\tilde{\Delta}u)}$$

Remark

If the $\tilde{Z}_i$'s are uncorrelated the fair loading amounts to the variance principle. The multiple of the variance which must be loaded is derived from the company portfolio, capitalization level and return objective:

$$(\rho-\rho_0)\cdot u \cdot Var^{-1}(\tilde{\Delta}u).$$

4.2 Portfolio Optimization

The excess profit of the company is

$$\tilde{\Delta}u - \rho_0 u = \sum_{i=1}^{m} \alpha_i \cdot (E(\tilde{X}_i) + \ell_i - \tilde{X}_i) + \sum_{j=1}^{m} \beta_j \cdot (\ell_j' - \tilde{X}_j') - (\tilde{R}_L - \rho_0)\cdot L$$
$$+ \sum_{i=1}^{n} (\tilde{R}_j - \rho_0)\cdot A_j$$

and our objective is to maximize the risk return ratio of the company

$$r = \frac{\rho(u) - \rho_0}{\sigma(u)} = \frac{E(\tilde{\Delta}u) - \rho_0 \cdot u}{\sigma(\tilde{\Delta}u)}.$$

In a first step we have to maximize the risk return ratio of the underwriting and loss reserve subportfolio through reinsurance buying. This leads to more homogeneous and less catastrophy exposed portfolios and hence to higher risk return ratios of the subportfolios. This process is discussed in section 2 Underwriting Risk.

We now turn to the second step which consists in the optimization of the global portfolio, i.e. in maximizing the above risk return ratio as a function of the α's, β's and A's. We first review the above general model for the excess profit of the company. It is not realistic to assume that the company can at the beginning of every financial year set a new retention β_j for each loss reserve risk. The reinsurance market for loss reserves is very limited. We shall therefore assume that the company cedes the same quota share $1-\alpha_i$ of each underwriting risk over many years and that it cedes no specific loss reserve quota share.

More formally we make the following

Assumption 7

$m = m'$, $\alpha_i = \beta_i$ $i=1,\dots,n$.

In addition we shall also assume that

$\tilde{R}_1 = \tilde{R}_L$.

This is no restriction of generality. It simply amounts to the convention that the first asset category is a bond portfolio with the same maturity profile as the expected maturity profile of the loss reserves. This excess profit of the company now reads

$$\tilde{\Delta}u - \rho_0 u = \sum_{i=1}^{m} \alpha_i \cdot (E(\tilde{X}_i) + \ell_i + \ell_i' - \tilde{X}_i - \tilde{X}_i') + (\tilde{R}_1 - \rho_0) \cdot (A_1 - L) + \sum_{j=2}^{n} \tilde{R}_j \cdot A_j$$

where $L = \sum_{i=1}^{m} \alpha_i L_i$ and L_i is the amount of discounted loss reserves pertaining to subportfolio i.

We introduce the following notation

$$\underline{x}' = (\alpha_1,\dots,\alpha_m,A_1-L,A_2,\dots,A_n)$$

$$\underline{\mu}' = (\ell_1+\ell_1',\dots,\ell_m+\ell_m',R_1-\rho_0,\dots,R_n-\rho_0)$$

where $R_j = E(\tilde{R}_j)$

$$\Sigma = \mathrm{Cov}(-(\tilde{X}_1+\tilde{X}_1') \dots , -(\tilde{X}_m+\tilde{X}_m'),\tilde{R}_1,\dots,\tilde{R}_n).$$

The optimization problem now reads

$$r = \frac{\underline{\mu}'\cdot\underline{x}'}{(\underline{x}'\Sigma\underline{x})^{1/2}} = \max_{\underline{x}\,\varepsilon\,\beta}! \;, \; \beta = \{\underline{x} \mid \alpha_i\varepsilon[0,1]\}$$

Remarks

1. Since the expected maturity profile of the loss reserves depends on the retentions α_i, we have $\tilde{R}_1 = \tilde{R}_1(\bar{\alpha})$. However the influence of the α_i's on $\tilde{R}_1$ is very small indeed – in practical situations as will be seen below it is nonexistent – we therefore ignore this slight complication.

2. We restrict the reinsurance agreements to genuine quota shares. The company is not allowed to take a short position in any insurance subportfolio –which would be unrealistic – or to increase its share of any insurance subportfolio beyond 100% – which would attract important acquisition costs –.

3. There are no restrictions on the amounts invested in any asset category. In particular the company is allowed to take short positions in certain asset categories. In order for any portfolio to be feasible the amount of liabilities must exceed the amount of assets

$$u + \sum_{i=1}^{m} \alpha_i L_i \geq \sum_{i=1}^{n} A_j$$

If this is a true inequality, the assets corresponding to the excess liabilities can be invested in the risk free asset. This amounts to a restriction in the choice of the amount of capital

$$u \geq \sum_{i=1}^{n} A_j - \sum_{i=1}^{m} \alpha_i L_i$$

We refer to the right hand side of the inequality as to the **amount of net invested assets.**

4. Within the framework of our model we can simultaneously optimize the reinsurance policy and the investment policy of the company. The model allows for a symmetrical treatment of the insurance risks and of the asset risks.

Theorem

We assume that Σ is a regular matrix

1. The unrestricted optimum, i.e. the vector $\underline{x}$ which maximizes

$$r = \frac{\underline{\mu}'\underline{x}}{(\underline{x}'\Sigma\underline{x})^{1/2}}$$

is given by

$$\underline{x} = c \cdot \Sigma^{-1}\underline{\mu}$$

(By definition $\underline{x}$ is only defined up to a constant factor c.)

The unrestricted optimal risk return ratio is equal to

$$r_{max} = (\underline{\mu}'\Sigma^{-1}\underline{\mu})^{1/2}$$

2. $\underline{x}$ is the unrestricted optimum if and only if all the actual loadings are equal to the fair loadings

Proof

1. We have to maximize

$$r = \frac{\underline{\mu}'\underline{x}}{(\underline{x}'\Sigma\underline{x})^{1/2}}$$

equating to derivatives with respect to x_i to zero, we obtain

$$\frac{dr}{dx_i} = \frac{\mu_i(\underline{x}'\Sigma\underline{x})^{1/2} - \underline{\mu}'\underline{x}\cdot\frac{1}{2}(\underline{x}'\Sigma\underline{x})^{-1/2}\cdot 2(\sum_j \sigma_{ij}x_j)}{\underline{x}'\Sigma\underline{x}} = 0 \quad i=1,\dots,m+n$$

where $(\sigma_{ij}) = \Sigma$.

After rearranging terms

$$\mu_i(\underline{x}'\Sigma\underline{x}) = (\underline{\mu}'\underline{x}) \sum_j \sigma_{ij}x_j, \quad \text{all } i$$

$$\underline{\mu} = k\cdot\Sigma\cdot\underline{x}$$

and since Σ is regular

$$\underline{x} = c\cdot\Sigma^{-1}\underline{\mu}$$

Plugging in the above definition of $\underline{x}$ we obtain

$$r_{max} = \frac{c\cdot\underline{\mu}'\Sigma^{-1}\underline{\mu}}{(c\underline{\mu}'\Sigma^{-1}\cdot\Sigma\cdot c\Sigma^{-1}\underline{\mu})^{1/2}} = (\underline{\mu}'\Sigma^{-1}\underline{\mu})^{1/2}$$

2. All the actual loadings are equal to the fair loadings if and only if the following equations are satisfied

$$\alpha_i(\ell_i+\ell_i') = k\cdot \mathrm{Cov}(-\alpha_i(\tilde{X}_i+\tilde{X}_i'),\tilde{\Delta}u) \qquad i=1,\dots,m$$

$$(A_1-L)(R_1-\rho_0) = k\cdot \mathrm{Cov}(\,(A_1-L)\tilde{R}_1,\tilde{\Delta}u\,)$$

$$A_j(R_j-\rho_0) = k\cdot \mathrm{Cov}(\,A_j\tilde{R}_j,\tilde{\Delta}u\,) \qquad j=2,\dots,n$$

Using the above notation, this is equivalent to

$$x_i\mu_i = k\cdot \mathrm{Cov}(x_i\tilde{Z}_i,\tilde{\Delta}u) \quad i=1,\dots,m+n$$

for an appropriate choice of $\tilde{Z}_i$.

Hence

$$\mu_i = k \cdot \sum_j \Sigma_{ij} x_j \quad i=1,\dots,m+n$$

$$\underline{\mu} = k \cdot \Sigma \cdot \underline{x}$$

which proves the 2nd statement of the theorem

q.e.d

Remarks

1. The general optimization problem with restrictions

$$r = \frac{\underline{\mu}' \cdot \underline{x}}{(\underline{x}' \Sigma \underline{x})^{1/2}} = \max_{\underline{x} \epsilon \beta}!$$

is equivalent to

$$\underline{\mu}' \underline{x} - \frac{\gamma}{2} \cdot \underline{x}' \Sigma \underline{x} = \max_{x \epsilon \beta}! \quad \text{for any } \gamma$$

which is a quadratic optimization problem with restrictions. It is a standard problem in finance theory, see for instance W.F. Sharpe (1970). The equivalence of the two problems is seen from the fact that in both cases the following conditions must be satisfied

$$\mu_i - \gamma \sum_j \sigma_{ij} x_j = 0 \quad i=1,\dots,m+n$$

subject to the restrictions $\underline{x} \epsilon \beta$.

2. From remark 1 it follows that our optimization criterion is equivalent with $E(\tilde{\Delta} u) - \frac{\gamma}{2} \operatorname{Var}(\tilde{\Delta} u) = \max_{x \epsilon \beta}!$ for all γ

i.e. it is equivalent to a maximization of the quadratic utility

3. The 3rd statement of our theorem is a justification for our capital allocation formula.

4. The theorem is a generalisation of the theorem of section 2.4.

4.3 Asset Risk

We now turn to a simple numerical example. The company has two insurance risks (underwriting and loss reserve risks) which correspond to the different customer segment of the company. The risks and returns are as follows

Insurance Subportfolio	Risk	ℓ	σ	$\frac{\ell}{\sigma}$
Private customers	$\tilde{X}_1$	5	20	25%
Industrial customers	$\tilde{X}_2$	16	40	40%

It is assumed that the two insurance subportfolios are uncorrelated. The discounted loss reserves are as follows

$$L_1 = L_2 = 500$$

Note that we do not give the premium income since it is irrelevant. There are four different asset categories with risks and returns as defined below

Asset Category	Risk	$R_i - \rho_0$	σ	$\frac{R_i - \rho_0}{\sigma}$
Bond portfolio with medium term duration ($\tilde{R}_1 = \tilde{R}_L$)	$\tilde{R}_1$	1%	4%	25%
Bond portfolio with long term duration	$\tilde{R}_2$	2%	6%	33%
Equity Portfolio	$\tilde{R}_3$	10%	20%	50%
Real Estate Portfolio	$\tilde{R}_4$	8%	20%	40%

The correlation matrix of the different asset categories is as follow

$$\mathrm{Corr}(\tilde{R}_i,\tilde{R}_j) = \begin{bmatrix} 1 & 0.9 & 0.4 & 0.4 \\ & 1 & 0.4 & 0.4 \\ & & 1 & 0.4 \\ & & & 1 \end{bmatrix}$$

It is assumed that insurance risks and asset risks are uncorrelated

$$\mathrm{Corr}(\tilde{X}_i,\tilde{R}_j) = 0 \quad \text{for all i,j.}$$

We have

$$\underline{\mu}' = (\ 5,\ 16,\ 0.01,\ 0.02,\ 0.1,\ 0.08\ )$$

$$\Sigma = \begin{bmatrix} 400 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1600 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.0016 & 0.00216 & 0.0032 & 0.0032 \\ 0 & 0 & 0.00216 & 0.0036 & 0.0048 & 0.0048 \\ 0 & 0 & 0.0032 & 0.0048 & 0.04 & 0.016 \\ 0 & 0 & 0.0032 & 0.0048 & 0.016 & 0.04 \end{bmatrix}$$

And it is easily seen that the unrestricted solution

$$\underline{x} = c \cdot \Sigma^{-1} \underline{\mu}$$

is a solution which satisfies the condition $\alpha_i \ \epsilon \ [0,1]$ for i=1,2. Thereby c is chosen in such a way that $\alpha_i \epsilon [0,1]$ for i=1,2 and that the amount of business retained by the company is maximized.

The optimal portfolio of the company is thus

Insurance Subportfolio	Retention α_i	Expected Profit $\alpha_i \ell_i$	Contribution to overall variance $\mathrm{Cov}(-\alpha_i \tilde{X}_i, \tilde{\Delta} u)$
Private customers	1	5	400
Industrial customers	0.8	12.8	1024

Asset category	Amount invested A_j	Expected Profit $A_j(R_j-\rho_0)$	Contribution to overall variance $\mathrm{Cov}(A_j\tilde{R}_j, \tilde{\Delta} u)$
medium bond (A_1-L)	–785	–7.85	–628.4
long bond	587	11.75	939.9
equities	156	15.64	1251.5
real estate	90	7.18	574.5
Grand Total		44.52	3561.6

The salient feature of the optimal portfolio are as follows

- the company cedes a 20% quota share of its industrial business.
- the total discounted net loss reserves amounts to 900. The company invests A_1=115 in the bond portfolio with the same return $(\tilde{R}_1=\tilde{R}_L)$ as the discounted loss reserves. All in all it keeps a very substantial short position in this asset $(A_1-L = -785)$. Asset liability matching is non optimal.
- The company takes a substantial position $(A_2=587)$ in the bond portfolio with the longer duration and the higher expected return. (In practice most insurance companies have a duration of assets which exceeds the duration of liabilities.)
- The company invests a significant amount of its assets in equities and

real estate (27% of its liabilities).

- The amount of net invested assets $\sum_j A_j - \sum_i \alpha_i L_i$ is 48. In order for the optimal portfolio to be feasible the amount of capital must exceed this amount of excess assets. This is hardly a restriction since $\sigma(\tilde{\Delta}u) = 59.68$.
- Statement 3 of our theorem

  $$x_i\mu_i = k \cdot \mathrm{Cov}(\tilde{Z}_i, \tilde{\Delta}u), \quad \text{for all } i$$

  is verified for $k = 0.0125$.
- The contribution to overall variance from asset risks (2137.5) is considerably higher than the corresponding quantity from insurance risks (1424). This is in line with practical experience.

The risk return ratio of the optimal portfolio is

$$r = \frac{44.25}{59.68} = 0.75.$$

It is interesting to compare this quantity with the maximum risk return ratio obtained from the insurance portfolio alone. To be more precise, by asset liability matching (A_1=L) and by investing the assets corresponding to the capital of the company into the risk free asset category, we can fully eliminate the financial risk. We are then left with a portfolio consisting of the two insurance risks described above. It is easily seen that the optimum consists then in a 20% quota share cession of the industrial business and that the maximum risk return ratio is r_1=0.47. Thus assuming the asset risk leads to a substantial increase of the risk return ratio of the company. We now show that this statement is true in general.

We rewrite the expression for the excess profit of the company derived in section 4.

$$\tilde{\Delta}u - \rho_0 u = E(\tilde{S}) + \ell + \ell_1 - (\tilde{S} + \tilde{\Delta}L_1) - (\tilde{R}_L - \rho_0)L + \sum_{j=1}^{n_2} (\tilde{R}_j - \rho_0)A_j$$
$$= \tilde{Z} + (\tilde{R} - \rho_0)A$$

where

$$\tilde{Z} = E(\tilde{S}) + \ell + \ell_1 - (\tilde{S} + \tilde{\Delta}L_1)$$

$$\tilde{R} = R_1 \cdot \frac{A_1 - L}{A} + \sum_{j=2}^{n} \tilde{R}_j \frac{A_j}{A} \qquad \text{with } A = \sum_{j=1}^{n} A_j - L$$

$\tilde{Z}$ is the insurance risk, i.e. the sum of the underwriting risk and of the loss reserve development risk. $\tilde{R}$ is the rate of return of the financial risk and A is the amount of net invested assets. We introduce the following notation

$$\ell_z = E(\tilde{Z}) = \ell + \ell_1, \qquad \sigma_z^2 = \mathrm{Var}(\tilde{Z})$$

$$\delta_R = E(\tilde{R}) - \rho_0, \qquad \sigma_R^2 = \mathrm{Var}(\tilde{R}), \qquad \kappa = \mathrm{Corr}(\tilde{Z}, \tilde{R})$$

where $\mathrm{Corr}(\tilde{X}, \tilde{Y})$ denotes the correlation between the random variables $\tilde{X}$ and $\tilde{Y}$.

The following theorem expresses the overall risk return ratio as a function of the insurance risk return ratio and of the financial risk return ratio.

Theorem

Let $\kappa \neq \pm 1$. The overall risk return ratio

$$r(A) = \frac{\rho - \rho_0}{\sigma} = \frac{E(\tilde{\Delta}u) - \rho_0 u}{\sigma(\tilde{\Delta}u)} = \frac{\ell_z + \delta_R A}{\sqrt{\sigma_z^2 + (\sigma_R A)^2 + 2\kappa\sigma_z(\sigma_R A)}}$$

is maximized for the following amount of net invested assets

$$A = \frac{\ell_z}{\delta_R} \frac{\left(\frac{\delta_R}{\sigma_R}\right)^2 - \kappa \frac{\ell_z}{\sigma_z} \frac{\delta_R}{\sigma_R}}{\left(\frac{\ell_z}{\sigma_z}\right)^2 - \kappa \frac{\ell_z}{\sigma_z} \frac{\delta_R}{\sigma_R}}$$

and the corresponding risk return ratio is

$$r = r(A) = \left[\frac{\left(\frac{\ell_z}{\sigma_z}\right)^2 + \left(\frac{\delta_R}{\sigma_R}\right)^2 - 2K\frac{\ell_z}{\sigma_z}\frac{\delta_R}{\sigma_R}}{1-K^2} \right]^{1/2}$$

Proof

We have

$$E(\tilde{\Delta}u) - \rho_0 u = \ell_z + \delta_R \cdot A, \qquad \sigma^2(\tilde{\Delta}u) = \sigma_z^2 + \sigma_R^2 A^2 + 2K\sigma_z\sigma_R \cdot A$$

it follows that

$$r(A) = \frac{\ell_z + \delta_R A}{\sqrt{\sigma_z^2 + (\sigma_R A)^2 + 2K\sigma_z\sigma_R A}} =: \frac{\delta(A)}{\sqrt{V(A)}}$$

where $\delta(A)$ is $E(\tilde{\Delta}u) - \rho_0 \cdot u$ and $V(A)$ is $\sigma^2(\tilde{\Delta}u)$ considered as function of A.

Putting the derivative of r with respect to A equal to zero, we obtain

$$r'(A) = \frac{\delta'(A) \cdot V(A)^{1/2} - \delta(A)\, \frac{1}{2}\, V(A)^{-1/2}\, V'(A)}{V(A)} = 0$$

$$\delta'(A)V(A) - \frac{1}{2}\,\delta(A)V'(A) = 0$$

$$\delta_R(\sigma_z^2 + \sigma_R^2 A^2 + 2K\sigma_z\sigma_R A) = (\ell_z + \delta_R A)(\sigma_R^2 A + K\sigma_z\sigma_R)$$

$$A = \frac{\delta_R\sigma_z^2 - K\ell_z\sigma_z\sigma_R}{\ell_z\sigma_R^2 - K\delta_R\sigma_z\sigma_R} = \frac{\ell_z}{\delta_R}\, \frac{\left(\frac{\delta_R}{\sigma_R}\right)^2 - K\frac{\ell_z}{\sigma_z}\frac{\delta_R}{\sigma_R}}{\left(\frac{\ell_z}{\sigma_z}\right)^2 - K\frac{\ell_z}{\sigma_z}\frac{\delta_R}{\sigma_R}}$$

which proves the first statement of the theorem. In order to evaluate $r(A)$, we introduce the following notation

$$r_1 = \frac{\ell_z}{\sigma_z} \qquad r_2 = \frac{\delta_R}{\sigma_R}$$

and we restate the expression for A

$$A = \frac{\sigma_z}{\sigma_R}\,\frac{r_2 - Kr_1}{r_1 - Kr_2}$$

Thus obtaining

$$V(A) = \sigma_z^2 + \sigma_z^2\left(\frac{r_2-\mathcal{K}r_1}{r_1-\mathcal{K}r_2}\right)^2 + 2\kappa\sigma_z^2\frac{r_2-\mathcal{K}r_1}{r_1-\mathcal{K}r_2}$$

$$V(A) = \frac{\sigma_z^2}{(r_1-\mathcal{K}r_2)^2}\left((r_1-\mathcal{K}r_2)^2 + (r_2-\mathcal{K}r_1)^2 + 2\mathcal{K}\,(r_2-\mathcal{K}r_1)(r_1-\mathcal{K}r_2) \right)$$

$$V(A) = \frac{\sigma_z^2}{(r_1-\mathcal{K}r_2)^2}\left(1-\mathcal{K}^2\right)\cdot\left(r_1^2 + r_2^2 - 2\mathcal{K}r_1r_2\right)$$

$$r(A) = \frac{r_1-\mathcal{K}r_2}{\sigma_z}\,\frac{\ell_z + \delta_R\,\frac{\sigma_z}{\sigma_R}\,\frac{r_2-\mathcal{K}r_1}{r_1-\mathcal{K}r_2}}{\sqrt{(1-\mathcal{K}^2)\,(r_1^2 + r_2^2 - 2\mathcal{K}r_1r_2)}} = \sqrt{\frac{r_1^2 + r_2^2 - 2\mathcal{K}r_1r_2}{1-\mathcal{K}^2}}$$

which proves the theorem.

q.e.d.

Remarks

1. From the theorem it is seen that the maximum risk return ratio r is independent of u. As a consequence any risk return pair (σ^*,ρ^*) on the straight line $\rho - \rho_0 = r\cdot\sigma$ can be achieved by the company through an appropriate choice of the amount of capital $u = \frac{\sigma(\tilde{\Delta}u)}{\sigma^*}$, subject only to $u \geq \sum_{i=1}^{n_2} A_i - L$.
2. From the proof of the theorem it is easily seen that for $\mathcal{K} = \pm 1$ we have $A = \mp\frac{\sigma_z}{\sigma_R}$ and $V(A) = 0$ i.e. the risk if fully eliminated.
3. For $\mathcal{K} = 0$ we have

$$A = \frac{\sigma_z}{\sigma_R}\,\frac{\left(\frac{\delta_R}{\sigma_R}\right)}{\left(\frac{\ell_z}{\sigma_z}\right)} \quad\text{and}\quad r(A) = \sqrt{\left(\frac{\ell_z}{\sigma_z}\right)^2 + \left(\frac{\delta_R}{\sigma_R}\right)^2}$$

and it is seen that the assumption of asset risk leads to a considerably higher risk return ratio. In practice we have $\kappa \simeq 0$ and the statement is thus true for all practical situations.

4.5 Example

We now turn to a more realistic example. The insurance portfolio of the company is broken down into four subportfolios corresponding to different lines of business and to different customer segments. The risks and returns of the combined underwriting and loss development risks are as follows

Insurance Subportfolio	Risk	P	L	σ	ℓ	$\frac{\ell}{\sigma}$
Motor	$\tilde{X}_1$	50	75	2.5	0.5	20%
Homeowners	$\tilde{X}_2$	20	10	3.2	0.8	25%
Industrial Fire	$\tilde{X}_3$	10	5	4	1	25%
General Third Party Liability	$\tilde{X}_4$	10	20	4	1.5	37.5%
		90	110		3.8	

L denotes the amount of loss reserves.

The premium volume is given for purely illustrative purposes. It is not used below. The ratio between standard deviation and premium volume as well as the ratio between loss reserves and premium are chosen in a realistic way. It is assumed that the motor and the homeowners portfolio are both exposed to storm and are therefore positively correlated.

$$\mathrm{Corr}(\tilde{X}_1,\tilde{X}_2) = 0.20$$

The other correlations between insurance risks stem from the influence of the economic cycle and are treated below.

The different asset categories are as in the example of section 4.3.

Asset Category	Risk	$R_i - \rho_0$	σ	$\frac{R_i - \rho_0}{\sigma}$
Bond portfolio with medium term duration ($\tilde{R}_1 = \tilde{R}_L$)	$\tilde{R}_1$	1%	4%	25%
Bond portfolio with long term duration	$\tilde{R}_2$	2%	6%	33%
Equity Portfolio	$\tilde{R}_3$	10%	20%	50%
Real Estate Portfolio	$\tilde{R}_4$	8%	20%	40%

The correlation matrix of the different asset categories is as follow

$$\mathrm{Corr}(\tilde{R}_i, \tilde{R}_j) = \begin{bmatrix} 1 & 0.9 & 0.4 & 0.4 \\ & 1 & 0.4 & 0.4 \\ & & 1 & 0.4 \\ & & & 1 \end{bmatrix}$$

During a boom phase of the economic cycle interest rates and therefore investment income from bonds are high, but so is the inflation rate which leads to an increased loss amount of the motor and of the general third party liability portfolio. Therefore we assume

$$\mathrm{Corr}(-\tilde{X}_1, \tilde{R}_1) = \mathrm{Corr}(-\tilde{X}_1, \tilde{R}_2) = -0.2$$

$$\mathrm{Corr}(-\tilde{X}_4, \tilde{R}_1) = \mathrm{Corr}(-\tilde{X}_4, \tilde{R}_2) = -0.2$$

and

$$\mathrm{Corr}(\tilde{X}_1,\tilde{X}_4) = 0.2$$

When the economy goes into recession, equities and real estate depreciate, industrial fire results worsen – due to arson – and motor results improve – because people drive less –.Thus

$$\mathrm{Corr}(-\tilde{X}_1,\tilde{R}_3) = \mathrm{Corr}(-\tilde{X}_1,\tilde{R}_4) = -0.2$$

$$\mathrm{Corr}(-\tilde{X}_3,\tilde{R}_3) = \mathrm{Corr}(-\tilde{X}_3,\tilde{R}_4) = 0.2$$

and

$$\mathrm{Corr}(\tilde{X}_1,\tilde{X}_3) = -0.2$$

In summary we have the following correlations

	$-\tilde{X}_1$	$-\tilde{X}_2$	$-\tilde{X}_3$	$-\tilde{X}_4$	$\tilde{R}_1$	$\tilde{R}_2$	$\tilde{R}_3$	$\tilde{R}_4$
$\tilde{X}_1$	1	0.2	−0.2	0.2	−0.2	−0.2	−0.2	−0.2
$-\tilde{X}_2$	0.2	1	0	0	0	0	0	0
$-\tilde{X}_3$	−0.2	0	1	0	0	0	0.2	0.2
$-\tilde{X}_4$	0.2	0	0	1	−0.2	−0.2	0	0

Thus

$$\underline{\mu}' = (0.5\ ,\ 0.8\ ,\ 1\ ,\ 1.5\ ,\ 0.01\ ,\ 0.02\ ,\ 0.10\ ,\ 0.08)$$

$$\Sigma = \begin{bmatrix} 6.25 & 1.6 & -2 & 2 & -0.02 & -0.03 & -0.1 & -0.1 \\ 1.6 & 10.24 & 0 & 0 & 0 & 0 & 0 & 0 \\ -2 & 0 & 16 & 0 & 0 & 0 & 0.16 & 0.16 \\ 2 & 0 & 0 & 16 & -0.032 & -0.048 & 0 & 0 \\ -0.02 & 0 & 0 & -0.032 & 0.0016 & 0.00216 & 0.0032 & 0.0032 \\ -0.03 & 0 & 0 & -0.048 & 0.00216 & 0.0036 & 0.0048 & 0.0048 \\ -0.1 & 0 & 0.16 & 0 & 0.0032 & 0.0048 & 0.04 & 0.016 \\ -0.1 & 0 & 0.16 & 0 & 0.0032 & 0.0048 & 0.016 & 0.04 \end{bmatrix}$$

and it is easily seen that the unrestricted solution

$$X = c\, \Sigma^{-1} \underline{\mu}$$

is a solution which satisfies the conditions $\alpha_i \varepsilon [0,1]$ *for* $i = 1,2,..,4$. Choosing c in such a way as to maximize the amount of business retained by the company we obtain the following optimal solution

Insurance Subportfolio	Retention α_i	Expected Profit $\alpha_i \ell_i$	Contribution to overall Variance $Cov(-\alpha_i \tilde{X}_i, \tilde{\Delta}u)$
Motor	1	0.5	4.47
Homeowners	0.54	0.43	3.87
Industrial Fire	0.44	0.44	3.93
GTPL	0.81	1.21	10.82

Asset Category	Amount invested A_j	Expected Profit $A_j R_j$	Contribution to overall Variance $Cov(A_j \tilde{R}, \tilde{\Delta}u)$
medium bond (A_1-L)	−69.3	−0.69	−6.20
long bond	77.9	1.56	13.92
equities	15.9	1.59	14.21
real estate	8.5	0.68	6.04
		5.72	51.07

The risk return ratio is 0.80, the amount of net invested assets is 33.0 and the amount of net loss reserves is 98.3.

By perfect asset liability matching and by investing the equity into the risk free asset one can fully eliminate the asset risk. The vector of expected returns and the covariance matrix of the pure insurance risk are respectively

$$\mu_0' = (0.5 , 0.8 , 1 , 1.5)$$

and

$$\Sigma_0 = \begin{bmatrix} 6.25 & 1.6 & -2 & 2 \\ 1.6 & 10.24 & 0 & 0 \\ -2 & 0 & 16 & 0 \\ 2 & 0 & 0 & 16 \end{bmatrix}$$

and from the theorem of section 4.3 we know that the maximum risk return ratio which can be achieved in such a situation is

$$r = (\mu_0' \Sigma^{-1} \mu_0)^{1/2} = 0.53$$

which is considerably lower than risk return ratio obtained above. Thus, in this example too, it is seen that the assumption of asset risk leads to a considerable improvement of the risk return ratio of the portfolio.

Through quota share cessions the company has reduced the expected profit of its insurance portfolio from 3.8 to 2.58, i.e. it forgoes a substantial amount of profit in order to maximize its risk return ratio. As a comparison, we now look at the optimal portfolio assuming that the company cedes no quota share. In that case, we have the following vector of expected returns

$$\mu_1' = (3.8 , 0.01 , ,0.02 , 0.1 , 0.08)$$

and covariance matrix

$$\Sigma_1 = \begin{bmatrix} 51.69 & -0.052 & -0.078 & 0.06 & 0.06 \\ -0.052 & 0.0016 & 0.00216 & 0.0032 & 0.0032 \\ -0.078 & 0.00216 & 0.0036 & 0.0048 & 0.0048 \\ 0.06 & 0.0032 & 0.0048 & 0.04 & 0.016 \\ 0.06 & 0.0032 & 0.0048 & 0.016 & 0.04 \end{bmatrix}$$

And the optimal solution excluding quota share cessions is

	$\alpha \mid A$	Expected Profit	Contribution to overall Variance
Insurance Portfolio	1	3.8	50.18
medium bond (A_I–L)	–104.2	–1.04	–13.75
long bond	114.0	2.28	30.10
equities	21.8	2.18	28.83
real estate	10.8	0.87	11.44
		8.09	106.79

The risk return ratio is now r = 0.78 which is only slightly lower than the optimal risk return ratio of 0.80. In practical circumstances an insurance company may prefer the above solution with the much higher expected profit of 8.09 (vs 5.72) to the optimal solution even if this entails a slight decrease of the risk return ratio. The optimization method we have derived is nevertheless valuable since it provides us with a benchmark, the optimal portfolio, against which to measure any given portfolio.

Concluding Remarks

The optimization method we have derived is a generalisation of Markowitz's portfolio optimisation method to finance and insurance risks. It has the advantage to allow a symmetrical treatment of insurance and finance risks and to allow a simultaneous optimisation of the whole portfolio. However when it comes to practical applications of the method the following must be taken into account:

1. Whilst one could in principle look individually at each policy in the company's portfolio and at each asset in the financial markets, this

would hardly be a tractable method. One has therefore to build insurance subportfolios (e.g. along lines of business and customer segments), to optimize those subportfolios individually (e.g. via surplus and excess of loss reinsurance as illustrated in section 2 and to build an optimal global portfolio via appropriate quota share cessions. Similarly one can view the portfolio of financial risks as positions in different funds – bonds, equities, real estate, ... – including a short position in the bond portfolio which replicates the liabilities of the company. The process is therefore a two steps optimization process and the result will depend in particular on the sub portfolio structure which has been chosen.

2. There are fundamental differences between financial risks and insurance risks: the transaction costs related to financial assets are negligible whereas acquisition costs for insurance portfolios are very high. It is possible to take short positions in financial assets but not in insurance portfolios. This is reflected in the conditions that $\alpha_i \epsilon [0,1]$ for all i. As consequence the optimal global portfolio of a given insurance company will heavily depend on its existing gross insurance portfolio. Therefore the optimal portfolios of different companies are in general <u>not</u> collinear (as opposed to optimal asset portfolios within the frame of CAPM).

Literature

– R. Schnieper, (1997), SCOR Notes,Capital Allocation and Solvency Testing

– W.F. Sharpe, (1970), Portfolio Theory and Capital Markets, McGraw–Hill