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PORTFOLIO SELECTION AND MATCHING: A SYNTHESIS

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ABSTRACT

This paper considers a general framework for the selection of assets to meet the liabilities of a life insurance or pension fund. This general framework contains the mean-variance efficient portfolios of modern portfolio theory as a special case. The paper also demonstrates how the portfolio selection and matching approach of Wise (1984a, 1984b, 1987a, 1987b) and Wilkie (1985) fits into this general framework. The matching portfolio is derived as a special case, and is also shown to have implications for determining the central value of the liabilities.

KEYWORDS

Investment; Matching; Modern Portfolio Theory; Risk

1. INTRODUCTION

1.1 The portfolio selection problem for life insurance and pension funds has received attention in the actuarial literature over recent years. Wilkie (1985) and Wise (1984a, 1984b, 1987a, 1987b) have examined the portfolio selection problem with a view to establishing an allocation of assets which best meets specified liabilities. Wise (1984a, 1984b) was concerned mainly with the matching of fixed liabilities and considered an asset allocation, referred to as the unbiased match, which minimised the variance of ultimate surplus for a mean ultimate surplus of zero. Wilkie (1985) demonstrated how the Wise approach could be incorporated into a mean-variance efficient portfolio framework, and introduced the price of the asset portfolio as an additional factor in the portfolio selection decision. Wise (1987a) showed how the more general framework developed by Wilkie related to his matching portfolio. This paper aims to show how the Wise approach (1984a, 1984b), as extended by Wilkie (1985) and further examined by Wise (1987a, 1987b), fits into a more general portfolio selection framework.

1.2 In the Wise–Wilkie model the asset holdings are chosen based on the mean and variance of ultimate surplus. Ultimate surplus is the difference between the accumulated asset cash flows and the accumulated liability cash flows as at a fixed time horizon, which is taken as the date of the final liability cash flow. In their model, interest rates for investing asset and liability cash flows are stochastic. All cash flows of assets and liabilities are accumulated to a horizon date at an interest rate which varies in each period between cash flows, and the mean, variance and covariance of these accumulated asset and liability cash flows are determined. Asset holdings which are to be held by the fund at the start of the period, and kept constant throughout the time horizon, are chosen so as to minimise the variance of ultimate surplus for a fixed expected value of ultimate surplus (or maximise the expected value for any fixed variance) for any given price of the asset portfolio. When the price of the portfolio is incorporated, then the aim is to minimise the price for any given mean and variance of ultimate surplus. Wise (1984a) demonstrates how the effect of inflation on assets and liabilities can be incorporated into the analysis.

1.3 Wise (1984a, 1984b) is primarily concerned with the issue of the matching of assets to fixed liabilities. As noted by Wise, matching of liability cash flows is usually associated with the management of interest rate risk. Milgrom (1986) also discusses the asset allocation required for the immunisation and matching of liabilities. Matching can be considered as a special case of the more general portfolio selection problem, with a special condition imposed on the assets held in the portfolio. In Wise's case the condition is that the variance of ultimate surplus is to be minimised and the expected value of ultimate surplus is to be zero. In contrast, portfolio selection models use a more general condition as a basis for selecting an optimal asset portfolio. Matching is concerned with the selection of assets which most closely resemble the liability cash flows, whereas portfolio selection is concerned with selection of assets which attain an optimal level of risk. Determining a matching asset portfolio has important implications for valuing non-traded liabilities. If the cash flows from a non-traded liability can be replicated exactly with traded assets, then the value of the liability should equal the market value of the replicating asset portfolio. Taylor (1990) shows how matching asset portfolios can be used to determine expected or central values of liabilities. These ideas are demonstrated further in this paper. If an exact matching portfolio cannot be constructed, then the liabilities cannot be valued only by reference to the asset values using a replicating portfolio. Investors do not generally adopt matching strategies unless they prefer not to bear any investment risk, regardless of the additional return from bearing this risk.

1.4 This paper shows how the Wise–Wilkie model is a special case of a more general formulation of the portfolio selection problem. This more general model can incorporate risk measures such as those proposed by Clarkson (1989, 1990) and can readily incorporate constraints such as positive holdings of assets and solvency constraints, which are covered in some detail in Wise (1984a) and Wilkie (1985). McCabe & Witt (1980) also incorporate a solvency constraint to consider pricing and regulatory effects for non-life insurers. The general formulation of the portfolio selection problem is also covered in Sherris (1990). The general optimisation approach suggested and used in this paper is not original, but appears to be new to the actuarial literature. Just as Wilkie (1985) based his analysis on the mean-variance approach so common to the finance literature in its early days, this paper also bases its analysis on a more general approach that has been explored in the finance literature in, amongst others, Mossin (1968), Samuelson (1969) and Breeden (1987).

1.5 The general portfolio selection problem is first formulated as an optimisation problem for a single period. A particular objective function in this optimisation problem is then shown to result in the Wise–Wilkie case. The equivalence between the Wise–Wilkie case and the more general portfolio selection problem is discussed. The Wilkie (1985) extension of the Wise model introduces the price, or market value, of the asset portfolio into the portfolio selection problem. The role of the value of the asset portfolio and its effect on the portfolio selection problem is discussed further in this paper.

1.6 The paper concludes by discussing the extension of the single-period model to the multiperiod case. The Wise–Wilkie model is, despite its apparent multiperiod formulation, a single period model, since it does not allow for revision of the asset allocation for the fund through time. The practical implementation of an immunisation strategy, as first discussed by Redington (1952), will require a rebalancing of the asset allocation through time, as would any option-based matching strategy for liabilities with maturity guarantees. This is a practical shortcoming of the Wise–Wilkie model. A brief discussion of the extension of the Wise–Wilkie model to a multiperiod setting was outlined in Yang (1988). The multiperiod extension briefly discussed in this paper also mentions how a more realistic model of interest rates than that used in the Wise–Wilkie model can be incorporated into the matching and portfolio selection model.

1.7 Computer techniques will generally be the most efficient method of determining the asset allocations derived from the general approach of this paper. It is only in special cases, such as the Wise–Wilkie model, that analytical solutions can be determined. The numerical analysis of the Wise–Wilkie model is extensively covered in Wilkie (1985) and the details will not be repeated in this paper. The multiperiod approach can readily be incorporated into computer based asset/liability models, which are becoming such an important actuarial management tool for life insurance and pension funds.

2. THE SINGLE PERIOD PORTFOLIO SELECTION PROBLEM

The Initial Fund

2.1 The portfolio selection problem will first be considered for the following one-period situation. The initial or start of period value of assets available to meet the liability will be denoted by A and will be assumed to consist of a fixed component of C and a variable component of K. The fixed component of C belongs to the fund members or claimants and the variable component belongs to the fund sponsors. In a life insurance fund, C would be determined as the expected or market value of the liability. In such a fund, K would be the provision for adverse deviations which would usually be provided as risk capital or equity. For a pension fund, C would be regarded as the expected value of claim payments under the rules of the fund and K the margin added to this by the actuary in establishing the contribution to the fund either explicitly or, as is more often the case, implicitly by using conservative assumptions. The total of C and K will be taken as the total value of liabilities determined by the actuary (or perhaps a regulator) for which assets of at least this value must be held by the fund at the start of the period. The initial amount of assets can therefore be written as:

$$A=(1+p)\times C$$

where $p=K/C$ and can be interpreted as the solvency margin. The amount of assets A corresponds to the price of the portfolio P in the Wise–Wilkie model.

The Asset Model

2.2 For simplicity, it will be assumed that there are two assets available which, for a dollar (pound) invested at the beginning of the period, provide a cash flow at the end of the period of R_1 for asset 1 and R_2 for asset 2. R_i is therefore one plus the rate of return on asset i, which will be assumed to have a mean of E_i, a variance of V_i and a covariance with asset j of C_{ij}. In the Wise–Wilkie model it is only the mean, variance and covariance of assets that determine the asset allocation, so that it is not necessary to specify cash flows for these assets. Notice that the R_i used here are equal to the Wise–Wilkie accumulated cash flows divided by the price of asset i, so that we are using returns per unit of currency (pound) which is a standard approach in investment models. As a result, the values for E_i, V_i and C_{ij} above are determined by taking the corresponding Wise–Wilkie values and dividing by P_i, P_i^2 and P_iP_j, respectively.

2.3 Let w_i be the proportion of the assets which is invested in asset i. These w_i will be the main decision variables, and represent the asset allocation, or portfolio selection, for the fund. There is no loss of generality in considering only two assets, since the extension to multiple assets is relatively straightforward. The two-asset case is also algebraically and computationally easier for illustrative purposes. In practice we could assume that asset 1 provides a certain return and corresponds to a 'risk-free' asset, and asset 2 provides a stochastic return and corresponds to a 'risky' asset, but, at this stage, this adds nothing to the formulation of the problem. These w_i are proportions of the fund invested in asset i and are different to the x_i in the Wise–Wilkie papers, which are the units of asset i in the fund. To obtain the w_i values in this paper, it is necessary to multiply the Wise–Wilkie x_i values by the price of the asset P_i and divide by the price of the portfolio $P=P_1x_1+P_2x_2$.

Surplus

2.4 The surplus at the end of the period will be a random variable, which will depend on the random amount of the liability cash flow and the random rates of return on the risky assets, as well as the (endogenous) decision variables w_1 and w_2. The liability value will be assumed to be a random variable. This is consistent with the Wise–Wilkie formulation, since the accumulated liability cash flows in their model is a random variable. The source of randomness in the liability value in the formulation presented here is not confined to that arising from the random earnings rate on the fund, which is the case in most of the examples in the Wise–Wilkie papers. In this sense the Wise–Wilkie model is quite general in the randomness that can be allowed for in the liability value. The end of period (ultimate) surplus is given by:

$$S = A(w_1R_1 + w_2R_2) - L$$

where the mean of L will be denoted by E_L, its variance by V_L and its covariance with the rate of return on asset i by C_{Li}. This formulation is identical to that of the Wise-Wilkie model, but our adjusted notation makes explicit the role of the value of the initial assets, or price of the portfolio.

2.5 This expression for S can be rewritten as:

$$S = C(1+p)\left[w_1R_1 + w_2R_2 - \left(\frac{1}{1+p}\right)R_L\right]$$

where $R_L = L/C$ can be considered as the (random) growth rate of the liability from its start of period expected value of C or the equivalent of the rate of return on the liabilities. This alternate formulation will not be used in this paper. It is given to illustrate how a rate of return on liabilities can be readily incorporated into the model, and to demonstrate the effect of the solvency margin. If the price of the portfolio is allowed to vary as in the Wilkie extension of the original Wise model, then this implies that the solvency margin is being varied. The fund sponsors might wish to minimise the amount of K which they provide to support the liabilities, as suggested by the Wilkie preference function. In practice, such actions are controlled by regulators and actuaries, who place constraints on the amount required as solvency margin. It is also not apparent that fund sponsors would always wish to minimise the amount of capital that they contribute to the fund. If the rate of return on this capital, for the level of risk arising from both the asset and liability variability, is greater than that available on other investments, then fund sponsors might rationally invest more capital rather than less. The optimal size of the fund assets is a complex issue that is influenced by many factors, and which cannot be easily addressed within the simple framework of this paper.

Insolvency

2.6 The probability of insolvency can be incorporated into the model by considering the ultimate surplus value under two 'states' of the fund. If the fund is solvent then $S \geqslant 0$. In this event the providers of the risk capital K (the fund sponsors) will be entitled to the full surplus S and the claimants (fund members) will be paid the liability payment then due of amount L. In the case where the fund is insolvent then $S < 0$. In this event the providers of the risk capital will usually be protected by limited liability and will receive (and contribute) nothing at the end of the period, and the claimants will receive $L + S < L$. Note that the limited liability of the providers of the risk capital results in the claimants' payments at the end of the period having an option payoff structure in the form of minimum $(L, L+S)$. This is an important feature of a life insurance or pension fund model which is mentioned in Wilkie's extension of the original Wise approach. In practice, the distinction between the claims on the fund by fund sponsors and fund members is not as clear cut as assumed above. However, in a single period model this allocation of the ultimate fund is the most sensible.

2.7 The effect of this limited liability is that the value of C implicitly depends on the asset allocation strategy adopted, since the expected value of the claimants' payments will depend on whether or not there are sufficient funds to pay them in full. This means that C must be determined based on knowledge of the optimal asset allocation. It will also have to be assumed that the fund does not act against the interests of the claimants beyond that already allowed for in the value of C, which could occur if the fund adopted a more 'risky' asset allocation strategy than that implicit in C, in which event the claimants will suffer the downside risk of such a strategy, but not benefit from the additional surplus on the upside. Such actions can be limited by incorporating a constraint on the probability of insolvency on the feasible asset allocation strategies. In practice, this is done through regulatory constraint or through actuarial control. Although this factor is not discussed in detail in this paper, it is a very important factor that should be taken into account by actuaries in the valuation of liabilities in practice. The probability of insolvency is a major consideration of risk theory. Wilkie (1985) discusses the incorporation of a solvency constraint in the Wise–Wilkie model.

Optimal Asset Allocation

2.8 In the single-period case, the general optimal asset allocation problem can readily be formulated as a simple optimisation problem. Typically the constraints will involve inequalities, and Kuhn–Tucker conditions will be used to determine the solution. Lambert (1985) provides details on optimisation for the problems considered in this paper. Wise (1984a, Appendix A) uses the Kuhn–Tucker conditions to derive the general algebraic form of the optimal matching portfolios. In general the optimisation problem can be placed in the form:

$$\underset{\{x_i=1,\,n\}}{\text{maximise}}\, f(\boldsymbol{x})$$

where $\boldsymbol{x}$ is a vector of n decision variables subject to:

$$x_k \geqslant 0 \qquad \text{for certain } k$$

$$g_j(\boldsymbol{x}) \leqslant b_j \qquad \text{for } j = 1, 2, \ldots, m.$$

2.9 The solution to this problem is given by the Kuhn–Tucker conditions, which are determined by first forming the Lagrangean:

$$L = f(x) - l^T[g(x) - b]$$

where l is a vector of Lagrange multipliers for each of the m constraints. The unique maximum is then given by Lambert (1985, 128):

$$\frac{\partial L}{\partial x_i} = 0 \text{ for } i \text{ not equal to } k \text{ (i.e. no positive constraint on } x_i)$$

$$x_k \geqslant 0 \quad \frac{\partial L}{\partial x_k} \leqslant 0 \text{ and } x_k\left(\frac{\partial L}{\partial x_k}\right) = 0 \text{ for all } k$$

$$l_j \geqslant 0 \quad \frac{\partial L}{\partial l_j} \leqslant 0 \text{ and } l_j\left(\frac{\partial L}{\partial l_j}\right) = 0 \text{ for } j = 1, 2, \ldots, m.$$

2.10 The asset allocation is determined by selecting that asset allocation (and possibly the solvency margin at the start of the period) which maximises the expected value of an appropriate risk weighting function of the end of period surplus, subject to any constraints which may be imposed. Thse constraints would include any requirement for all asset holdings to be positive, so that short selling (and hence borrowing by disallowing the short selling of any risk free asset) would be excluded, and any requirement that the probability that the surplus be negative (the probability of insolvency) be less than or equal to a pre-specified figure.

2.11 The problem to be solved for the single period case can be written as:

$$\underset{\{w_1, w_2\}}{\text{maximise}} \; E[U(S)]$$

subject to:

$$w_1 + w_2 = 1 \quad \text{(budget constraint)}$$

$$F_s(0) \leqslant q \quad \text{(probability of insolvency constraint)}$$

where $F_s(\cdot)$ is the cumulative distribution function of ultimate surplus S. For a 'positive' asset allocation the additional constraints $w_i \geqslant 0$ would be required. The budget constraint in this problem corresponds to Wilkie's price equation. The probability of insolvency constraint could be replaced with a constraint that the value of the assets exceeds the expected value of the liabilities by a specified percentage, so that a minimum solvency margin would then be assumed to apply, as is the case in some countries. In this case the constraint would become $p \geqslant p^*$ where p^* is the minimum solvency margin.

2.12 In order to ensure a unique maximum, it is necessary to impose some conditions on the form of $E[U(\cdot)]$. In particular it is necessary to assume that it is at least monotonic, twice differentiable and quasi-concave. This latter condition requires that the matrix of second partial derivatives (the Hessian) of the objective function be negative semi-definite. In other words, the determinant of the Hessian has to be less than or equal to zero. These conditions can be interpreted as conditions applying to the utility or risk function, such that more surplus is preferred to less, but at a decreasing rate.

Utility or Risk Function

2.13 The most important part of the optimisation problem is the choice of the function $U(\cdot)$. This is a function which incorporates the risk preferences of the fund sponsors, who are assumed to set the asset allocation strategy for the fund. In general it can be a complex function, in which case a numerical technique will be required to solve for the optimum asset allocation. This function is usually referred to as a utility function in the economic, finance and decision theory literatures. For more detail on the assumptions underlying the use of expected utility, readers are referred to Chapter 7 of Jarrow (1988). The risk measure given in Clarkson (1989, 1990) is, for all practical purposes, identical to expected utility using a complicated form of utility function. For many utility functions it will be necessary to use numerical optimisation techniques to determine the optimal asset allocation.

2.14 In the formulation of the portfolio selection problem given above, the assumption is made that the fund sponsors give weight to negative surplus values in the choice of asset allocation. This should only be the case if the fund sponsors were financially affected by the negative surplus. The limited liability feature mentioned previously could mean that negative values will not be taken into account, since claimants bear the negative surplus values in reduced claims/benefits and not the fund sponsors. In practice, the asset allocation decision will need to take into account the claimants' welfare in some form or other. This issue is a complex one, since the interests of the fund sponsors and claimants are to some extent in conflict in setting the asset allocation. Fund sponsors, with limited liability, might prefer a more risky investment approach than claimants, who are more likely to prefer that their claims/benefits are as immune as possible from asset risk. These conflicts will also differ between life insurance and pension funds. This problem can be considered in the context of determining a pareto optimal asset allocation, which takes into account the welfare of both fund sponsors and claimants. This issue is not discussed further in this paper, but a pareto optimal approach, using concepts that are well developed in the agency theory literature, could well be the best framework to consider this issue further.

3. A SPECIAL CASE—THE WISE–WILKIE MODEL

3.1 In order to derive analytical solutions to the optimisation problem formulated in Section 2, it is necessary to impose some structure on $U(\cdot)$. The requirement for 'positive' portfolios and the solvency constraint will not be considered here, since Wilkie (1985) provides details on these constraints. In the Wise–Wilkie model, assuming that the price of the portfolio is fixed, it is only the mean and variance of ultimate surplus that affect the asset allocation. Such a result can be obtained by imposing the assumption that the end-of-period returns on assets and on the liability are jointly normally distributed. It would also be sufficient to assume that the surplus is normally distributed without imposing distributional assumptions on assets and liabilities. This would ensure that, for any utility function $U(\cdot)$, the objective function would depend only on the mean and variance of ultimate surplus.

3.2 Note that the end-of-period surplus is a linear function of the returns on the assets and on the liability so that, if these are assumed to be jointly normal, then the ultimate surplus will be normally distributed. The $U(S)$ function can be expanded in a Taylor series expansion around its mean, and expectations taken. In this expectation only mean and variance terms will appear.

Risk Tolerance

3.3 In order to show explicitly how risk attitudes are taken into account in the model, it will be assumed that the utility function is exponential and that the ultimate surplus is normally distributed. Hence utility of end of period surplus is given by $U(S) = -\exp\{-S/r\}$, where r is the risk tolerance of the fund. The value of r reflects the extent to which the fund avoids (or prefers if r is negative) higher risk investment allocations. The risk tolerance is the inverse of the risk aversion of the fund. (For more on risk tolerance and aversion measures for utility functions, see Pratt (1964).) A typical value of r for a fund manager might be around 25, as suggested in Sharpe & Tint (1990). Fund sponsors, regulators and actuaries might have their own views as to the risk tolerance which is appropriate for any particular fund.

3.4 In order to use a mean-variance criterion to determine optimal asset allocation strategies when surplus is not normally distributed, it is necessary to assume that investors have quadratic utility functions, since only in this case will expected utility be a function of only the mean and variance of the ultimate surplus for general distributions of ultimate surplus. Using quadratic utility to justify such a model is not desirable, because the requirements for a maximum are inconsistent for large enough values of S. Wise (1984a, Appendix C) discusses stochastic models for interest rates and inflation. In the Wise–Wilkie examples the interest rate is modelled as a binomial random variable in each period between cash flows. Whatever the distributional assumptions which are made for asset returns and liability values, it is important to recognise that the mean-variance criterion might implicitly be based on the assumption of a quadratic utility function. The alternative is to use a sensible utility function such as the exponential, and incorporate the normality assumption for the distribution of surplus at least in order to obtain analytical results. In practice, computer based asset/liability models will allow a more general specification of the distribution of returns and of the utility function in the optimisation problem.

Optimisation

3.5 The portfolio selection problem can now be written as:

$$\underset{\{w_1,\, w_2\}}{\text{maximise}}\; E[-\exp\{-S/r\}]$$

subject to $w_1+w_2=1$, where $S=A(w_1R_1+w_2R_2)-L$ is assumed to be normally distributed with mean E and variance V given by:

$$E = Aw_1E_1 + Aw_2E_2 - E_L$$

$$V = A^2w_1^2V_1 + A^2w_2^2V_2 + V_L + 2A^2w_1w_2C_{12} - 2Aw_1C_{1L} - 2Aw_2C_{2L}.$$

3.6 The function to be maximised can be recognised as equivalent to the negative of the moment generating function of the normal distribution given by:

$$E[\exp\{tS\}] = \exp\{Et + Vt^2/2\}$$

with $t=-(1/r)$. The problem can therefore be reformulated as minimising the logarithm of the original objective function or minimising $\{E(-1/r)+V/2r^2\}$. This is identical to maximising $\{E-V/2r\}$. The constraint that the total asset allocation proportions sum to one, is handled by substituting $w_2=1-w_1$ (or $w_1=1-w_2$) into the objective function.

Optimal Asset Allocation

3.7 Differentiating the objective function with respect to w_1 (or w_2) and setting the derivative to zero, gives the optimal asset proportions as a function of the initial assets A. Doing this gives:

$$w_1 = \frac{(C_{1L} - C_{2L}) + A(V_2 - C_{12}) + r(E_1 - E_2)}{A(V_1 - 2C_{12} + V_2)}$$

$$w_2 = \frac{(C_{2L} - C_{1L}) + A(V_1 - C_{12}) + r(E_2 - E_1)}{A(V_1 + V_2 - 2C_{12})} = 1 - w_1.$$

These are readily verified to be identical to the equations given in Wise (1987a, 117–118) for the portfolio with fixed price P. The preference for risk versus return in the Wise–Wilkie case is denoted by $1/\mu$ with our risk tolerance r equal to $1/2\mu$. This implies that for values of r of 25, which is supposed to be typical for a fund manager, the equivalent value for μ in the Wise–Wilkie model would be 0·02.

Sensitivity to A *and* r

3.8 It is of interest to note how the asset proportions vary as the size of the fund varies and as the risk tolerance parameter varies. The expression for w_1 can be differentiated with respect to A and r and the sign of these derivatives determined. Doing this gives:

$$\frac{\partial w_1}{\partial A} = \frac{-[(C_{1L} - C_{2L}) + r(E_1 - E_2)]}{A^2(V_1 - 2C_{12} + V_2)}$$

and

$$\frac{\partial w_1}{\partial r} = \frac{(E_1 - E_2)}{A(V_1 - 2C_{12} + V_2)}.$$

In general, it is not possible to say whether or not the proportion of the fund invested in asset 1 will increase or decrease as the fund size increases, since it depends on the risk tolerance of the fund, the difference between the expected returns on each asset and the difference between the covariances of each asset with the liability. In the case where the covariances of each asset with the liability are equal, the proportion invested in the higher returning asset will decrease as the fund size increases. The proportion of the fund invested in the higher returning asset always increases as the risk tolerance increases.

Price of the Portfolio

3.9 The Wise–Wilkie model incorporates the price into the objective function by assuming a simple trade-off between price and mean and variance of ultimate surplus. The trade-off would not appear to be as simple as Wilkie (1985) implies. As noted earlier, the value of initial assets will, in practice, be determined by solvency requirements and also by the relative attractiveness to the fund sponsors of investing funds into the insurance or pension fund. The lower the price of the portfolio the lower the expected value of assets available to meet the liability, and hence the higher the probability of insolvency. A solvency constraint will ensure that the price of the portfolio will be at a high enough level for the actuary or regulators to feel comfortable that the fund will meet its liabilities.

A Wise Example

3.10 As mentioned previously, the notation used in this paper is different to that used in the Wise–Wilkie papers. To illustrate the notational differences, and the numerical equivalence of the results obtained, Table 1 has been derived using the same parameters as in Table 3 of Wise (1987a, 118). The means, variances and covariances have been converted into the rate of return form used in this paper to get:

Means	$E_1 = 1{\cdot}29979$	$E_2 = 1{\cdot}29243$	$E_L = 327{\cdot}810$
Variances	$V_1 = 0{\cdot}00014357$	$V_2 = 0{\cdot}000006157$	$V_L = 5{\cdot}5563$
Standard Deviations	$\sigma_1 = 0{\cdot}011982$	$\sigma_2 = 0{\cdot}002481$	$\sigma_L = 2{\cdot}357181$
Covariances	$C_{12} = 0{\cdot}000027579$	$C_{1L} = 0{\cdot}0262003$	$C_{2L} = 0{\cdot}0058487$
Correlation Coefficients	$r_{12} = 0{\cdot}9276$	$r_{1L} = 0{\cdot}9276$	$r_{2L} = 1{\cdot}0$

Table 1. *Asset allocations for Wise's (1987a) Table 3 example*

A	r	w_1	w_2	E	V
200	0	0·84941	0·15059	−68·07	0·73
200	0·5	1·04408	−0·04408	−67·79	0·88
200	1·0	1·23874	−0·23874	−67·50	1·31
400	0	0·31145	0·68855	190·08	0·39
400	0·5	0·40878	0·59122	190·37	0·54
400	1·0	0·50612	0·49388	190·65	0·97
1000	0	−0·01133	1·01133	964·54	0·00
1000	0·5	0·02761	0·97239	964·82	0·15
1000	1·0	0·06654	0·93946	965·11	0·58

A Revised Example

3.11 The Wise–Wilkie example is based on the means, variances and covariances of the accumulated asset and liability cash flows. In the single-period framework it will be more meaningful to use revised values for the returns on the assets. In Table 2, asset 1 is taken as a low risk, low variability asset and asset 2 taken as a high return, high variability asset with the asset returns having a correlation coefficient of 0·4. The asset parameters have been selected as representative of actual asset returns, variances and covariances of a fixed-interest investment (asset 1) and an ordinary share investment (asset 2). The expected value and variance of the liability is the same as the Wise–Wilkie examples. The correlation coefficients of the assets with the liability value have been determined to be consistent with the Wise–Wilkie values. The parameters for Table 2 are:

Means	$E_1 = 1{\cdot}05$	$E_2 = 1{\cdot}15$	$E_L = 327{\cdot}810$
Variances	$V_1 = 0{\cdot}0001$	$V_2 = 0{\cdot}0225$	$V_L = 5{\cdot}5563$
Standard Deviations	$\sigma_1 = 0{\cdot}01$	$\sigma_2 = 0{\cdot}15$	$\sigma_L = 2{\cdot}357181$
Covariances	$C_{12} = 0{\cdot}00060$	$C_{1L} = 0{\cdot}009429$	$C_{2L} = 0{\cdot}353577$
Correlation Coefficients	$r_{12} = 0{\cdot}4$	$r_{1L} = 0{\cdot}4$	$r_{2L} = 1{\cdot}0$

Table 2. *Asset allocations for revised asset returns*

A	r	w_1	w_2	E	V
200	0	0·94296	0·05704	−116·67	3·0
200	0·5	0·93127	0·06873	−116·44	3·1
200	1·0	0·91959	0·08041	−116·20	3·5
200	25	0·35884	0·64116	−104·99	295·1
400	0	0·98316	0·01684	92·86	13·0
400	0·5	0·97732	0·02268	93·10	13·2
400	1·0	0·97148	0·02852	93·33	13·5
400	25	0·69110	0·30890	104·55	305·1
1000	0	1·00728	−0·00728	721·46	85·6
1000	0·5	1·00495	−0·00495	721·70	85·7
1000	1·0	1·00261	−0·00261	721·93	86·0
1000	25	0·89046	0·10954	733·14	377·6

3.12 From Table 2 it can be seen that, as the amount of initial assets available to meet the liability increases, there is an increase in the proportion of the fund invested in asset 1. This result might seem counterintuitive, since as the size of the fund increases it might be expected that the proportion invested in the riskier security should increase rather than decrease. This is a result from using the mean-variance approach. From Table 2 it can also be seen that, for any given size of the fund, there is an increased proportion invested in asset 2 as the risk tolerance parameter increases. This latter result is understandable, since asset 2 is a riskier security than asset 1, and the fund will prefer to hold more of asset 2 as it becomes more risk tolerant. These results agree with the signs of the derivatives derived in § 3.8.

Solvency Margin

3.13 The trade-off between the level of solvency margin and initial assets can be considered by examining the probability of insolvency resulting from any given optimal investment strategy. Table 3 gives the size of initial fund and optimal asset allocation which meets a 1% probability of solvency constraint for a range of risk tolerance parameters. In order to maintain optimality, the level of initial fund is varied. The size of the fund is chosen so that, for the optimal asset allocation, the probability that the surplus will be negative is 0·01 or $E \geqslant 2{\cdot}36V$.

Table 3. *Asset allocations with 1% probability of insolvency for revised asset returns*

A	r	w_1	w_2	E	V
339	25	0·63178	0·36822	41·06	301·3
324	10	0·82932	0·17074	17·49	55·1
319	5	0·89977	0·10023	10·50	19·8

Unbiased Match

3.14 The unbiased match as covered in Wise (1984a, 1984b), requires that the expected value of surplus be zero. For the two-asset case with a fixed initial asset value, this requirement results in only one portfolio which meets this requirement. This is so since the asset allocation must meet the following two constraints:

$$A(w_1E_1 + w_2E_2) - E_L = 0$$

and

$$w_1 + w_2 = 1.$$

Solving these two simultaneous equations gives the unbiased match for the case of two securities with:

$$w_1 = \frac{E_L - AE_2}{A(E_1 - E_2)}$$

and

$$w_2 = \frac{E_L - AE_1}{A(E_2 - E_1)}.$$

In general, with more than two securities the unbiased match will require the choice of asset proportions which minimises the variance of ultimate surplus. Table 4 gives the unbiased match for the revised example used earlier in this paper for Tables 2 and 3 for a range of initial fund values. The corresponding risk parameter is given for the case where the unbiased match would be considered an optimal portfolio. The case with $r = 0$ is obtained by simply substituting zero for r in the optimal asset proportion formula. This case can be considered as representing the situation where the fund sponsors do not wish to bear any risk in any circumstances, and corresponds to the risk characteristics necessary to justify a matching investment strategy.

Table 4. *Optimal unbiased match for the example of Table 2*

A	r	w_1	w_2	E	V
200	250	−4·88882	5·88882	0	29115
311	0	0·97171	0·02829	0	8
400	−199	3·30559	−2·30559	0	18481

Matching and the Valuation of Liabilities

3.15 If it were possible to select assets that matched the liability exactly, then the value of the untraded liability could be determined by reference to the value of the exactly matching traded asset portfolio. In order to consider this matter we will alter asset returns in the revised Wise example, so that asset 1 is a risk-free security and asset 2 continues to be perfectly correlated with the liability. The parameters for this matching example will therefore be:

Matching asset portfolio parameters

Means	$E_1=1{\cdot}05$	$E_2=1{\cdot}15$	$E_L=327{\cdot}810$
Variances	$V_1=0{\cdot}0$	$V_2=0{\cdot}0225$	$V_L=5{\cdot}5563$
Standard Deviations	$\sigma_1=0{\cdot}0$	$\sigma_2=0{\cdot}15$	$\sigma_L=2{\cdot}357181$
Covariances	$C_{12}=0{\cdot}0$	$C_{1L}=0{\cdot}0$	$C_{2L}=0{\cdot}353577$
Correlation Coefficients	$r_{12}=0{\cdot}0$	$r_{1L}=0{\cdot}0$	$r_{2L}=1{\cdot}0$

Matching Portfolio

3.16 Since the return on asset 1 is now a constant and the return on asset 2 is perfectly correlated with L, we can write L as a linear combination of R_1 and R_2 so that:

$$L=\alpha_1 R_1+\alpha_2 R_2.$$

The values for the α_i are seen to be the amounts invested in asset i in an exactly replicating portfolio. These values can be determined by equating the mean and variance of the liability and the matching asset portfolio. Doing this gives:

$$\alpha_2=\frac{\sigma_L}{\sigma_2}$$

$$\alpha_1=\frac{1}{E_1}\left(E_L-\frac{\sigma_L}{\sigma_2}E_2\right).$$

3.17 For the example given above we have:

$$\alpha_2=\frac{2{\cdot}357181}{0{\cdot}15}=15{\cdot}7145$$

$$\alpha_1=\frac{1}{1{\cdot}05}(327{\cdot}81-15{\cdot}7145\times 1{\cdot}15)=294{\cdot}9888$$

so that the matching portfolio contains 5·06% in the perfectly correlated asset (asset 2) and 94·94% in the risk free asset (asset 1).

Value of Liability

3.18 The value of the untraded liability can be determined as the value of the matching asset portfolio. Since the returns on assets 1 and 2 are per unit of currency (pound) invested, the present values of these assets are just 1, so that the present value of the liability is:

$$\begin{aligned}\text{present value of } L &= \alpha_1 \times \text{present value of } R_1 + \alpha_2 \times \text{present value of } R_2 \\ &= \alpha_1 \times 1 + \alpha_2 \times 1 \\ &= \frac{1}{E_1}\left[E_L - \frac{\sigma_L}{\sigma_2}(E_2 - E_1)\right]\end{aligned}$$

which for the example is $294{\cdot}9888 + 15{\cdot}7145 = 310{\cdot}70$.

Optimality of Matching Portfolio

3.19 The optimal asset proportions can be rewritten for the case of a risk free asset, a perfectly correlated asset and with a risk tolerance parameter of zero as:

$$w_1 = \frac{AV_2 - C_{2L}}{AV_2} = 1 - \frac{r_{2L}\sigma_2\sigma_L}{A\sigma_2\sigma_2} = 1 - \frac{\sigma_L}{A\sigma_2}$$

$$w_2 = \frac{\sigma_L}{A\sigma_2}.$$

When these are substituted into the expression for the expected value of surplus, the value of the initial fund A, which results in an unbiased match can be determined as:

$$A = \frac{1}{E_1}\left[E_L - \frac{\sigma_L}{\sigma_2}(E_2 - E_1)\right]$$

Hence, the exactly matching portfolio is an optimal portfolio for a fund with a risk tolerance parameter of zero and with the value of initial assets in the fund equal to the value of the liabilities.

3.20 The single period model of Section 2 can be used to consider the trade-off between increased (or decreased) solvency margin and ultimate surplus by specifying the objective function as:

$$\underset{\{w_1, w_2\}}{\text{maximise}}\ E[U(-K) + U(S)/(1+b)]$$

where K is the initial solvency margin ($=p \times C$) and b represents the fund sponsors' time preference for solvency margin. The solution to this problem will incorporate the initial fund assets into the problem to reflect the correct tradeoff between fund assets and surplus. This introduces the multiperiod case which is briefly outlined in the next section.

4. THE MULTIPERIOD CASE

4.1 The Wise–Wilkie model attempts to model cash flows on assets and liabilities at discrete time periods during the time horizon of the liabilities. Interest rates are allowed to vary in each sub-period, and the random interest rate is a source of variation in accumulated liability values at the ultimate date along with inflation effects. The optimal asset portfolio that is determined is fixed at the start date, and is not varied as actual interest rates and inflation effects are realised over sub-periods. In other words, their model does not allow for rebalancing or revision of asset holdings through time, which should be a desirable feature of a dynamic portfolio selection model. As the time period to the ultimate date of the model reduces, the total amount of uncertainty in the accumulated assets and liabilities, and hence ultimate surplus, will reduce, and a revised optimal asset allocation should apply for this shorter remaining time period. To allow for portfolio revisions through time, a multiperiod model is required. The extension of the models in this paper is briefly outlined in this section. More details on the multiperiod approach can be found in Breeden (1987), Mossin (1968), and Samuelson (1969). This is also an area of current research by the author.

Asset Model

4.2 In order to extend the model in Section 2 to a multiperiod setting, it is also necessary to consider a more extended model of asset returns than allowed for in the Wise–Wilkie model. To do this we can consider two classes of assets. One class of assets will be identical to those in the earlier model, which provide a return in the form of a total unknown (random) payment at the end of each period, comprising capital gain and dividends. Per unit of currency (pound) invested these returns, denoted by R_{it}, will vary in each time period t. It is possible to allow these returns to be correlated to each other in any specific time period as well as across time periods, so that, in general, autocorrelation of returns can be modelled. This return model is suitable for equity assets such as shares and property, which provide random returns over future periods.

4.3 The other class of assets will provide fixed and known cash flows over the life of the asset, but whose holding period return over any sub-period can be random. Holding period returns over individual periods will be random on this class of assets except for the final period prior to maturity. These assets have fixed maturity dates T, and the price of the asset will be determined by valuing the individual cash flows at the spot interest rates applying for the time period to receipt of each of the cash flows. These spot rates, at each point of time, represent the term structure of interest rates, and are required to model and price assets with fixed and known cash flows. The spot rate at time t for a time to receipt of a cash flow of T periods, denoted by S_{tT}, will vary over time. This return model is suitable for fixed-interest assets such as government securities and mortgages.

4.4 At the current time, all S_{tT} values are known for all times to receipt, $T=1$

to the ultimate date of the asset (or liability) cash flows or determined from the yield to maturities on fixed interest securities. Values at future points of time are random for each of these S_{tT} variables. Conditions do need to be imposed on the relationship between these random spot rates through time, to ensure that there are no inconsistencies because of the existence of arbitrage opportunities. This is an area of considerable current research on models of the term structure of interest rates by both actuaries and financial economists.

Optimisation

4.5 The asset allocation problem can be written for the multiperiod case as:

$$\text{maximise}\left[U(-K_0) + \sum_{t=0}^{t=T} E\{U(S_t)/(1+b)^t\}\right]$$

where the surplus at each future time period is the net amount available to be withdrawn by the fund sponsors at time t, after providing sufficient assets K_t to support the liabilities over each subsequent time period. This objective function assumes that utility of surplus at each time period is discounted by a factor $(1+b)^t$, and then summed by the fund sponsors in order to determine the optimal asset allocation and the trade-off across time periods. This objective function can be maximised by considering its maximum value from time s to the horizon date, denoted by $V(s)$, and recognising that the maximum can be rewritten using Bellman's principle of optimality as:

$$V(s) = \text{maximum}[U(-K_s) + EU(S_{t+1})/(1+b)] + V(s+1)/(1+b).$$

The optimal values in each period can then be solved using dynamic programming by working backwards from the ultimate period. The ultimate period is simply a single period problem, as covered earlier in the paper.

4.6 In the multiperiod case, the constraints will need to be imposed for each period in the form of a budget constraint and, where considered appropriate, constraints will be required to ensure positive holdings of assets and to limit the probability of insolvency. It will also be necessary to incorporate the valuation of liabilities into the model, since this is required to determine the surplus and solvency of the fund. Such factors can be readily incorporated into computer-based asset/liability models for life insurance and pension funds.

4.7 The multiperiod approach is not explored further in this paper, since restrictive assumptions are required in order to obtain analytical results. Asset/liability models are the basis of the multiperiod approach. This paper has illustrated how an optimisation criterion can be incorporated into such models to reflect the risk attitudes appropriate for a fund. It is also important that the asset models, particularly for fixed-interest assets, are based on the term structure models that have been developed in the finance and, increasingly, the actuarial literatures.

5. CONCLUSION

5.1 This paper has attempted to show how the portfolio selection and matching problem fits into a more general framework than appears in the existing actuarial literature. This more general framework allows for the extension of the portfolio selection problem to a true multiperiod model, and can allow the incorporation of risk preferences in quite a general form. In order to implement this general multiperiod approach, as outlined in this paper, it will require a computer based asset/liability model and the use of numerical optimisation techniques. Analytical solutions are not so easy to derive in the more general framework.

5.2 The main body of the paper illustrates the relationship between existing approaches in the actuarial literature and that outlined in the early part of this paper. The Wise-Wilkie model was shown to be a special case. This makes more explicit the assumptions underlying a mean-variance approach to the portfolio selection problem and, it is hoped, illustrates the potential of the general optimisation approach for actuarial work.

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