# PRICING FOR RISK IN FINANCIAL TRANSACTIONS

1998 GENERAL INSURANCE CONVENTION
AND
ASTIN COLLOQUIUM

GLASGOW, SCOTLAND: 7-10 OCTOBER 1998

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**Paper submitted to the GISG for the**
**Brian Hey Prize**

GISG/ASTIN Joint Meeting
GLASGOW
October 1998

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## SUMMARY

This paper considers the pricing of uncertain cash flows, which includes those arising in insurance and reinsurance, using the proportional hazards (p-h) transform pricing basis defined by Wang (1995). This basis satisfies all the desirable properties of a sound pricing principle including sub-additivity and layer additivity and is a generalisation of the classic Standard Deviation Principle of Risk Theory, which appears to be valid when the underlying distribution has fixed skewness. The p-h basis deals with all distributions, including empirical ones, by taking account of all their moments in its formulation.

This framework does not depend on any financial or economic theory (e.g. MPT) or model (e.g. CAPM) but associates a risk loaded price to an index $\rho \geq 1$. Pricing cycles are simply changes in this index. Market sentiment may also impact this index.

This paper outlines and illustrates the use of the p-h basis for a range of insurance and reinsurance examples and then investigates the characteristics of the resulting risk loadings for a range of underlying distributions. The identification of the implicit risk aversion level (RAL), or index $\rho$, is investigated for a number of financial transactions, including simple games of roulette, a gamble on a lottery and an investment in premium bonds. A speculative attempt is also made to identify a historic asset market risk free rate and index $\rho$. The Black-Scholes option pricing formula is also investigated and found to be pricing at the risk aversion index $\rho = 1$, which is an expected value with no risk loading.

Finally a number of potential applications are discussed, such as identification of 'appropriate risk discount rates' in valuation type exercises, such as embedded, appraised and shareholder value calculations, the setting of target product profit margins, capital allocation and portfolio optimisations.

# Table of Contents

# 1. INTRODUCTION

## *1.1. Pricing Cash Flows in Financial Transactions*

The pricing of a stream of uncertain cash flows is a fundamental challenge for any financial institution, be it a life or non-life insurance or reinsurance company, a lending institution or an investor in corporate bonds. In principle, as these institutions need to produce a profit, or service capital, they need to charge something in excess of the expected present value of the cash flow stream after expenses.

It is generally accepted that cash flows with the same expected present value warrant progressively higher market prices as the 'uncertainty, variability or risk' inherent in these cash flows increases. Finding a simple measure of this variability for such pricing purposes has, however, proved to be a very difficult challenge.

The actual price, in relation to the expected value, will depend on the level of uncertainty (risk) and the risk aversion of the buyer (investor). In a highly risk averse environment the maximum price the investor can demand can approach the maximum possible value of the cash flows. Exceptionally, the investor may obtain benefits from an exchange which reduces the variability of his portfolio in which case the transaction may be done at expected value. In other words, the price has to fall somewhere between the expected and maximum values, depending on the level of uncertainty and the risk tolerance of the investor.

In practice, the pricing basis has to be capable of producing values in this range and also satisfy some other desirable conditions in order to avoid arbitraging situations. The most challenging of these conditions is the preservation of layer additivity most commonly found in excess of loss reinsurance pricing.

This paper outlines the proportional hazards (P-H) pricing basis (Wang 1995) which satisfies all the desirable conditions and is, in addition, relatively simple to apply in practice. All that is required to calculate the risk adjusted price at any given Risk Aversion Level (RAL) is the survival function, S(x), of the underlying distribution and the integration of the proportional hazard transform of this distribution, given by $S(x)^{(1/\rho)}$, over the range 0 to infinity. This is fairly easily done in a spreadsheet by numerical integration.

The paper illustrates the use of the basis for a variety of situations and explores the implicit RAL's of some transactions where the both the market price and the survival function are known, or can be estimated, before considering potential further applications of the framework.

## *1.2. Actuarial (Risk Theory) Premium Pricing Principles*

Insurance works by pooling policies together in sufficiently large numbers so that the Central Limit Theorem begins to apply resulting in the reduction of variability in the aggregated losses arising from these policies.

Although this diversification, which arises from the assumed claims independence of the underlying policies, may substantially reduce the portfolio variability, it does not eliminate it totally and consequently the insurer needs to charge something in excess of the expected losses and expenses for taking on this residual exposure to losses in excess of those expected and in order to compensate the shareholders who provide the capital (solvency) necessary for the transactions to take place. There may also be other factors he needs to consider, such as changes in economic conditions and the legal environment, that may increase the portfolio variability further.

The stochastic element of the insurance claims process, or Risk Theory, continues to be an area of intense study. In this framework, the expected losses are, confusingly, called the 'risk premium'. Ignoring the impact of the cash flow timings and expenses, it is evident that the insurer needs to charge more than these 'risk premiums' to remain solvent. This loading of the risk premiums, or safety loading, as well as the inherent variability of the portfolio loss experience, can then be shown to play a key part in the security or solvency of the insurer as measured by probabilities of ruin.

Whilst this is a theoretically appealing and sometimes elegant theory, it is far too simplistic in the real world where, for example, the solvency of an insurer is primarily driven by external factors, such as changes in market asset values, pricing cycles with, perhaps, catastrophe losses being the major stochastic claims component. Note that the use of the term risk so early in the development of this theory is very misleading as it describes expected outcomes and takes no account of variability in these outcomes. To avoid confusion we will refer to this expected value as the pure premium.

The relationship between the premium required, the safety loading and the pure premium can be expressed very simply by the following equation:

**Required Premium = Pure Premium + Safety Loading**

It is more convenient to express this relationship as follows

$$\mathbf{P_\lambda = (1 + \lambda) * E(P)}$$

where the $\mathbf{E(P)}$ is the expected loss or pure premium and $\mathbf{P_\lambda}$ is the required loaded premium at some safety loading coefficient $\lambda$.

The search for a 'premium principle' has resulted in a large number of proposals with, perhaps, the 'expected value', 'standard deviation' and 'variance' principles being the most common. These may be expressed as follows:

| | |
|---|---|
| **Expected Value Principle:** | $P_\lambda = (1 + \lambda) * E(P)$ |
| **Standard Deviation:** | $P_\lambda = E(P) + \beta * SD(P)$ |
| **Variance Principle:** | $P_\lambda = E(P) + \gamma * Var(P)$ |

As a generalisation of these principles we can express the safety loading as a linear combination of these moments as follows: (see Daykin et al 1994)

$$\textbf{Safety Loading } \lambda = \lambda_1 * E(P) + \lambda_2 * SD(P) + \lambda_3 * Var(P)$$

None of these principles satisfy all of the basic conditions expected of a sound basis for pricing for risk (see Sundt 1984). In the case of the variance principle, or a basis that includes a variance term, we are also adding two values of different dimensions (£ and $£^2$) which seems somewhat strange as well as inconsistent.

The lack of success in this key search for a sound pricing basis has hindered real progress in actuarial pricing methodology. Practical solutions have been developed, such as Profit Testing in Life Assurance, but invariably these rely on subjective selections of notional capital amounts or risk discount rates.

Whilst neither of these approaches is wrong, as we will see in Section 6, we need to find a more objective basis for determining these values rather than relying on experience, judgment and perceived market wisdom. The P-H methodology provides such a framework.

The identification of a robust premium principle and the potential uses of such a framework is the main focus of this paper. We seek a sound basis for adjusting expected values to allow for the riskiness, or variability, in the outcome. Ideally we would like to be able to say that the basis is the unique solution to this quest. It is currently the only such solution, but proving its uniqueness is far beyond the scope of this paper.

## 2. THE PROPORTIONAL HAZARDS PREMIUM BASIS

### *2.1. The Survival Function Of Random Variable*

The Survival Function S(x) of a random variable X, is the probability that the value exceeds x. In other words,

$$S(x) = \text{Probability}\,(X > x) = 1 - \text{Probability}\,(X \le x) = 1 - F(x)$$

where F(x) is the Distribution Function. The S(x) starts at 1 when x = 0 and decreases to zero at the variable approaches its maximum value, M.

For non-negative random variables X, which includes all the cases we would meet in a pricing environment, the expected or mean value of X, E(X), is given by integral of the survival function over the range from zero to infinity.

$$E(X) = \int_0^\infty S(x)\,dx$$

Proof: Let f(x) and F(x) be the density and distribution functions respectively, then

$$E(X) = \int_0^\infty x\,f(x)\,dx = \int_0^\infty x\,\frac{d}{dx}(F(x))dx = \left[xF(x)\right]_0^\infty - \int_0^\infty (1-S(x))\,dx$$

$$= \left[xF(x)\right]_0^\infty - \left[x\right]_0^\infty + \int_0^\infty S(x)\,dt = \int_0^\infty S(x)\,dt \quad \text{as } F(x) \to 1 \text{ as } x \to \infty$$

It is helpful to visualise this result as the mean of the variable is equal to the area under the survival function curve. This observation facilitates all the subsequent calculations, particularly where the results are obtained from a simulation.

### *2.2. Proportional Hazards Risk Adjusted Premiums*

Given a random variable X with survival function S(x), define a new random variable Y whose survival function, S(y), is given by

$$S(y) = (S(x))^{1/\rho}, \text{ where } \rho \ge 1.$$

The mapping $\Pi_\rho : X \to Y$ is called the proportional hazards (P-H) transform which is a well known transformation in statistics, attributed to Cox. Note that $S(y) \ge S(x)$.

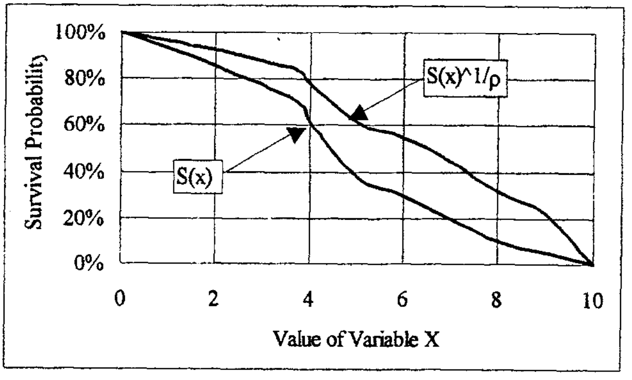
Wang (1995) defines the P-H risk-adjusted premium for a policy or portfolio with expected losses described by a positive, random variable X as the mean of the transformed distribution given as follows:

$$\Pi_\rho(X) = E[\Pi_\rho(X)] = \int_0^\infty (S(x))^{1/\rho}\, dx$$

This formula enables us to calculate the risk adjusted premiums at any value of $\rho$ once we have obtained or estimated the underlying survival function either by integration of an analytic function or by numerical integration. In either case we need to ensure that we are integrating over the complete range (0 to $\infty$) and not just the range of our variable X.

The following chart shows a simplified survival function for a variable and its P-H transform. The areas under the two lines give the mean (risk premium) and the risk-adjusted premium respectively. The area between the two lines can be interpreted as the additional premium required to pay for the variability at the particular RAL $\rho$.

**Figure 2.1. The Survival Function and its P-H Transform**




We note that we can calculate these values fairly easily for any value of $\rho$, once we have the underlying survival distribution. The easiest way to estimate this function is to develop a model of the cash flows and then use simulation to estimate the survival function. Once we have the survival function we can simply use numerical integration to produce risk-loaded premiums at various RAL values. Numerical integration is easily implemented in a spreadsheet and a data-table can then be set-up to show the risk adjusted premiums for different values of $\rho$.

## *2.3. Basic Properties of the P-H risk-adjusted premium*

The p-h risk adjusted premium basis satisfies all the properties required of a sound or desirable (actuarial) pricing principle, including linearity, layer additivity and sub-additivity. These are outlined below. Details and proofs can be found in Wang (1995).

**Positive Loading and No Rip-Off**

As the transformed distribution survival function S(y) is greater than the underlying S(x) for all x, clearly the expected value of Y will exceed the expected value of X for all $\rho > 1$. This is the so-called Positive Loading property.

The mean of this transformed distribution also increases as $\rho$ increases and it can be shown that as $\rho$ tends to infinity the transformed mean tends to the maximum value of the underlying distribution. In other words the mean of the transformed distribution can take any value from the mean to the maximum value of the underlying distribution, as the 'risk-aversion' index $\rho$ increases from 1 to infinity. This is the so-called No Rip-off property.

**Preservation of Stochastic Order**

The two variables P and Q are said to be stochastically ordered if the survival function of one is at least equal to the survival function of the other for all values from 0 to infinity. In such cases, the risk adjusted premiums, at a given $\rho$, will always preserve this order. So, if $S_P(t) \leq S_Q(t)$ for all $t \geq 0$, then $\Pi_\rho(P) \leq \Pi_\rho(Q)$ for all $\rho \geq 1$.

**Linearity and scale invariance**

Linearity means that $\Pi_\rho(\alpha X + \beta) = \alpha \Pi_\rho(X) + \beta$. Apart from avoiding obvious problems with scale conversions, such as currency translation, this property is useful as it corresponds with what happens with a quota share treaty, where we see that $\Pi_\rho(X) = \Pi_\rho(\alpha X) + \Pi_\rho(\{1-\alpha\}X)$.

**Layer additivity**

Layer additivity is a crucial property when we consider what happens in the much more complex case of excess of loss reinsurance. Here our premium principle has to preserve layer costs, that is the pricing of two adjacent layers should equal the price of the one combined layer for a given value of $\rho$. If this is not the case then the premium principle would imply existence of arbitrages which would be difficult to justify.

Let $L(a,b)$ denote the excess of loss layer from a to b, (i.e. (b-a) excess of a). Suppose that we have two adjacent layers $L(a,b)$ and layer $L(b,c)$. These two layers can then be combined to a single layer $L(a,c)$. If we denote the risk adjusted premium for the layer $L(a,b)$ at a risk aversion level $\rho$ by $\Pi_\rho(L(a,b))$ then under the PH transform we have:

$$\Pi_\rho(L(a,c)) = \Pi_\rho(L(a,b)) + \Pi_\rho(L(b,c))$$

This is a particularly useful and powerful result and can be validated by simulation. The proof can be found in Wang (1995). Anyone tempted to check this result should bear in mind that in case of limited reinstatements the combined layer will provide more cover and so should attract higher premiums.

## 2.4. *Sub-additivity and Synergy Value*

It can be shown that, for two non negative random variables, X and Y, the risk-adjusted premium for the combined portfolio, X+Y, at any given RAL $\rho$, described by $\Pi_\rho(X+Y)$, satisfies the following inequality:

$$\Pi_\rho(X+Y) \leq \Pi_\rho(X) + \Pi_\rho(Y) \qquad \rho \geq 1$$

where $\Pi_\rho(X)$ and $\Pi_\rho(Y)$ denote the risk-adjusted premiums for the individual portfolios. This is a key property and the one that the variance principle fails to satisfy. (see Sundt 1984). Note that if it it does not hold, then we can reduce the cost of insurance by dividing the cover into two or more sections.

**Synergy Values**

The sub-additivity property expresses in symbols what we would normally describe as risk diversification. This observation leads naturally to the following definition.

Let **Synergy** be defined by $\Sigma_\rho(X,Y)$ where.

$$\Sigma_\rho(X,Y) = \Pi_\rho(X) + \Pi_\rho(Y) - \Pi_\rho(X+Y)$$

From the sub-additivity inequality we see that $\Sigma_\rho(X,Y)$ is positive for all $\rho \geq 1$.

This apparent 'saving' from the aggregation of the portfolios is what is commonly described as **synergy value** although it is rare for synergy value to be defined objectively. Our formal definition not only defines synergy on an objective basis but also shows how this value is, as we would expect, dependent on the RAL level, $\rho$, at which we are pricing. This has uses and implications in many areas involving 'valuations' and 'optimisations'.

# 3. SOME BASIC RESULTS OF THE P-H PRICING BASIS

## *3.1. Risk Loads for Normally distributed variables*

The following table shows the ratio of risk adjusted values to the underlying mean, or the Risk Aversion Factors (RAF), for various values of a Normal variate with varying coefficient of variation. As the cumulative distribution function does not have a closed form, these values were calculated by numerical integration.

**Table 3.1 Risk Aversion Factors ($RAF_\rho$) for a Normal Variable.**

| | Coefficient of Variation | | | | | | |
|---|---|---|---|---|---|---|---|
| RAL | 5% | 7.5% | 10% | 12.5% | 15% | 20% | 25% |
| 1 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| 1.2 | 1.008 | 1.013 | 1.017 | 1.021 | 1.025 | 1.034 | 1.042 |
| 1.4 | 1.016 | 1.024 | 1.032 | 1.040 | 1.048 | 1.064 | 1.080 |
| 1.6 | 1.023 | 1.034 | 1.046 | 1.057 | 1.069 | 1.092 | 1.115 |
| 1.8 | 1.029 | 1.044 | 1.058 | 1.073 | 1.088 | 1.117 | 1.146 |
| 2 | 1.035 | 1.053 | 1.070 | 1.088 | 1.105 | 1.140 | 1.175 |
| 3 | 1.059 | 1.088 | 1.118 | 1.147 | 1.176 | 1.235 | 1.294 |
| 5 | 1.090 | 1.135 | 1.180 | 1.225 | 1.270 | 1.359 | 1.449 |

Let us now define the Standardised Risk Loadings ($SRL_\rho$) at RAL ρ, as the ratio of ($RAF_\rho$ -1) to the coefficient of variation. The values are shown below.

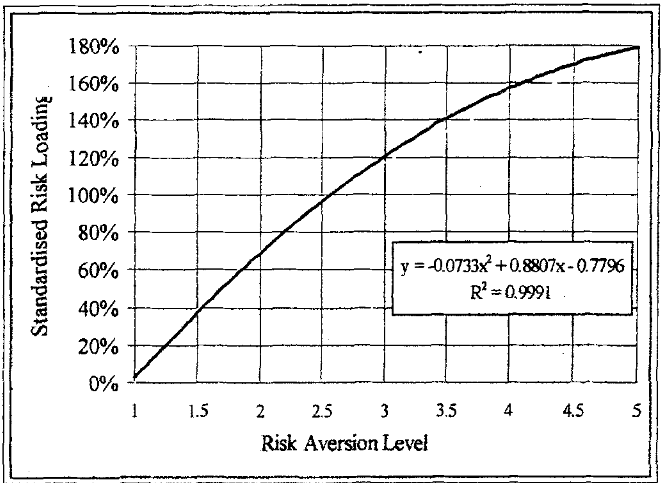
**Table 3.2. Standardised Risk Loadings ($SRL_\rho$) for a Normal Variable**

| | Coefficient of Variation | | | | | | |
|---|---|---|---|---|---|---|---|
| RAL | 5% | 7.5% | 10% | 12.5% | 15% | 20% | 25% |
| 1 | 0% | 0% | 0% | 0% | 0% | 0% | 0% |
| 1.2 | 17% | 17% | 17% | 17% | 17% | 17% | 17% |
| 1.4 | 32% | 32% | 32% | 32% | 32% | 32% | 32% |
| 1.6 | 46% | 46% | 46% | 46% | 46% | 46% | 46% |
| 1.8 | 58% | 58% | 58% | 58% | 58% | 58% | 58% |
| 2 | 70% | 70% | 70% | 70% | 70% | 70% | 70% |
| 3 | 118% | 118% | 118% | 118% | 118% | 118% | 118% |
| 5 | 180% | 180% | 180% | 180% | 180% | 180% | 180% |

The above table shows that for the range of coefficients of variation and RAL tabulated the SLR's are dependent only on the RAL ρ. This is a restatement of the Standard Deviation Principle discussed in Section 2.1. as we can see by restating this from **$P_\lambda$ = E(P) + β * SD(P)** to **$P_\rho$ = E(P) ( 1 + $\lambda_\rho$ * Coefficient of Variation).**

The p-h framework simply enables us to derive the shape of this $\lambda_\rho$ as we vary ρ. The following chart shows this relationship.

**Figure 3.1 Risk Aversion Loads for Normal Variables**




The relationship between the SRL and the RAL can be approximated fairly accurately by a quadratic curve. This provides a simple means of estimating the RAF at any RAL, ρ, at least in the range $1 \le \rho \le 5$, simply from the coefficient of variation of the distribution, as long as the coefficient of variation does not exceed 25% so that almost all the values are positive.

For example, the RAF, at ρ = 1.7, for a Normal variable with a coefficient of variation of 8% can be calculated as follows:

$$RAF = 1 + 0.08 * (-0.0733*1.7^2 + 0.8807*1.7 - 0.7796) = 1.0405$$

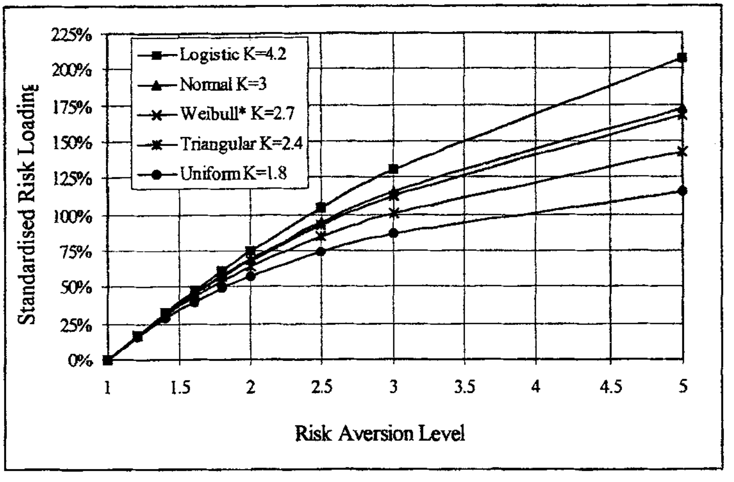
**These results provide justification for the use of the standard deviation as a measure of risk is cases where the distribution can be assumed to be normal and positive.**

## 3.2. *Symmetric distributions and risk loadings*

We saw in the previous section that in the case of the Normal distribution, the risk adjustment, at a given RAL, is proportional to the coefficient of variation. This property appears to hold for all the other symmetric distributions tested and can be demonstrated mathematically for some of these distributions. This is left to the reader to test and explore further.

The following chart shows the resulting relationships between the SRL and the RAL for a number of these standard symmetric distributions as well the 'symmetric' Weibull (skewness of zero). The kurtosis of this Weibull is 2.7 slightly below that of the Normal distribution which has a kurtosis of 3.

**Figure 3.2. Risk Aversion, Symmetry and Kurtosis**




The labels in the chart identify the kurtosis of each of these distributions and chart demonstrates that increasing the kurtosis appears to increase the risk loading, all else (lower moments) being equal.

This simple example suggests that the P-H framework incorporates a kurtosis component in its risk loadings, and that at least the first three moments of the distribution are significant contributors to the risk loading.

## *3.3. Risk Loadings for a Distribution with Fixed Skewness*

The Extreme Value, or Gumbel, distribution finds application in a many diverse areas of study. It is best described by its distribution function:

ExtremeValue ( a, b) : $F(x) = \exp(-\exp(-(x - a)/b)$

where x ranges over the real line and b is positive( b >0).

The mean of this distribution is equal to ( a + .577 b) and the standard deviation is equal to $(\pi * b/\sqrt{6})$. The skewness and kurtosis are both invariant, at 1.14 and 5.4 respectively. (See Evans, Hastings & Peacock (1993)).

The following table shows the risk adjustment factors on the mean at various RAL values for a range of coefficients of variation obtained by numerical integration.

**Table 3.3 Risk Loads for the Extreme (Gumbel) Distribution**

| RAL | Risk Loading Factors | | | | Standardised Risk Loading Factors | | | |
|---|---|---|---|---|---|---|---|---|
| | Coefficient of Variation | | | | Coefficient of Variation | | | |
| | 5.2% | 10.2% | 19.6% | 24.0% | 5.2% | 10.2% | 19.6% | 24.0% |
| 1 | 1 | 1 | 1 | 1 | 0 | 0 | 0 | 0 |
| 1.2 | 1.01 | 1.02 | 1.038 | 1.046 | 0.192 | 0.192 | 0.192 | 0.192 |
| 1.4 | 1.02 | 1.039 | 1.074 | 1.09 | 0.376 | 0.376 | 0.376 | 0.376 |
| 1.6 | 1.029 | 1.057 | 1.109 | 1.133 | 0.555 | 0.555 | 0.555 | 0.555 |
| 1.8 | 1.038 | 1.075 | 1.143 | 1.175 | 0.73 | 0.73 | 0.731 | 0.731 |
| 2 | 1.047 | 1.093 | 1.177 | 1.216 | 0.903 | 0.903 | 0.903 | 0.903 |
| 2.2 | 1.056 | 1.11 | 1.21 | 1.257 | 1.073 | 1.073 | 1.073 | 1.073 |
| 2.4 | 1.065 | 1.127 | 1.243 | 1.297 | 1.241 | 1.241 | 1.241 | 1.241 |
| 2.6 | 1.074 | 1.144 | 1.276 | 1.337 | 1.408 | 1.408 | 1.408 | 1.408 |
| 2.8 | 1.083 | 1.161 | 1.308 | 1.377 | 1.574 | 1.574 | 1.574 | 1.574 |
| 3 | 1.091 | 1.178 | 1.341 | 1.417 | 1.738 | 1.738 | 1.738 | 1.738 |

We can see from the left portion of this table that the risk loading increases as we increase the RAL and the coefficient of variation, as we would expect. The right portion of this table shows the result of 'standardising' these loadings by the coefficient of variation and produces a somewhat surprising result. As we found in the case of the 'symmetric' distributions there appears to be a relationship between the coefficient of variation and the RAF for each RAL also for the Extreme Value distribution. Note also that for the range of RAL shown in the Table, the standardised percentage loads can be approximated by a simple linear form : $y = 0.865x - 0.8399$.

## *3.4. Risk Loads for a Distribution with Variable Skewness*

The Weibull distribution, with scale parameter b and shape parameter c, has a particularly convenient Survival function (See Evans et al (1993)).

Weibull ( b, c) : S(x) = exp (- x/b) ^c

We can see that the PH Transform is also a Weibull described by W ( b, c/ρ ). The mean of the Weibull is given by b * Γ [ (c + 1)/ c ) ] and so the ratio of the risk adjusted to the expected value (or RLF) will be independent of the b parameter. The coefficient of variation and kurtosis are also independent of this parameter. The following table shows the risk adjustment factors on the mean at various RAL values for a range of coefficients of variation.

**Table 3.4 Risk Loadings for the Weibull Distribution (variable skewness)**

| RAL | Risk Loading Factors | | | | Standardised % Risk Load | | | |
|---|---|---|---|---|---|---|---|---|
| | Coefficient of Variation | | | | Coefficient of Variation | | | |
| | 12.0% | 36.3% | 52.3% | 100.0% | 12.0% | 36.3% | 52.3% | 100.0% |
| 1 | 1.000 | 1.000 | 1.000 | 1.000 | 0% | 0% | 0% | 0% |
| 2 | 1.072 | 1.260 | 1.414 | 2.000 | 60% | 72% | 79% | 100% |
| 3 | 1.116 | 1.442 | 1.732 | 3.000 | 97% | 122% | 140% | 200% |
| 4 | 1.149 | 1.587 | 2.000 | 4.000 | 124% | 162% | 191% | 300% |
| 5 | 1.175 | 1.710 | 2.236 | 5.000 | 146% | 196% | 236% | 400% |

We can see from the above table, as was the case in the previous examples, that the risk loads increase as we increase the RAL and the coefficient of variation. In this instance the standardised percentage risk loadings increase with both the RAL and the coefficient of variation and the earlier apparent relationships between the risk loading and the coefficient of variation no longer holds. We are, however, tempted to speculate:

## *3.5. Conjecture:*

***The Standard Deviation principle, given in Section 1.2, holds for distributions with zero or constant skewness. In other words for a given Risk Aversion Level level $\rho$, $RAL_\rho$, the Risk Aversion Factor $RAF_\rho$ can be expressed as: $RAF_\rho = 1 + \lambda_\rho$ * Coefficient of Variation, where $\lambda_\rho$ is a function of $\rho$.***

***This would provide support for the use of the standard deviation as a measure of risk where the underlying distributions have fixed skewness, which includes all the symmetric ones, and also identifies the limitations of this common basis.***

## 4. P-H PRICING IN INSURANCE AND REINSURANCE

### *4.1. Pricing a simple insurance product and diversification*

Consider a simple contract of insurance for breakdown cover of a household appliance and assume that we know both the frequency and claim severity distributions fairly accurately.

Suppose that the annual frequency is 10% (per annum) and that the claims size distribution is described reasonably well by a truncated exponential distribution with minimum \$30, shape \$30 and maximum claim amount of \$1000. The expected average claim cost is therefore \$60 and the annual expected loss is \$6.

Now consider a household, or a small hotel, that has a number of these appliances. In order to demonstrate 'risk diversification' we will assume that we have 1, 5, 10 and 50 such appliances on cover. The expected annual costs are then \$6, \$30, \$60 and \$300 respectively.

The following table shows the estimated risk adjusted costs at various RAL's for each of these policies assuming that the frequency is Poisson distributed. The calculations have been performed by a simulation and numerical integration.

**Table 4.1 Risk Loaded Premiums and Factors for various size portfolios**

| | | | | |
|---|---|---|---|---|
| Mean Simulated | 6.0 | 29.7 | 59.6 | 299.8 |
| Skewness | 4 | 1.8 | 1.3 | 0.6 |

| | Risk Loaded Premiums | | | | Risk Loading Factors $1+\lambda$ | | | |
|---|---|---|---|---|---|---|---|---|
| | Number of Appliances | | | | Number of Appliances | | | |
| RAL | 1 | 5 | 10 | 50 | 1 | 5 | 10 | 50 |
| 1 | 6.0 | 29.7 | 59.6 | 299.8 | 1.00 | 1.00 | 1.00 | 1.00 |
| 1.2 | 9.6 | 38.1 | 71.3 | 324.7 | 1.59 | 1.28 | 1.20 | 1.08 |
| 1.4 | 13.7 | 46.5 | 82.5 | 347.5 | 2.26 | 1.56 | 1.39 | 1.16 |
| 1.6 | 18 | 54.6 | 93.2 | 368.7 | 2.98 | 1.84 | 1.56 | 1.23 |
| 1.8 | 22.5 | 62.7 | 103.4 | 388.5 | 3.73 | 2.11 | 1.74 | 1.30 |
| 2 | 27.2 | 70.6 | 113.1 | 407.1 | 4.50 | 2.37 | 1.90 | 1.36 |
| 2.5 | 38.8 | 89.5 | 135.4 | 448.9 | 6.42 | 3.01 | 2.27 | 1.50 |
| 3 | 50.0 | 107.1 | 155.0 | 485.1 | 8.28 | 3.60 | 2.60 | 1.62 |
| 5 | 87.1 | 164.8 | 212.6 | 590.5 | 14.42 | 5.54 | 3.57 | 1.97 |

Let us assume, for illustration purposes, that the actual premium per appliance is $22.5 We can see from the RAL Table above that this indicates that the pricing, from the policyholders perspective, is at an RAL of around 1.8 and the premium is 3.73 times risk premium (or expected cost for the year).

For a household with five similar appliances (with the same breakdown characteristics) the risk loaded premium at a RAL of 1.8 is $62.7, or 2.11 times risk premium. For a housing association with 50 such appliances the risk adjusted premium reduces to just 1.3 times the risk premium at the same RAL of 1.8.

The example assumes that the experience of each of these appliances is identically distributed and independent of the other machines. It is not unknown for a particular model of an appliance to develop a problem with a component and there is, therefore a potential aggregation of risk that may need to be considered in practice.

The table also shows the reduction in the risk loading at a given RAL as the number of appliances increases. We can see, for example, that if the insurer prices these at an RAL of, say, 1.8 based on a portfolio of 50 such machines he will be loading his expected costs ($6 per machine) by 30%. At such a loading factor the individual policyholders are being charged at an equivalent RAL of around 1.1, that is in the absence of expense charges. The fact that they are paying considerably more is due to these charges which, in essence, utilise the 'diversification' benefits that the insurance pooling achieves and turn it into expenses and commissions incurred in facilitating the pooling. In the case of such low premium cases both these items may substantially exceed the underlying risk premium.

The informed policyholder needs to consider whether having a large number of low risk premium policies, such as extended warranty, appliance breakdown cover, pet insurance, dental insurance and so on, makes economic sense given the sort of analysis above. There may also be a RAL, such as 1.8, over which the informed policyholder may be reluctant to purchase such cover and run the 'risk' themselves.

## *4.2. Pricing Corporate Bonds (Securitised Catastrophe Exposures)*

The recent developments in the securitisation of insurance risk, particularly the issue of catastrophe linked bonds, have led to comparisons between the pricing methodologies of traditional catastrophe reinsurers and (bond) rating agencies.

In the simplest case, 'act of god' bonds behave in exactly the same way as the more traditional corporate bonds. On the whole, the bonds will perform without any problem. There is, however, a chance of default in which case part or all of the capital may be lost. In the case of a 1-year bond of $100, the survival distributions will look very similar. There will be a probability, $p_1$ that the bond does not repay fully and a probability $p_2$ that all the capital is lost and, generally, some smooth line joining these two points. These two probabilities and the shape of the curve joining them describe the survival function and provide all we need to price these bonds in a P-H framework. The nature or origin of the bond is immaterial.

In the case of the ground-breaking USAA/Residential Re issue in 1997, information was provided in the prospectus on the estimated losses at various probabilities, from well below the trigger point ($1billion from the ground-up (fgu) losses) up to the top limit of the coverage ($1.5 billion fgu).

For example, the probability of any loss of capital ($1billion fgu) was just under 1% and the probability that all the capital would be lost ($1.5billion fgu) was just below 0.4%. The probability of losing half the capital ($1.25billion fgu) was estimated at 0.7%. The implicit survival function can be approximated well by an exponential of the form:

$$S(x) = 0.01 \exp( - (.5*\ln(.01/.004))x) \qquad \text{for } x= 0 \text{ to } 0.5 \text{ (\$billion)}$$

or $$S(x) = 0.01 \exp( -1.833x) \qquad \text{for } x= 0 \text{ to } 0.5 \text{ (\$billion)}$$

With our convenient choice of approximate survival function, we can derive a formula for the P-H risk-loaded premium at any RAL $\rho$ fairly easily by high school integration. The result is given by the following formula:

$$\Pi_\rho(x) = 0.01^{(1/\rho)} * (\rho / 1.833) * ( 1 - \exp ( - 0.9165/\rho) )$$

The mean loss of capital ( at $\rho = 1$ ) is then 0.0033 (billion) or 0.66% of the notional $500m of cover if covered by a single full issue. The actual element of the issue that had its capital exposed to such a loss was issued at LIBOR plus 567 basis points or 5.67% ROL (Rate On Line) in reinsurance terminology which, according to the above formula, indicates a RAL $\rho$ = 1.77. The LIBOR component simply compensates the bond holders at a 'risk free' rate.

Traditional catastrophe excess of loss reinsurance appears to be priced currently (1998 renewals) at levels consistent with RAL of around 1.5 to 1.7, although some reinsurers would hotly dispute any assertion that these prices are anything above expected values, which in p-h methodology equates to $\rho = 1.0$.

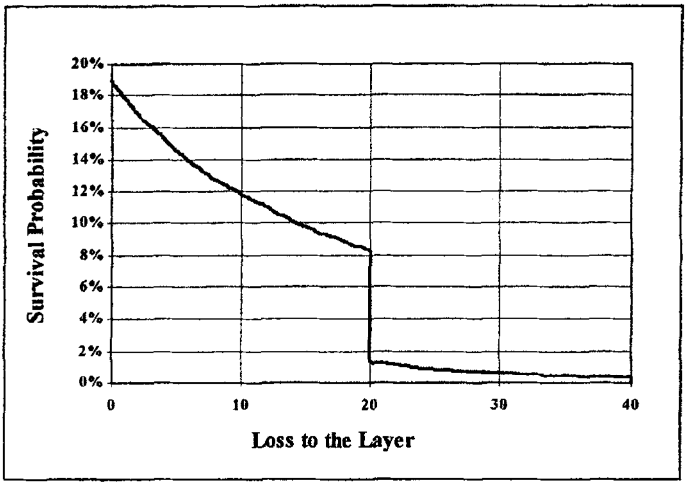
The above analysis indicates that the element of the 1997 USAA deal, where the capital was at risk, was done at a fairly high price as measured by the underlying RAL, with the investors receiving a very attractive return. In a recent second issue for 1998, the cost, for what appears to be very similar cover, is reported to be 400 basis points over LIBOR compared to the 576 basis points of the 1997 issue. Assuming that the underlying probabilities have not changed, the new issue has an implicit $\rho$ of 1.57 which, whilst below the 1.77 of the earlier issue, is still fairly high and still comparable with the current implicit levels in the more traditional catastrophe reinsurance market which is also seeing some price reductions.

Ultimately, the success of the ART market will depend on the pricing of these instruments at much lower RAL's which, in an options pricing environment, should approach values near expected cost or RAL $\rho = 1$ (see Section 5.7). At around $\rho=1.2$ the price, in basis points, would be around twice expected cost which would significantly undercut the traditional reinsurance pricing and still provide any investors, who find these bonds attractive for diversification purposes, a relatively high return.

## *4.3. Pricing Catastrophe Excess of Loss Reinsurance*

In the case of an excess of loss contract, the survival function may have a number of distinct elements or parts depending on the number of available reinstatements. In the case of catastrophe reinsurance one reinstatement at cost is usually available. This means that the amount of 'cover' available, before reinstatement premiums, is twice the layer amount. The following chart illustrates a typical survival function for an excess of loss layer, for \$20m excess \$30m, with one reinstatement.

**Figure 4.1 Survival Function for an Excess of Loss Layer with Reinstatement**




We note that for such covers, with a single reinstatement, the survival function has two distinct parts, with a break in the middle, where there is a measurable probability of a single event exhausting the available cover for an individual event. Looking at the extreme right of the chart we also note that there is a measurable probability that the cover is totally exhausted.

The easiest way to derive these distributions is by simulation. For example a Poisson or negative binomial distribution may be used for the frequency of the events and a lognormal, Pareto or Burr distribution for the severity of these events. The simulation then compounds these distributions and generates the number of events and their costs for a large number of one year periods. The reinsurance cover can then be matched to these events to estimate the likely recoveries under these assumptions for any number of layers of reinsurance.

Once we have the survival distribution for the layer we can calculate the risk loaded premiums at various RAL's by numerical integration.

In the chart above the event generating process was modelled by a Poisson frequency of 0.5 and a Pareto severity with scale 10 and shape 1.26. The following table shows the estimated costs of the layers, with unlimited free reinstatements, for two adjacent layers of reinsurance as well as that of the combined single layer, all derived from a simulation of 10,000 periods using these parameters.

**Table 4.2 Risk Adjusted Premiums for Excess of Loss Layers**

| | Layer 1 | Layer 2 | Combined |
|---|---|---|---|
| Limits | \$30xs\$30 | \$40xs\$60 | \$70xs\$30 |
| Simulated Cost | 2.346 | 1.484 | 3.830 |

| RAL | Risk Loaded Premiums | | |
|---|---|---|---|
| 1 | 2.346 | 1.484 | 3.830 |
| 1.2 | 3.706 | 2.619 | 6.309 |
| 1.4 | 5.209 | 3.970 | 9.135 |
| 1.6 | 6.803 | 5.469 | 12.192 |
| 1.8 | 8.451 | 7.065 | 15.395 |
| 2 | 10.127 | 8.719 | 18.679 |
| 2.5 | 14.325 | 12.935 | 26.976 |
| 3 | 18.395 | 17.062 | 35.066 |
| 5 | 32.176 | 30.993 | 62.524 |

The observant reader will note that the sum of the first two risk loaded premiums does not exactly match that calculated for the combined layer, especially as we increase the RAL. This apparent discrepancy is the result of the inaccuracies involved with any simulation. Also, as we increase the RAL to the sort of values at the end of this table the values at the tail of the distribution are given much more weight and so have an increasing influence on the results. This can be overcome by more simulations or some smoothing of these extreme values.

Note that the ability to estimate the implicit RAL at which the market is pricing these layers, by comparing derived risk loaded premiums with market premiums adjusted for expenses, provides a good index of market pricing hardness or softness especially when retention levels are changing.

## 4.4. *Pricing and Impact of a Stop Loss Reinsurance*

Consider a portfolio whose loss ratio can be described by the following empirical distribution of the loss ratio survival function S(x):

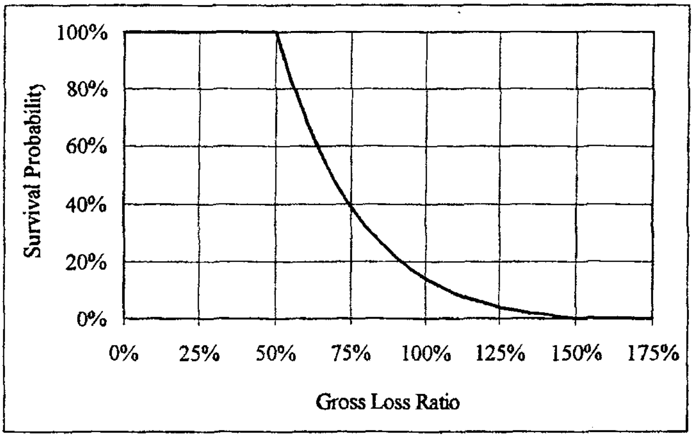
$S(x) = 1$, x=0, 50%

$S(x) = 6.26 * \exp(-3.62*x)-0.026$, x=50% - 150%

$S(x) = 0$ for x> 150%

The survival function is shown in the diagram below. As is clear from the description above, the loss ratio lies in the range 50% to 150%.

**Figure 4.1 Survival Function of the Gross Loss Ratio of Class XYZ**




We can calculate the expected or mean loss ratio by integrating the survival function:

$$E(x) = \int_0^{\infty} S(t)dt = 0.5 + \int_{0.5}^{1.5} (6.26 * \exp(-3.62*t)-0.026)\, dt = 0.749 = 74.9\%$$

Now let us suppose that expenses amount to 18% and that, therefore, on average the business makes a 7.1% 'profit' but clearly with a high degree of variation. We can estimate the risk aversion level implicit in the mean pricing level of this portfolio by estimating the RAL at which the risk-adjusted loss ratio is 82%.

The following table shows the risk-adjusted values at various risk aversion levels (RAL) calculated by numerical integration for the gross account, for a stop loss reinsurance for 50% excess 100% of the loss ratio and for the net account after this reinsurance, assuming this is priced at a RAL of 1.6. For simplicity it is assumed that the reinsurer will use the same underlying loss ratio distribution to price this stop loss.

**Table 4.3. Risk Adjusted Gross Loss Ratios**

| | P-H Risk Adjusted Loss Ratios | | |
|---|---|---|---|
| RAL | Gross | 50% xs 100% | Net after RI at RAL 1.6 |
| 1.0 | 74.9% | 2.6% | 79.6% |
| 1.1 | 77.1% | 3.3% | 81.0% |
| 1.2 | 79.1% | 4.0% | 82.3% |
| 1.3 | 81.1% | 4.8% | 83.5% |
| 1.4 | 82.9% | 5.6% | 84.5% |
| 1.5 | 84.7% | 6.4% | 85.5% |
| 1.6 | 86.4% | 7.2% | 86.4% |
| 1.7 | 88.1% | 8.0% | 87.3% |
| 1.8 | 89.7% | 8.8% | 88.1% |
| 2.0 | 92.6% | 10.4% | 89.5% |

We can see from the above that the implicit gross RAL is somewhere between 1.3 and 1.4. The table shows that the reinsurers expected loss is 2.6% of the original gross premium. However, if he prices his business at a RAL of 1.6 then he will require a premium of 7.2% of the original gross premium.

We can now consider the implications for the insurer after this reinsurance which we will assume he views as an additional claims cost. His net losses will now be limited to 100% of the original gross premium plus the 7.2% cost of the stop loss cover. As the mean recovery from the reinsurance is equal to 2.6% points, his expected average result with the reinsurance, against the original gross premium, is now (after rounding) 79.5%, that is the original gross ratio of 74.9% less the reinsurance benefit plus of 2.6% plus the cost of this reinsurance, which we assumed is 7.2%.

The insurer still has his expenses of 18% and so, after the stop loss his expected average profit margin is reduced from 7.1% to just 2.5%. We see from the above table that his implicit, after reinsurance, RAL has now reduced from the original gross value of 1.35 to below 1.2. Whether stop loss can still be considered to be the 'optimal form' of reinsurance is left to the reader to investigate now that we have a more appropriate pricing basis than the variance type principle.

## 5. IDENTIFYING RAL IN PRACTICE

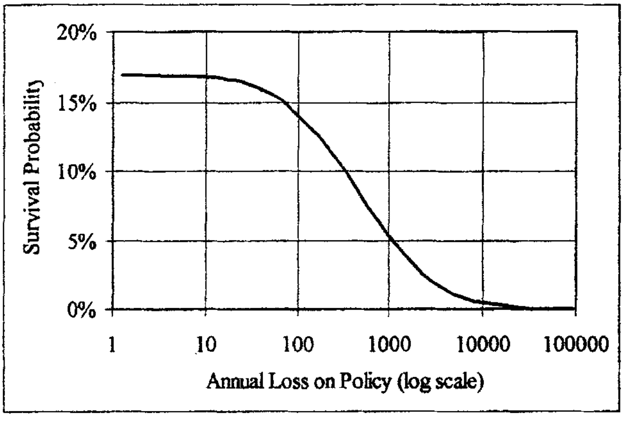
### 5.1. *Risk Aversion Level of a Motor Policyholder*

Although motorists are required by law to have third party insurance to cover them in case they injure someone, most drivers generally buy more than the basic cover to protect themselves against theft of their vehicle and damage they may cause to it. They are averse to the potential financial exposure which may run into many thousands of dollars and are prepared to pay an amount in excess of their annual expected cost to take on the year on year uncertainty.

Ignoring the impact of investment earnings for simplicity, we can associate a RAF to any loss ratio simply as the ratio of premiums to expected losses. For example, if the loss ratio is 70% then the policyholder pays $100 to cover an annual expected loss of $70 and so is being charged at an implicit Risk Aversion Factor of 100/70=1.43.

In order to estimate the underlying RAL for the policyholder we need to make an assumption about the annual loss distribution. We will assume for the purposes of illustration that the probability of a claim in a year is 18.5% and is Poisson distributed and that the loss size distribution follows a lognormal distribution with parameters 6 and 1.55. The mean is then $1,341 and standard deviation $4,252. Simulation is used to derive the survival function of the aggregate annual losses for the policy year and this is shown in the chart below.

**Figure 5.1. Survival Function for Losses on a Motor policy**



The average annual simulated loss, from a 10,000 year simulation is $256.1 with a standard deviation of $1,745, skewness of 24 and a kurtosis of 930.

The maximum annual loss simulated was just below $100,000. Note that as this distribution is very skew we have used a logarithmic scale for the x-axis. We can see, however, that the chance of a loss in a year is around 18% and that the chance of annual losses exceeding $1000 is 5%.

Once we have the survival function we can calculate the Risk Adjusted Premiums (RAP) at various Risk Aversion Levels (RAL). The table below shows a range of these values together with the Risk Aversion Factors (RAF) or multiples defined as the ratio of these risk adjusted premiums to the expected losses.

**Table 5.1. Risk Aversion Table for the Policyholder**

| RAL | RAP | RAF | L RATIO |
|---|---|---|---|
| 1 | 256.1 | 1.00 | 100% |
| 1.02 | 276.9 | 1.08 | 92% |
| 1.04 | 298.9 | 1.17 | 86% |
| 1.06 | 322.0 | 1.26 | 80% |
| 1.08 | 346.3 | 1.35 | 74% |
| 1.1 | 371.8 | 1.45 | 69% |
| 1.12 | 398.5 | 1.56 | 64% |
| 1.14 | 426.5 | 1.67 | 60% |
| 1.175 | 478.7 | 1.87 | 54% |
| 1.2 | 518.3 | 2.02 | 49% |
| 1.25 | 604.0 | 2.36 | 42% |
| 1.3 | 698.3 | 2.73 | 37% |
| 1.5 | 1164.9 | 4.55 | 22% |

This table also shows the loss ratio the insurance company would expect if it charged the RAP. We can see from the table that if the insurer prices these policies to achieve a 70% loss ratio then the policyholder RAL is, approximately, 1.105. When insurance prices drop so that the insurer's loss ratio increases to 80% the policyholder RAL drops to 1.06. In other words we could use the RAL to monitor the pricing cycle.

The insurer is not however using either of these RAL's to price his business as his annual motor portfolio survival function will look very different from the individual policy one. Risk diversification, or the law of large numbers in this case, will ensure that the portfolio losses for the year will have a near-normal distribution with a very small coefficient of variation. In essence, the policyholder excess premium has gone to pay for the commissions and expenses of facilitating this risk diversification.

For the insurer, therefore, the RAF should reduce, for a given RAL, as the portfolio increases in size and the relative uncertainty in the annual loss amount reduces in relation to the expected. In other words, the insurers risk load reduces as the number of policies (actually the expected claims) in his portfolio increases. In practice, the variability of the insurer's annual losses is influenced much more by competition, pricing cycles and the occurrence of weather events rather than any stochastic variability from the underlying policy frequency and severity assumptions at policy level.

These other sources of variability, as well as market forces, will determine the implicit RAL at which the insurer actually prices his policies. These levels may well vary by product as well as vary by policy within product categories. The p-h methodology should enable the insurer to make informed decisions on these difficult practical choices.

## *5.2. Risk Aversion Level for a High Risk (Young) Motorist*

The poor young driver has traditionally provided the most popular description of a 'high risk policyholder'. Whilst inexperience, the urgency to go faster and playfulness do result in much higher claims frequencies and severities for these drivers compared to their mothers, fathers or sisters, the only obvious conclusion from these observations is that the expected annual losses from these policies will be higher and so higher premiums are warranted. Higher frequency of claim usually means reduced variability and so the amount of risk adjustment that is appropriate for these policies in relation to the lower frequency ones is by no means clear cut. The expected higher severity, particularly from bodily injury cases, does however, have the opposite effect from the increased frequency as it increases the variability.

It should be possible, with enough data, to repeat the calculations of the previous example for a young driver and a mature driver so as to compare the RAF at various RAL. Some care is required in such an exercise as the two sets of parameters will probably have different estimation errors as we tend to have a lot more older policyholders than younger ones and so are likely to have a lot more information on which to estimate and validate our model and parameter assumptions.

## 5.3. *Risk Aversion and Roulette*

In a simple game of (European) Roulette the gambler bets on one of two colours, with an expected probability of winning back twice his stake of 18/37. The mean outcome, for a £1 stake, is then 36/37, slightly below his stake.

The survival function, for the bank, is very simple in this instance as there are only two possible outcomes, zero, with probability 19/37 and 2 with probability 18/37.

The 'risk adjusted' cost at RAL $\rho$ is then simply the solution to the equation:

$$2 * (18/37)^{\wedge}(1/\rho) = 1$$

which gives $\rho = (\ln(37) - \ln(18)) / \ln(2) = 1.03953$

Betting on a single number changes the odds and the RAL. Here the gambler bets £1 and collects £36 with probability 1/37.

The 'risk adjusted' cost at RAL $\rho$ is then simply the solution to the equation:

$$36 * (1/37)^{\wedge}(1/\rho) = 1$$

which gives $\rho = \ln(37) / \ln(36) = 1.007646$

These examples illustrate how these calculations can be extended to more complex games, where a number of outcomes is possible. We will consider one such example below.

## 5.4. *Risk Aversion and the UK National Lottery*

To play the UK National Lottery ones selects six numbers from 1 to 49. On the specified date, a Saturday or a Wednesday evening, a machine picks seven balls at random from a set of 49. To win the top prize one has to pick all six of the first six balls selected. To win a second prize the player has to have any five of the first six balls selected plus the seventh ball. Third prize winners have to have any five of the first six balls, fourth prize winners any four of the first six balls and, finally, fifth prize winners need three of the first six balls selected by the machine.

The prize money is 45% of the money collected for the particular draw, with top winners sharing approximately a third of the prize money. The fifth prizewinners collect £10 each, which usually means that in aggregate they share a third of the prize money. The theoretical odds and relative prize money, which assumes a total sale of all possible combinations of some 13,983,816 unique selections without any multiples is given by the organisers as shown in the table below, which also shows the Survival Function for this distribution.

**Table 5.2. National Lottery Survival Function**

| No of Tickets | Winning Amount £ | Survival Prob |
|---|---|---|
| 13724690 | 0 | 1 |
| 245330 | 10 | 0.018530421 |
| 13537 | 62 | 0.000986569 |
| 252 | 1500 | 1.85214E-05 |
| 6 | 100000 | 5.00579E-07 |
| 1 | 2000000 | 7.15112E-08 |

The expected or mean win is £0.45 for each £1 selection. We can thus estimate the 'Risk Aversion Level' implicit in this game, by solving for it. In this case we find that the RAL is 1.0801. In actual games, as people tend to prefer certain numbers and combinations of numbers, the expected winnings tend to vary somewhat from the theoretical values. In three weeks of December 1997, the actual outcomes of the Saturday draws had implicit or hindsight RAL between 1.071 and 1.077.

When the top prize is not won the organisers add the related amount to the next weekly draw, sometimes guaranteeing the amount of the top prize. The draw on the 22nd December 1997 was a 'roll-over' draw with a jackpot of £25m. This was shared by two tickets. The effective RAL for this special draw, post event, was just over 1.04.

## *5.5. Risk Aversion and Premium Bonds*

Premium Savings Bonds are a UK Government security issued by the Treasury in units of £1. A draw is held every month and each £1 bond has (at the time of writing) a chance of 1 in 18,000 of winning a tax free prize from £50 to £1,000,000. Rules govern the allocation of prize money but the overall prize money is equivalent to a tax-free return of 5% per annum.

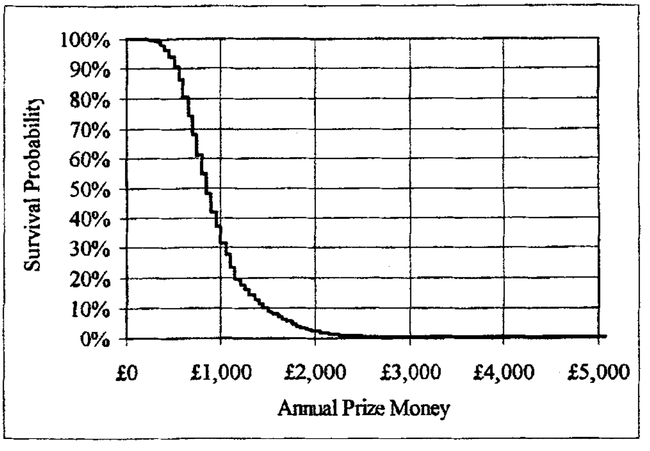
The maximum holding is currently £20,000 and it is estimated that the April 1998 draw will distribute over £40m in prize money. The following table shows the number of prizes expected to be distributed in the April 1998 draw. This indicates that the total number of premium bonds at issue at this time is around 9.7 billion.

**Table 5.3. Premium Bonds monthly prize distribution**

| Prize £ | Number |
|---|---|
| 50 | 396346 |
| 100 | 106028 |
| 500 | 7302 |
| 1000 | 2434 |
| 5000 | 122 |
| 10000 | 62 |
| 25000 | 25 |
| 50000 | 12 |
| 100000 | 6 |
| 1000000 | 1 |

The actual return that an investor will achieve in any one year will vary from nothing at all to £12m if he happens to win the top prize every month - a chance of 1 in 9.7billion to the power of 12. The following chart shows the survival probability for the return an investor with the maximum holding of £20,000 can expect, ignoring interest on any prize money won during the year for simplicity.

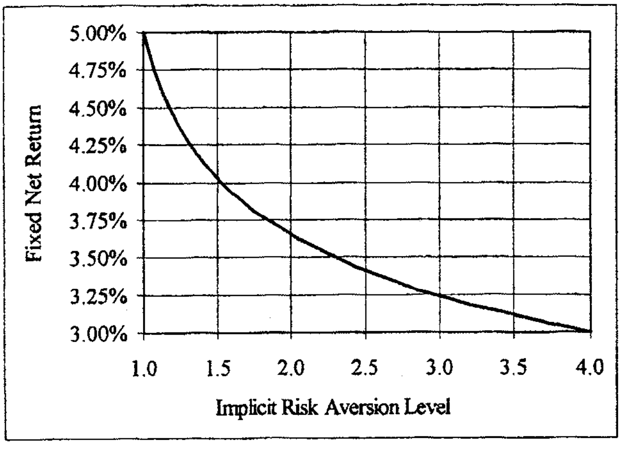
**Figure 5.2. Survival Function of annual prize money on 20,000 Bonds**




The expected return is 5% of £20,000, or £1,000 with a fairly skew distribution with an estimated standard deviation of around £1,600 and skewness of over 30. Clearly, the chart does not extend to the ultimate limit (£1,000,000).

As this is however, a risky investment return we are looking at the 'transformed' distribution of risk adjusted returns. Let us consider how we can use the approach to compare these returns with alternative fixed returns. For example, if our investor puts his £20,000 into an equally secure Government Option Bond he can obtain just over 4.0% tax free with a marginal tax rate of 40%. We can estimate the implicit RAL that investing in Premium Bonds corresponds to by reversing the transform until the mean value of the un-transformed distribution matches the fixed return of £800 or 4%. The RAL obtained by this process turns out to be just over 1.5. The following chart shows the implicit RAL's for various fixed net returns.

**Figure 5.3. Premium Bond Risk Aversion Net Returns**




The implicit RAL will depend on the level of net fixed returns available to the investor and these can and do vary from time to time. At the beginning of April 1998 an investor with a the top tax rate of 40% can obtain around 4.5% from a building society for an investment of £20,000 which reduces the implicit RAL to 1.17. Clearly for the investor, a higher implicit RAL the more desirable the investment in the risky security becomes. If the investor has an identified RAL then he can use the above chart to determine when it makes sense to switch his investments from fixed to variable returns depending on the relativities of the net returns available.

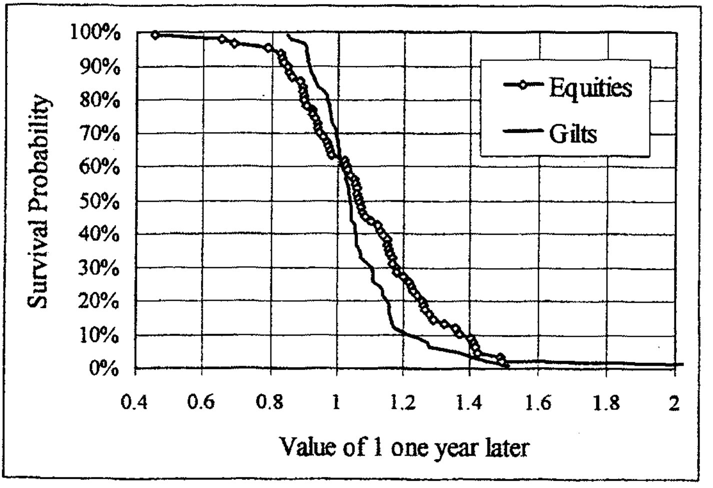
The p-h methodology provides a framework for comparing the risk-return characteristics of these 'investments'.

## 5.6. *Asset Returns, risk free returns and likely Market RAL*

Risk in asset returns is generally assumed to be proportional to the standard deviation of these returns. We will briefly consider the application of the p-h transform methodology to compare the annual total returns for UK Gilts and UK Equities over the period from 1919 to 1989 from the widely available BZW indices.

The following chart shows the survival function for these annual total returns over this period, as a ratio of the total value at the end to the value at the beginning of the year. The return is from both price changes and income. The mean annual returns during this period were just over 9% for Equities and 6.1% for Gilts, standard deviations were 25% and 13%, skewness 1.62 and 1.31 and kurtosis of 8.1 and 2.3 respectively.

**Figure 5.4. UK Total annual returns for equities and gilts 1919-1989**



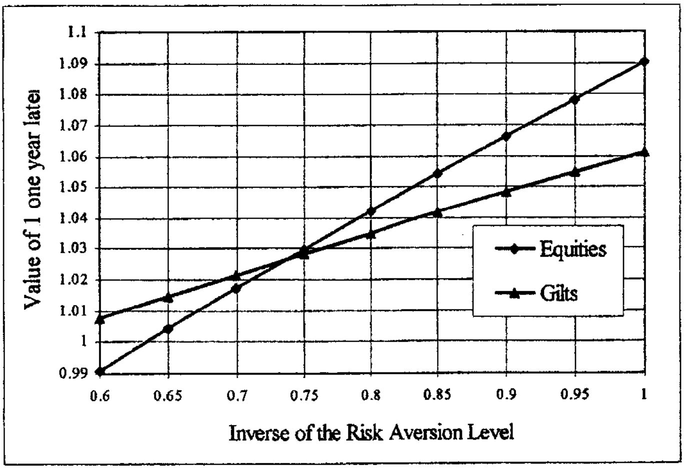

Let us recall that we are observing returns in the 'risk loaded' space and consider what happens if we attempt to take these risk adjusted returns back to the 'risk free space'. The following table shows the results of reversing the p-h transform.

**Table 5.4. Risk deflated Asset Returns 1919-1989**

| Inverse RAL | Equities | Gilts |
|---|---|---|
| 1 | 1.0901 | 1.0610 |
| 0.95 | 1.0783 | 1.0547 |
| 0.9 | 1.0665 | 1.0482 |
| 0.85 | 1.0544 | 1.0417 |
| 0.8 | 1.0423 | 1.0351 |
| 0.75 | 1.0299 | 1.0283 |
| 0.7 | 1.0172 | 1.0215 |
| 0.65 | 1.0042 | 1.0145 |
| 0.6 | 0.9907 | 1.0075 |

The following charts plots the values in the table above.

**Figure 5.5. Risk deflated mean returns of UK Equities and Gilts 1919-1989**




We can see from the chart that the two curves cross at a point which in this example happens to be around ( 0.74, 1.027).

At the cross over point the mean returns are equal and so we can interpret this to represent the 'original' or 'risk free' space where variability does not warrant any additional returns. We see that using this data we have estimated this 'period implicit risk free annual return' as 2.7%. Gilts attract an additional risk return of 3.3.% and Equities 6.1%.

But why stop here. We can see that during this period of time the 'market' was pricing for risk in these returns at an implicit RAL of 1.35 (1/0.74).

Unfortunately, time has not allowed any further testing of this approach on asset returns and clearly this is an area with significant potential applications. In the first instance more asset classes need to be tested to see how stable these estimated 'risk free returns' and market RAL are when returns from more than two classes are compared. Also the behaviour of returns over shorter periods of time periods warrant study. These comparisons may, then, have implications about the relative attractiveness of available returns and portfolio optimisations.

## *5.7. Option pricing and the Black-Scholes model*

The Black-Scholes option pricing formula is one of the most important formulae in financial economics. Although its derivation can appear daunting, and the result particularly complex, most of this complexity is due to the lack of a closed form for the cumulative function of the normal distribution.

A call option is the right to buy a given quantity of a particular stock at a specified price, say X, at a given future date, time t from the time of purchase. The holder will only exercise his option to buy if the stock price at expiry, P(t) say, is greater than the exercise price X, in which case the option is worth the difference between the price at expiry, that is P(t) - X. If $X \geq P(t)$ the option is of no (zero) value.

If we denote the value of the option at expiry, that is at time t, by C(t) we can express this value as follows:

$$C(t) = \max(0, P(t) - X)$$

Now let us assume that the risk free rate is r, and that the volatility of the particular stock, as measured by the standard deviation of logarithmic stock returns per unit time, is $\sigma$ and that no dividends are payable prior to the maturity of the option.

In simple terms, it is assumed that share prices follow a relationship of the form:

$P(t) = P(0) * \exp(m*t + \sigma \sqrt{t} * N(0,1))$ - where m is the mean log-return.

The Black-Scholes call option price C(t) is then given by - (see England 1994):

$$C(t) = P(0) * \Phi(h) - * \Phi(h - \sigma \sqrt{t})$$

where $\Phi(h)$ is the cumulative function of the Normal distribution and h is given by:

$$h = \{ \ln ( P(0) / X e^{-rt} ) / \sigma \sqrt{t} \} + \tfrac{1}{2} \sigma \sqrt{t}$$

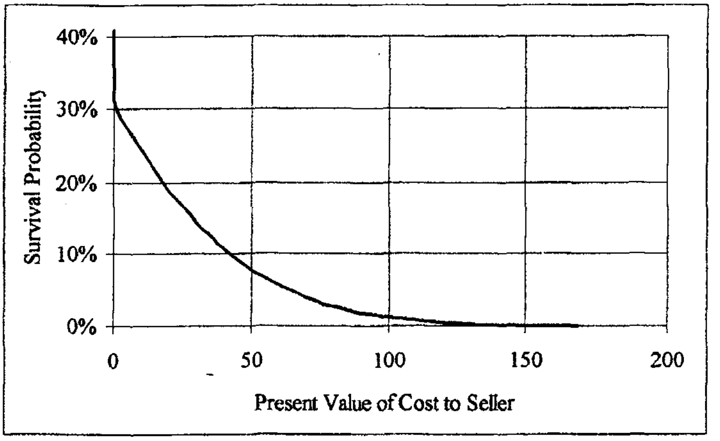
It is interesting to note that the option price seems independent of this mean log return. This is, perhaps, not quite so as market efficiency arguments seem to boil down to a relationship between the risk free rate, the mean return and the volatility. If we assume, for example, that market efficiency means that there is no benefit in delaying a purchase or sale of the 'risky' share and investing instead in risk free government bonds which return a rate of, say r, we see that the mean discounted present value, at the risk free rate, of the future price at time t has to be equal to the current price. In other words $r = m + \sigma^2/2$.

This also enables us to validate the B-S formula as an expected value formula without any loading for risk. In other words it is a price at a RAL $\rho = 1$. This can be easily demonstrated by numerical integration. This may seem surprising to those that believe that the formula 'prices for risk' since it includes the stock volatility term, $\sigma$, as one of its parameters. This volatility simply impacts the mean expected value of the stock price at time t, P(t) and will clearly impact the cost of the option. This cost is still, however, an expected value without any risk loading for the uncertainty inherent in the final outcome.

We can calculate the survival function of the present value of the cost to the seller of these options very easily once we have a model of the stock price behaviour. For example, let us assume that the stock price behaviour is as outlined above and that we have an initial price P(0)=500, the strike price X=550, time t = 1, volatility $\sigma$ is 10% and the risk free rate, r, 5%. The B-S formula gives a price of 10.87, using values for the Normal distribution given in a spreadsheet.

Using the relationship between the risk free rate, the stock mean log return and volatility, as given by: $r = m + \sigma^2/2$, we can easily calculate the survival function of these costs, which is shown in the following chart.

**Figure 5.6. Survival Function of present value of cost of the Option**




The actual area under this chart, estimated numerically using 2000 points, is 10.82 which is sufficiently close to the B-S formula value given the accuracy limitations in both calculations. The reader is encouraged to validate this result.

We can now investigate the impact of loading these B-S prices for the risk inherent in the variability of the cost to the seller (present value at the risk free rate) using the p-h transform basis. This is analogous to pricing an insurance policy that has a 30% probability of producing a claim (reading from the chart) with the amount of the annual loss described by the above survival function.

This enables us to calculate risk loaded option prices at any desired RAL $\rho$, which reduce to the B-S values when $\rho = 1$. We can go a stage further and tabulate these risk loaded option prices against both the RAL $\rho$ as well as another of the variables in the model, as shown in the tables below. This is yet another area where more research is warranted but time has not allow in this instance.

**Table 5.5. Risk Loaded Call Option Prices ( 500, 550, $\sigma$ = 10%, r = 5%)**

| RAL | Time to exercise of the Call Option in years | | | | | |
|---|---|---|---|---|---|---|
| | 0.25 | 0.5 | 0.75 | 1 | 1.5 | 2 |
| 1 = B-S | 0.51 | 3.06 | 6.69 | 10.82 | 19.81 | 29.23 |
| 1.05 | 0.62 | 3.48 | 7.46 | 11.92 | 21.50 | 31.46 |
| 1.1 | 0.74 | 3.93 | 8.24 | 13.03 | 23.20 | 33.68 |
| 1.15 | 0.87 | 4.40 | 9.05 | 14.16 | 24.90 | 35.89 |
| 1.2 | 1.00 | 4.88 | 9.87 | 15.29 | 26.60 | 38.09 |
| 1.25 | 1.15 | 5.37 | 10.70 | 16.44 | 28.30 | 40.28 |
| 1.3 | 1.30 | 5.88 | 11.55 | 17.58 | 30.00 | 42.46 |
| 1.35 | 1.47 | 6.39 | 12.39 | 18.74 | 31.69 | 44.62 |
| 1.4 | 1.64 | 6.92 | 13.25 | 19.89 | 33.37 | 46.77 |
| 1.5 | 2.00 | 8.00 | 14.98 | 22.20 | 36.71 | 51.01 |

**Table 5.6. Risk Loaded Call Option Prices ( 500, 550, t=1, r = 5%)**

| RAL | Volatility of underlying stock price | | | | | |
|---|---|---|---|---|---|---|
| | 2.5% | 5.0% | 7.5% | 10.0% | 12.5% | 15.0% |
| 1 | 0.17 | 2.52 | 6.39 | 10.82 | 15.50 | 20.30 |
| 1.05 | 0.21 | 2.85 | 7.10 | 11.92 | 17.00 | 22.22 |
| 1.1 | 0.25 | 3.20 | 7.81 | 13.03 | 18.52 | 24.16 |
| 1.15 | 0.30 | 3.55 | 8.54 | 14.16 | 20.05 | 26.11 |
| 1.2 | 0.35 | 3.92 | 9.29 | 15.29 | 21.60 | 28.08 |
| 1.25 | 0.40 | 4.29 | 10.04 | 16.44 | 23.15 | 30.06 |
| 1.3 | 0.46 | 4.67 | 10.79 | 17.58 | 24.71 | 32.04 |
| 1.35 | 0.52 | 5.06 | 11.55 | 18.74 | 26.27 | 34.02 |
| 1.4 | 0.59 | 5.45 | 12.32 | 19.89 | 27.83 | 36.00 |
| 1.5 | 0.73 | 6.25 | 13.85 | 22.20 | 30.94 | 39.95 |

More of these tables could be produced reasonably easily for further study of the implications of the various input parameters on the 'risk loading' of options. Once again this is somewhat beyond the scope of this paper.

# 6. POTENTIAL FURTHER APPLICATIONS OF THE P-H RAL

## 6.1. *Risk Discount Rates (RDR), WACC, RAROC .....*

When we consider the pricing of an insurance policy the main issue is the identification of the likely cash flows that will be incurred, their timing and the level of uncertainty around both the amounts of these payments and their timing.

When considered in such a framework, the underlying reasons for the cash flows are not particularly relevant. For example, we may be considering building a supermarket and need to test its financial viability given the likely costs of the development and the projected future profit stream and its uncertainty.

In the commercial world, these issues may be seen as unrelated or even opposite, primarily due to misunderstandings on the amounts of capital which may or may not be at risk (of being lost) if things go very wrong. In either case, some return above risk free rates is demanded for any capital (money) that is at risk of being lost. In reality, terms such as RDR (Risk Discount Rates), 'hurdle rates', RAROC (Risk Adjusted Returns On Capital), ROCAR (Return On Capital At Risk), WACC or Weighted Average Cost of Capital, and so on, are attempts to price the underlying cash flows for the level of risk involved. On the whole, these values are selected subjectively rather than objectively.

We will consider a number of examples below to demonstrate how the p-h transform basis provides a framework for developing objective values for these benchmarks.

## 6.2. *Claims reserves, prudential margins and risk discount rates*

Consider that we have estimated the future payments that will be made in respect of a block of insurance claims to be $1000 spread over a number of years according to an estimated payment pattern. Let us further assume that our estimation process has also enabled us to estimate the statistical distributions of these future payments, which is taken to be log-normal with payments in each year being independent of payments in any other year. For details on how one may actually arrive at such estimates the reader can start by reviewing some of the papers in Volume 2 of the Institute of Actuaries Claims Reserving Manual (1997) and Lowe (1994).

The following table shows how this very simple model may be set-up in a spreadsheet.

**Table 6.1. Claims payments simulation model**

| | Year 1 | Year 2 | Year 3 | Year 4 | Year 5 | Total |
|---|---|---|---|---|---|---|
| Mean Proj Paid | 400 | 300 | 150 | 100 | 50 | 1000 |
| SD as % of Mean | 10% | 15% | 20% | 30% | 50% | |
| Sim Value-LogNor | 418.2 | 316.8 | 190.0 | 76.5 | 55.3 | 1056.8 |
| Risk Free | 5% | | | | | |
| Disc Factor | 0.976 | 0.929 | 0.885 | 0.843 | 0.803 | |
| Present Value | 408.1 | 294.4 | 168.2 | 64.5 | 44.4 | 979.6 |
| | | | | | | |
| Risk Disc Rate | 2.49% | | | | | |
| Risk Disc Factor | 0.988 | 0.964 | 0.940 | 0.917 | 0.895 | Target |
| Risk Disc Value | 395.1 | 289.1 | 141.0 | 91.7 | 44.8 | 961.8 |

The simulated values are assumed to come from a lognormal distribution with the mean and standard deviation as shown for each of the year's expected payments. The top table shows a single simulation which happens to project total payments of $1056.8 with a present value of $979.6 at the assumed risk free rate of 5%. These calculations assume that payments take place at the mid-year.

The second part of the table shows the discounting of the expected payments at a different discount rate which has been allowed to vary so as to produce a target value for the total as shown in the table below.

Let us now look at the results of the simulation for the two key values in question, that is the undiscounted and discounted (at the risk free rate) future payments. These results are based on a minimum 2000 simulations. The number of simulations needed in practice to obtain stable results depends on the variability in the underlying distributions.

**Table 6.2. Simulation results summary**

| | Undiscounted | Discounted |
|---|---|---|
| Mean | 1000.0 | 926.4 |
| Std Deviation | 80.5 | 73.3 |
| Coeff Variation | 8.05% | 7.91% |
| Skewness | 0.287 | 0.271 |
| Kurtosis | 3.12 | 3.11 |

The results show that the simulated values do exhibit some skewness although the kurtosis is not that far away from that of the Normal distribution. Further investigation identifies the log-normal distribution as providing a reasonable fit to the data. Taking the discounted values, the log-normal distribution with the same mean and standard deviation as the distribution of simulated values has a skewness of 0.24 (cf 0.27) and kurtosis of 3.10 (cf 3.11). This is, perhaps, not too surprising given the model assumptions.

The following table shows the risk adjusted values of both the undiscounted and discounted future cash flows. It also shows the implicit Risk Discount Rates at each Risk Aversion Level derived by targeting the risk adjusted discounted values as indicated in the table above.

**Table 6.3. Risk adjusted discounted values and implicit RDR**

| RAL | Undiscounted Value | Discounted Value | Implicit RDR |
|---|---|---|---|
| 1 | 1000.0 | 926.4 | 5.00% |
| 1.2 | 1014.2 | 939.3 | 4.06% |
| 1.4 | 1027.1 | 951.0 | 3.23% |
| 1.6 | 1038.9 | 961.8 | 2.49% |
| 1.8 | 1049.9 | 971.8 | 1.82% |
| 2 | 1060.2 | 981.1 | 1.21% |
| 2.25 | 1072.1 | 992.0 | 0.50% |
| 2.5 | 1083.1 | 1002.1 | -0.13% |
| 3 | 1102.9 | 1020.3 | -1.24% |

The RDR in this example are lower than the underlying, assumed, risk free rate as the cash flows are payments and not receipts. Where the cash flows to be valued are future receipts then these RDR will exceed the underlying risk free rate.

In essence this 'valuation' of the future payments prices the reserve run-off. For example, an investor who demands returns consistent with a RAL of, say 2, would value this run-off at $981.1 assuming he concurs with all the other assumptions concerning the inherent variability in these cash flows, including the appropriateness of the assumed risk free rate.

We can also see that such a basis can be used by the managers of the business for setting discounted claims provisions with explicit and consistent 'Prudential Margins' as each percentile level of the final distribution corresponds to a RAL.

In practice, perhaps the easiest way to obtain the distribution of the present value of the run-off is by bootstrapping the payments using a chain ladder or similar underlying model. For more details on how this may be done, and how the results compare with a number of other stochastic reserving methods see Lowe (1994).

## *6.3. Appraised and Shareholder Values*

**Appraised Values and RDR**

The Appraised Value of a company is generally defined as the present value of the future cash flows 'valued at an appropriate (risk) discount rate'. In essence then as the uncertainty in these future cash flows (profits) increases the appraised value decreases. The selection of what an 'appropriate RDR' ought to be then soon degenerates into a fairly subjective argument about similar transactions, market conditions, benchmark rates appropriate for the industry and a number of other similar but generally subjective arguments.

The P-H methodology is clearly able to deliver such RDR's given that we can model the underlying uncertainty and have a means of 'selecting' the appropriate RAL at which to value these cash flows. The previous example demonstrates how this may be done in practice and there is no need to repeat any of these explanations. This approach also highlights the need to understand the inherent variability in the future cash flows in order to value these in a consistent pricing framework. Stochastic modelling is here to stay and together with such pricing principles will revolutionise these valuations. Questions on variability of other inputs, such as future risk free rates, simply become another challenge for the modeller in constructing, validating and calibrating the model to be used in the valuations. At least, the existence of such a model will facilitate sensitivity testing of the results against the input assumptions and thus provide additional and potentially very useful information to management.

Questions on the choice of appropriate RAL to use in a particular instance are equivalent to the 'strength' of the valuation. Clearly, a low value of RAL may be selected to defend an unwelcome bid as it will produce a higher value. Note, however, that more aggressive profit projections or more aggressive strategies may also imvolve increases in the 'uncertainty' in the projected profit streams. The p-h framework will correct for this whilst a more subjective basis is unlikely to do so.

**Shareholder Value Analysis**

A related concept to Appraised Value and one that is gaining momentum at present, is that of Shareholder Value Added or SVA. In its simplest form this is the 'value of the cash flows generated by the business conducted during the period' where these cash flows are valued at some appropriate RDR.

This concept is particularly useful to management as it enables them to identify the areas of their business which are producing returns that exceed those expected for the amount of risk or variability inherent in these returns, or 'creating SV' and those that may be destroying it or producing returns inferior to those required to compensate for their inherent variability or level of risk.

The main problem with these calculations is the choice of this illusive RDR appropriate for the calculations at such a level, which is exactly the same problem described above. The solution is then simply the same as that outlined above. Clearly, for maximum value to management, the modelling will identify the key components or sources of variability and these can then be targeted for monitoring purposed. Companies that recognise the potential benefits from such an approach and are then able to implement it successfully should be better placed to achieve their objectives, create and perhaps also sustain competitive advantage.

The following table is based on the results of the claims run-off example above but treating the projected payments as risk adjusted returns, that is the transformed distribution. The RAL are then inverted to calculate the means of the underlying distributions for 'pricing' purposes and for identifying the implicit risk discount rates.

**Table 6.4. Risk 'deflated' discounted values and implicit RDR**

| RAL | INV RAL | Undisc Value | Disc Value | Implicit RDR |
|---|---|---|---|---|
| 1 | 1.000 | 1000.0 | 926.4 | 5.00% |
| 1.2 | 0.833 | 986.9 | 914.5 | 5.89% |
| 1.4 | 0.714 | 976.5 | 905.1 | 6.62% |
| 1.6 | 0.625 | 968.0 | 897.3 | 7.23% |
| 1.8 | 0.556 | 960.9 | 890.8 | 7.75% |
| 2 | 0.500 | 954.8 | 885.2 | 8.20% |
| 2.25 | 0.444 | 948.2 | 879.2 | 8.70% |
| 2.5 | 0.400 | 942.5 | 874.1 | 9.13% |
| 3 | 0.333 | 933.2 | 865.6 | 9.85% |

This approach can thus be used to identify the appropriate RDR to apply in these evaluations. Further analysis would, however, be beyond the objectives of this paper.

The explicit derivation of these 'appropriate' risk discount rates should be contrasted with the alternative approach often adopted in practice of using a so-called WACC or Weighted Average Cost of Capital. This may incorporate some beta adjustment on the cost of equity capital arising from overestimation of returns, given the beta, by the Capital Asset Pricing Model. This known limitation of the CAPM does not seem that surprising in a P-H Risk loading context, given the non-linearity between the P-H risk adjusted values and the standard deviation as a measure of risk. We have also identified instances where this framework (CAPM) may work, that is with distributions with zero or, perhaps, fixed skewness. This, potentially, has implications for a significant amount of these theories and their use in determining return objectives and portfolio optimisations but is, once again, beyond the scope of this paper.

## *6.4. Allocation of Capital to Classes of Business*

Efficient Capital Allocation is a very popular Conference Title, whatever efficiency may mean. The reasons for this are fairly easy to see. In the absence of a usable definition and quantification of 'risk' the ability to allocate one's capital to the classes or products of the business provides an easy benchmark against which to measure the relative performance of these products or classes in a 'risk-return' exercise.

The reality is, however, that such allocations of capital to classes are generally based on some fairly basic and haphazard assumptions, such as x% for product A, y% for product B and so on, without any foundation or validation that they are in any way appropriate. Under such circumstances, the use of these values for determining pricing, comparing performance, setting targets or identifying value creation is very questionable and fraught with potential and real dangers.

This obsession with capital allocation often raises questions on the proportion of the available capital that is exposed to 'additional risk' and so warrants additional returns. At the most basic level, we could argue that the amount of capital required to cover 'risk' or uncertainty in any projected cash flow has to, at least, be sufficient to cover all eventualities. This would be very inefficient and as a consequence some lower level is considered appropriate, often determined by market practice and expectations. In certain instances, particularly for banks and US insurers, Risk Based Capital requirements have been devised and implemented by the regulators. These tend to be a mixture of art, science and some final scaling to produce answers that look believable or close to those the market generally can find acceptable. Further discussion of these aspects is beyond the scope of this paper. We will return to capital allocation at the end of the next section.

## *6.5. Required Returns for a Class of Business*

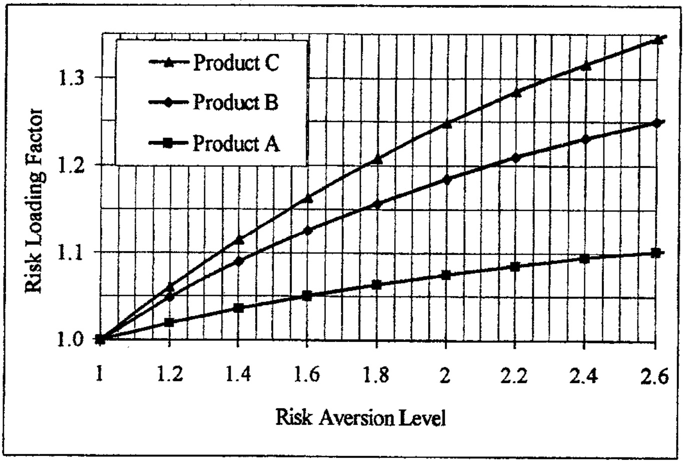
Since investors expect and demand returns on their capital in excess of risk-free returns and commensurable with the levels of perceived risk associated with the business this capital is invested in, the companies in which they invest have to generate sufficient profits to satisfy these demands. So the real issue with regard to product pricing or comparative performance measurement or the creation of value, is not capital allocation but profit allocation. We can now make some progress.

The p-h methodology provides us with a consistent basis for 'allocating' the overall company profit objective to profit centres. We will outline the process and demonstrate this with a simple example. The process starts with identifying the required overall profit objective of the company. This may be something the Directors determine in order to satisfy stakeholder expectations. Irrespective of the profit objective, there will be an implicit RAL associated with it at the overall level. We could, but do need to identify this level.

We will instead look at a simplified example for a company that has three classes of business, say Product A, Product B and Product C.

Let us suppose that we have investigated the profit or results variability of these classes, say as we discussed in Section 4.4, and identified the appropriate risk loading factors on expected costs for a range of RAL. The results of such an exercise are shown in the following chart. Here, it is assumed that the expected values are discounted for the value of money at the risk free rate.

**Figure 6.1. Risk loading factors for three Products A, B and C**




Now let us suppose that the business, by volume or premium, consists of 50% Product A, 30% Product B and 20% for Product C.

Let us also suppose that the required profit target, is 9.5% of the Premium. We can now identify the RAL at which the combined profit adds to this value. This is close to a RAL of 1.6. We can see from the chart that at this RAL the Product A business has to be priced to produce a 5% return (profit target), Product B a 12.5% return and Product C, which is clearly the high risk business in this portfolio, a high 17% return. For the whole portfolio the overall return is then 9.65% indicating that the implicit RAL required is slightly lower than 1.6. The overall profit objective was chosen to ease the explanation and should not be taken as indicative of appropriate values to be used in practice.

## Capital Allocation Revisited

We can now return to the capital allocation conundrum and provide a plausible, but totally redundant basis. Let us suppose that, in exactly the same way that a given overall profit objective is given as a starting point, we now have an amount of capital to service which then needs to be allocated to classes so as to receive the return commensurate with the 'risk' inherent in this class return.

Suppose that this capital was 40% of the premium and a return of some 24% from insurance was required, chosen to match the earlier objective for illustration purposes. We know from above that for this return of 9.6% overall, the Product A business needs to deliver 2.5% (5% on 50%), the Product B business the next 3.75% and Product C the final 3.4%. The Capital can then be allocated in proportion to these values so that the Product A gets 25.9% (2.5/9.65) of the 40%, or 10.3% of the overall which equates to 20.7% of its premium, Product B finishes with a 'capital allocation' of 51.1% of its premium and, finally, Product C attracts capital of 70.5% of its premium.

This is however pretty useless, or redundant, information as we had to have a solution to the required risk loadings to derive these answers. We also saw how it is much more sensible to concentrate on profit objectives in relation to premiums from which these profits are to be generated rather than try and set these by allocating some notional capital on which a fixed percentage return is required.

**Shareholder Value Creation**

The RAL route to 'profit allocation' also provides the means to another tricky and topical issue, that of identifying and measuring shareholder value creation. Without getting into any details of definitions and explanations it suffices to note that relative to our overall profit objective and its RAL allocation to products, a product that returns in excess of its allocated profit is creating value equal to this excess amount.

## *6.6. Optimisation of Portfolios*

Following on from the target setting we could turn and consider the available returns for each class given the derived target returns. This simple analysis will identify the classes that are under-performing and those that are over-performing in relation to our desired objectives/expectations.

Note, however, that changing the mix of business will change the experience of the overall new portfolio and its required profit expectation at the required risk load.

We could attempt to evaluate the impact of a marginal change in the amount of business from a class in both the overall portfolio and hence attempt to establish if there is some optimum size for the particular class of business. Note here also that as we change the size of business we may be changing its variability in relation to premium and so may require a higher load at the same risk aversion factor.

Although this is a complex process, it should be possible to develop a computer model to progess this sort of investigation, given that we start with a good appreciation of the underlying processes and can describe these and their correlations sufficiently in an appropriate manner.

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