

PRICING THE RISK OF A GENERAL INSURANCE
PORTFOLIO USING SERIES EXPANSIONS FOR
THE FINITE TIME MULTIVARIATE RUIN
PROBABILITY IN A FINANCIAL-ACTUARIAL RISK
PROCESS

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Pricing the risk of a General Insurance portfolio using series expansions for the finite time multivariate ruin probability in a financial-actuarial risk process

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ABSTRACT The risk involved in a General Insurance portfolio can be priced using the related concepts *Multivariate Ruin*

Probability-Annual Premium. In the present work, McLaurin expansion, with respect the arrival intensity of claims, for the finite time Multivariate Ruin Probability — considering the surplus just before and deficit at ruin time — is obtained in the context of a financial-actuarial model for the risk process, $\Psi_{t_0,u,x,y}^i(\lambda_0)$ and used to price a General Insurance portfolio in terms of its annual premium.

Finally, the concept of *Financial Autonomy Ratio*, $R_{t,u,x,y}^i(\lambda)$, is introduced as a measure of the strength of the investments' yields to prevent ruin.

1. INTRODUCTION: A PURE ACTUARIAL MODEL

Let us define an actuarial risk process in continuous time $\{Z_t\}_{t \geq 0}$ with U_k claim sizes and premium c per time unit,

$$Z_t = u + ct - \sum_{k=1}^{N_t} U_k \quad (1.1)$$

where u are the initial reserves and N_t the total number of claims up to time t (with a c.d.f. of the waiting times between claims $w(t)$) where λ_0 is the average number of claims in one year (or time units considered). Let B denote the distribution function of claim sizes U_k with mean μ^{-1} and $c = \lambda_0 \mu^{-1}(1 + \theta)$, where θ is the premium loading factor.

We can define $\tau = \inf \{w > 0 : Z_w < 0\}$ as the ruin time and $Y = -Z_\tau$ as the deficit at ruin time or severity of ruin and $X = Z_{\tau-}$ as the surplus just before the ruin.

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Let us now introduce the concept of *t-years deferred ultimate ruin probability* with initial reserves u , severity of ruin less than y and surplus before ruin less than x ,

$$P\{\tau > t_0, X \leq x, Y \leq y\} = {}_{t_0|}\Psi_{u,x,y}(\lambda_0) \quad (1.2)$$

$$x > 0 \quad y > 0$$

as the probability of the event consisting in that the stochastic process that models the reserves shall cross the ruin barrier, $Z_u < 0$ for the first time necessarily after t_0 years.

The probability of ruin with time span t and initial reserves u , severity of ruin less than y and surplus before ruin less than x can be expressed,

$$P\{\tau < t_0, X \leq x, Y \leq y\} = \Psi_{t_0,u,x,y}(\lambda_0)$$

and the ultimate ruin probability,

$$P\{\tau < \infty, X \leq x, Y \leq y\} = \Psi_{u,x,y}$$

It is obvious that,

$$\Psi_{u,x,y}(\lambda_0) = \Psi_{t_0,u,x,y}(\lambda_0) + {}_{t_0|}\Psi_{u,x,y}(\lambda_0) \quad (1.3)$$

Approximations to multivariate characteristics of Classical Ruin processes have been obtained in actuarial literature using different methodologies such as inversion of Laplace transforms for particular claim size distributions – Gerber, Goovaerts and Kaas(1987) and Dufresne and Gerber(1988) a), b)– or discretization of the claim size and time – Dickson(1989) and Dickson and Waters(1991,92) and Dickson(1993)– applying the so-called Panjer's recursive algorithm (Panjer(1981)). Lately, Frey and Schmidt (1996) also introduced the series expansions approach with respect the arrival intensity of claims, λ_0 , in the pure actuarial risk process.

In section 2, Theorem 1, using the pure actuarial risk process (1.1), we will obtain a series expansion for the t -years deferred ultimate multivariate ruin probability based on a recursive scheme. The former theorem can be used for any waiting times distribution between claims(d.f. $w(t)$).

The Classical case of risk theory will be studied, as a particular case within the framework stated before, in sections 3, 4 and 5. McLaurin expansion of the t -years deferred ultimate multivariate ruin probability with respect the arrival intensity of claims, λ , was obtained in section 4, Theorem 3, ${}_t|\Psi_{u,x,y}(\lambda)$. As a corollary of Theorem 3, in section 5, McLaurin expansion of the finite time multivariate ruin probability is also obtained, $\Psi_{t,u,x,y}(\lambda)$.

Later, we will introduce the effect of the rate of interest on the pure actuarial model and, therefore, consider a financial-actuarial risk process (6.1)

introduced in section 6. McLaurin expansion is also obtained for the finite time multivariate ruin probability, $\Psi_{t,u,x,y}^i(\lambda)$, in Theorem 4 (6.4), using again recursive formulas. We will also see that the former multivariate ruin probability is the sum of the *posterior* ($\Psi_{t,u,x,y}^{pi}(\lambda)$) and *simultaneous* ($\Psi_{t,u,x,y}^{si}(\lambda)$) financial-actuarial ruin probability, concepts introduced in the same section. In Theorem 4, the McLaurin expansion of $\Psi_{t,u,x,y}^{si}(\lambda)$ is also presented.

Finally, in section 7, the former result is used to price the risk of a General Insurance portfolio in terms of its annual premium ($P = \lambda_0 \mu^{-1}(1 + \theta)$) and we introduced the concept of *Financial Autonomy Ratio*, $R_{t,u,x,y}^i(\lambda)$ as a measure of the financial independence of a portfolio from its actuarial foundations; or in other words, a measure of the strength of the investments' yields to prevent ruin. A numerical illustration is presented in section 8.

2. SERIES EXPANSION FOR THE T-YEARS DEFERRED MULTIVARIATE ULTIMATE RUIN PROBABILITY

Let us now suppose, without any loss of generality, that the premium factor is 1, $c = 1$. This last fact means that we can consider the equivalent ruin probabilities

$$\begin{aligned}\Psi_{t,u,x,y}(\lambda) &\equiv \Psi_{t_0,u,x,y}(\lambda_0) \\ {}_t\Psi_{u,x,y}(\lambda) &\equiv {}_{t_0}\Psi_{u,x,y}(\lambda_0) \\ \Psi_{u,x,y}(\lambda) &\equiv \Psi_{u,x,y}(\lambda_0)\end{aligned}\quad (2.1)$$

defining

$$\begin{aligned}\lambda &= \frac{1}{(1 + \theta)} \\ t &= \lambda_0 t_0 (1 + \theta)\end{aligned}\quad (2.2)$$

Let us bear in mind that the security loading θ determines the value of the new parameters for the average number of claims in one year (λ) and time span (t). We should mention that λ can be considered as the arrival intensity of claims in the time re-scaled process.

Formulas 2.1 can be easily proved using a time re-scaling argument similar as, for instance, in Dickson and Waters (1991). For the reason stated above we will focus our attention in approximating $\Psi_{t,u,x,y}(\lambda)$ and ${}_t\Psi_{u,x,y}(\lambda)$.

We will prove in the following theorem that the t -years deferred multivariate ultimate ruin probability can be approximated, using total probability theorem, by a series expansion whose terms are defined recursively.

Theorem 1. *The t -years deferred multivariate ultimate ruin probability can be expressed using the following convergent series*

$${}_t\Psi_{u,x,y}(\lambda) = \sum_{i=0}^{\infty} {}_tA_{u,x,y}^i(\lambda) \quad (2.3)$$

where ${}_tA_{u,x,y}^i(\lambda)$ is the t -years deferred ruin probability with exactly i claims in the interval $(0, t]$

$$\begin{aligned} {}_tA_{u,x,y}^i(\lambda) &= P\{\tau > t, X \leq x, Y \leq y, N_t = i\} \\ i &= 0, 1, \dots \end{aligned}$$

and

$$\begin{aligned} {}_tA_{u,x,y}^0(\lambda) &= \Psi_{u+t,x,y}(\lambda) (1 - W(t)) \\ {}_tA_{u,x,y}^i(\lambda) &= \int_0^t \int_0^{u+s} {}_{t-s}A_{u+s-z,x,y}^{i-1}(\lambda) b(z) w(s) dz ds \\ i &= 1, 2, \dots \end{aligned}$$

Proof.

The first member of the family (no claims in $(0, t]$, $i=0$) could be written as the probability of the joint event formed by two independent events:

- no claims in that interval : $(1 - W(t))$
- Ultimate ruin with initial reserves $u+t$, starting at time point t ,

$${}_tA_{u,x,y}^0(\lambda) = \Psi_{u+t,x,y}(\lambda) (1 - W(t))$$

The second member is obtained from the first one integrating over the time and the claim size

$$\begin{aligned} {}_tA_{u,x,y}^1(\lambda) &= \int_0^t \int_0^{u+s} {}_{t-s}A_{u+s-z,x,y}^0(\lambda) b(z) w(s) dz ds \\ &= \int_0^t \int_0^{u+s} \Psi_{u+t-z,x,y}(\lambda) b(z) w(s) (1 - W(t-s)) dz ds \end{aligned} \quad (2.4)$$

and proceeding recursively using the same renewal argument

$${}_tA_{u,x,y}^i(\lambda) = \int_0^t \int_0^{u+s} {}_{t-s}A_{u+s-z,x,y}^{i-1}(\lambda) b(z) w(s) dz ds \quad i = 1, 2, \dots$$

As a consequence, using total probability theorem over the number of claims N_t leads us to

$${}_t\Psi_{u,x,y}(\lambda) = \sum_{i=0}^{\infty} {}_tA_{u,x,y}^i(\lambda)$$

Bearing in mind that the terms of the former series are probabilities, ${}_tA_{u,x,y}^i \in [0, 1]$, and

$$\lim_{t \rightarrow \infty} {}_tA_{u,x,y}^i(\lambda) = 0$$

as it is obvious because the larger the number of claims in $(0, t)$ the more likely the ruin to happen within that interval; the series is convergent. ■

3. THE CLASSICAL CASE

We will now consider the Classical case of Risk Theory, exponential waiting time between claims, $w(t) = \lambda e^{-\lambda t}$, as a particular case in our approach. The following Theorem states an interesting series expansion based on recursive relations for the so-called Classical case.

Theorem 2. *In the context of the Classical case of Risk Theory, t -years deferred multivariate ruin probability can be expressed*

$$\begin{aligned} {}_t\Psi_{u,x,y}(\lambda) &= \sum_{k=0}^{\infty} {}_tA_{u,x,y}^k(\lambda) \\ &= e^{-\lambda t} \sum_{k=0}^{\infty} \lambda^k C_{t,u,x,y}^k(\lambda) \end{aligned} \quad (3.1)$$

where, introducing the operator $\mathcal{I}_1$

$$\begin{aligned} C_{t,u,x,y}^0(\lambda) &= \Psi_{u+t,x,y}(\lambda) \\ C_{t,u,x,y}^j(\lambda) &= \int_0^t \int_0^{u+s} C_{t-s,u+s-z,x,y}^{j-1}(\lambda) b(z) dz ds \\ &= \mathcal{I}_1(C_{t,u,x,y}^{j-1}(\lambda)) \\ j &= 1, 2, \dots \end{aligned} \quad (3.2)$$

Proof. Let us proceed by induction

$$\begin{aligned} {}_tA_{u,x,y}^0(\lambda) &= \Psi_{u+t,x,y}(\lambda) (1 - W(t)) = e^{-\lambda t} \Psi_{u+t,x,y}(\lambda) \\ &= e^{-\lambda t} C_{t,u,x,y}^0(\lambda) \\ {}_tA_{u,x,y}^1(\lambda) &= \int_0^t \int_0^{u+s} {}_tA_{u,x,y}^0(\lambda) b(z) w(s) dz ds \\ &= \lambda e^{-\lambda t} \mathcal{I}_1(C_{t,u,x,y}^0(\lambda)) = \lambda e^{-\lambda t} C_{t,u,x,y}^1(\lambda) \end{aligned}$$

$$\begin{aligned} {}_tA_{u,x,y}^2(\lambda) &= \int_0^t \int_0^{u+s} {}_tA_{u,x,y}^1(\lambda) b(z)w(s) dzds \\ &= \lambda^2 e^{-\lambda t} \mathcal{I}_1(C_{t,u,x,y}^1(\lambda)) = \lambda^2 e^{-\lambda t} C_{t,u,x,y}^2(\lambda) \end{aligned}$$

in general

$$\begin{aligned} {}_tA_{u,x,y}^k(\lambda) &= \int_0^t \int_0^{u+s} {}_tA_{u,x,y}^{k-1}(\lambda) b(z)w(s) dzds \\ &= \lambda^k e^{-\lambda t} \mathcal{I}_1(C_{t,u,x,y}^{k-1}(\lambda)) \\ &= \lambda^k e^{-\lambda t} C_{t,u,x,y}^k(\lambda) \\ k &= 0, 1, 2, \dots \end{aligned}$$

■

4. McLaurin Expansion for the Classical Case

Using the statement of Theorem 2, McLaurin expansion of the t -years deferred multivariate ruin probability can be obtained with the following theorem

Theorem 3. *In the context of the Classical case of Risk Theory, the McLaurin expansion with respect the parameter λ of the t -years deferred multivariate ruin probability can be expressed*

$${}_t\Psi_{u,x,y} = \sum_{k=1}^{\infty} \lambda^k \left(\sum_{l=0}^{k-1} (-1)^l \frac{t^l}{l!} \mathcal{F}_{t,u,x,y}^{k-l} \right) \quad (4.1)$$

using the recursive scheme

$$\mathcal{F}_{t,u,x,y}^1 = F_{x,y}(u+t)$$

$$\begin{aligned} \mathcal{F}_{t,u,x,y}^k &= \int_0^t \int_0^{u+s} \mathcal{F}_{t-s,u+s-z,x,y}^{k-1} b(z) dz ds + F_{x,y} * G^{*(k)}(u+t) \\ k &= 2, 3, \dots \end{aligned} \quad (4.2)$$

Proof. Using the result of Theorem 2(3.1)

$${}_t\Psi_{u,x,y} = e^{-\lambda t} \sum_{k=0}^{\infty} \lambda^k C_{t,u,x,y}^k(\lambda)$$

and (3.2), it is clear that for evaluating the family of functions $C_{t,u,x,y}^k(\lambda)$ ($k = 0, 1, 2, \dots$) we need to obtain or approximate $\Psi_{z,x,y}(\lambda)$.

Following a similar argument as in Gerber et al. (1987), we can use the text-book renewal equation for the ultimate ruin

probability

$$\Psi_{z,x,y}(\lambda) = \frac{\lambda}{c} F_{x,y}(z) + \frac{\lambda}{c} \int_0^z \Psi_{z-\tau,x,y}(\lambda) dG(\tau)$$

where $G(w) = \int_0^w (1 - B(z)) dz$ and

$$F_{x,y}(w) = \int_w^{\max\{x,w\}} (B(z+y) - B(z)) dz$$

and in the case considered in this work, $c = 1$.

The following power series expansion using the Pollaczek-Khinchine formula that can be considered as a McLaurin expansion

$$\Psi_{z,x,y}(\lambda) = \sum_{n=1}^{\infty} \lambda^n F_{x,y} * G^{*(n-1)}(z) \quad (4.3)$$

Let us bear in mind that the ultimate ruin probability does not depend on the initial mean number of claims in the time unit λ_0 , considered.

Let us now define recursively the following family of functions

$$\begin{aligned} p_{t,u,x,y}^{(0,k)} &= F_{x,y} * G^{*(k-1)}(u+t) \\ p_{t,u,x,y}^{(i,k)} &= \mathcal{I}_1(p_{t,u,x,y}^{(i-1,k)}) \\ &= \int_0^t \int_0^{u+s} p_{t-s,u+s-z,x,y}^{(i-1,k)} b(z) dz ds \\ i &= 1, 2, \dots \end{aligned} \quad (4.4)$$

and we should mention that the members of this family $p_{t,u,x,y}^{(i,k)}$ do not depend on the parameter λ .

Using the definition of $C_{t,u,x,y}^0(\lambda)$ and (4.3), it is easy to prove that

$$\begin{aligned} C_{t,u,x,y}^0(\lambda) &= \Psi_{u+t,x,y}(\lambda) \\ &= \sum_{n=1}^{\infty} \lambda^n F_{x,y} * G^{*(n-1)}(u+t) \\ &= \sum_{n=1}^{\infty} \lambda^n p_{t,u,x,y}^{(0,n)} \end{aligned}$$

and with (3.2) and (4.4)

$$\begin{aligned} C_{t,u,x,y}^j(\lambda) &= \sum_{n=1}^{\infty} \lambda^n p_{t,u,x,y}^{(j,n)} \\ j &= 1, 2, \dots \end{aligned} \quad (4.5)$$

Substituting the former expressions in (3.1)

$$\begin{aligned} {}_t\Psi_{u,x,y} &= e^{-\lambda t} \sum_{k=0}^{\infty} \lambda^k \left(\sum_{i=1}^k p_{t,u,x,y}^{(k-i,i)} \right) \\ &= e^{-\lambda t} \sum_{k=1}^{\infty} \lambda^k \mathcal{F}_{t,u,x,y}^k \\ &= \sum_{k=1}^{\infty} \lambda^k \left(\sum_{j=0}^{\infty} (-1)^j \cdot \frac{(\lambda t)^j \mathcal{F}_{t,u,x,y}^k}{j!} \right) \end{aligned} \quad (4.6)$$

$$= \sum_{k=1}^{\infty} \lambda^k \left(\sum_{l=0}^{k-1} (-1)^l \frac{t^l}{l!} \mathcal{F}_{t,u,x,y}^{k-l} \right) \quad (4.7)$$

and the functions $\mathcal{F}_{t,u,x,y}^k$

$$\mathcal{F}_{t,u,x,y}^k = \sum_{i=1}^k p_{t,u,x,y}^{(k-i,i)}, \quad k = 1, 2, \dots$$

again do not depend on the parameter λ and because of (4.4) are defined recursively using (4.2). ■

5. FINITE TIME MULTIVARIATE RUIN PROBABILITY

Using the formula (1.3)

$$\Psi_{t,u,x,y}(\lambda) = \Psi_{u,x,y}(\lambda) - {}_t\Psi_{u,x,y}(\lambda)$$

it is clear that approximating the ultimate and the t -years deferred multivariate ruin probability will lead us to an approximation of the finite time multivariate ruin probability.

Moreover, McLaurin expansion of $\Psi_{t,u,x,y}(\lambda)$ can be obtained using (4.1) and (4.3),

$$\begin{aligned} \Psi_{t,u,x,y}(\lambda) &= \sum_{k=1}^{\infty} \lambda^k \left(F_{x,y} * G^{*(k-1)}(u) - \Xi_{t,u,x,y}^k \right) \\ &= \sum_{k=1}^{\infty} \lambda^k \Lambda_{t,u,x,y}^k \end{aligned} \quad (5.1)$$

where

$$\Lambda_{t,u,x,y}^k = F_{x,y} * G^{*(k-1)}(u) - \sum_{l=0}^{k-1} (-1)^l \frac{t^l}{l!} \mathcal{F}_{t,u,x,y}^{k-l} \quad (5.2)$$

$$k = 1, 2, \dots$$

and subsequently

$$\left[\frac{\partial^k \Psi_{t,u,x,y}(\lambda)}{\partial \lambda^k} \right]_{\lambda=0} = \Lambda_{t,u,x,y}^k \quad k = 1, 2, \dots \quad (5.3)$$

6. FINANCIAL-ACTUARIAL FINITE TIME MULTIVARIATE RUIN PROBABILITY

Let us now introduce the following financial-actuarial model as a financial generalization of the pure actuarial model introduced in section 1 (1.1)

$$Z_t^i = uv(t) + c \bar{s}_{t|} - \sum_{k=1}^{N_t} U_k \quad (6.1)$$

where

- $\delta(s) \quad s > 0$ Instantaneous rate of interest.
- $v(t) = e^{\int_0^t \delta(s) ds}$ Actualization factor
- $\bar{s}_{t|} = \int_0^t v(s) ds$ Final value of a Continuous annuity

Subsequently, initial reserves, u , earn interest continuously according to $\delta(s) \quad s > 0$ and the premium process can be modelled using the final value of a continuous annuity. We will suppose in the present work that the instantaneous rate of interest is deterministic.

We can now define $\tau^i = \inf \{w > 0 : Z_w^i < 0\}$ as the ruin time in the financial-actuarial model and $Y^i = -Z_{\tau^i}^i$ as the deficit at ruin time or severity of ruin and $X^i = Z_{\tau^i-}^i$ as the surplus just before the ruin; subsequently, the probability of ruin with time span t_0 and initial reserves u , severity of ruin less than y and surplus before ruin less than x can be expressed

$$P \left\{ \tau^i < t_0, X^i \leq x, Y^i \leq y \right\} = \Psi_{t_0,u,x,y}^i(\lambda_0)$$

It will be most interesting the introduction of the concepts of *simultaneous actuarial-financial probability of ruin* ($\Psi_{t_0,u,x,y}^{si}(\lambda_0)$) and *posterior actuarial-financial probability of ruin* ($\Psi_{t_0,u,x,y}^{pi}(\lambda_0)$)

$$\Psi_{t_0,u,x,y}^{si}(\lambda_0) = P \left\{ \tau = \tau^i < t_0, X^i \leq x, Y^i \leq y \right\}$$

$$\Psi_{t_0,u,x,y}^{pi}(\lambda_0) = P \left\{ \tau < \tau^i < t_0, X^i \leq x, Y^i \leq y \right\}$$

It is clear that

$$\Psi_{t_0,u,x,y}^i(\lambda_0) = \Psi_{t_0,u,x,y}^{si}(\lambda_0) + \Psi_{t_0,u,x,y}^{pi}(\lambda_0) \quad (6.2)$$

as long as pure actuarial ruin should occur before or at the same instant of actuarial-financial ruin.

Using the same argument as in section 2 (2.2)

$$\begin{aligned} \Psi_{t_0,u,x,y}^i(\lambda_0) &\equiv \Psi_{t,u,x,y}^i(\lambda) \\ \Psi_{t_0,u,x,y}^{si}(\lambda_0) &\equiv \Psi_{t,u,x,y}^{si}(\lambda) \\ \Psi_{t_0,u,x,y}^{pi}(\lambda_0) &\equiv \Psi_{t,u,x,y}^{pi}(\lambda) \end{aligned}$$

where

$$\begin{aligned} \lambda &= \frac{1}{(1 + \theta)} \\ t &= \lambda_0 t_0 (1 + \theta) \\ c &= 1 \end{aligned}$$

Again, as it was stated in section 2, the security loading θ determines the value of the new parameters for the average number of claims in one year (λ) and time span (t).

Theorem 4. *In the context of the Classical case of Risk Theory, the McLaurin expansions with respect the parameter λ of the financial-actuarial finite time multivariate ruin probability and simultaneous financial-actuarial finite time multivariate ruin probability are*

$$\Psi_{t_0,u,x,y}^i(\lambda_0) = \sum_{k=1}^{\infty} \lambda^k \left(\int_0^t \Lambda_{z,u+h(u,z),x,y}^{'k} dz \right) \quad (6.3)$$

$$\begin{aligned} &\Psi_{t_0,u,x,y}^{si}(\lambda_0) \\ &= \sum_{k=1}^{\infty} \lambda^k \int_0^t (\Lambda_{z,u,x-h(u,z),y+h(u,z)}^{'k} - \Lambda_{z,u,x-h(u,z),h(u,z)}^{'k}) dz \quad (6.4) \end{aligned}$$

where

$$h(u, z) = u(v(z) - 1) + (\bar{s}_z] - z)$$

and

$$\begin{aligned}\Lambda_{t,u,x,y}^k &= F_{x,u} * G^{*(k-1)}(u) - \sum_{l=0}^{k-1} (-1)^l \frac{t^l}{l!} \mathcal{F}_{t,u,x,y}^{k-l} \\ \Lambda'_{t,u,x,y} &= \frac{\partial \Lambda_{t,u,x,y}^k}{\partial t} \\ k &= 1, 2, \dots\end{aligned}$$

using the recursive formulas

$$\mathcal{F}_{t,u,x,y}^1 = F_{x,y}(u+t)$$

$$\begin{aligned}\mathcal{F}_{t,u,x,y}^k &= \int_0^t \int_0^{u+s} \mathcal{F}_{t-s,u+s-z,x,y}^{k-1} b(z) dz ds + F_{x,y} * G^{*(k)}(u+t) \\ k &= 2, 3, \dots\end{aligned}$$

Proof. Using a similar argument as Gerber and Shiu (1998) (expression (2.8) of the original paper)

$$\begin{aligned}P\{X \leq x, Y < y, \tau < s \mid u\} &= \Psi_{z,u,x,y}(\lambda) \\ &= \int_0^z \int_0^x \int_0^y f(s, a, b \mid u) db da ds\end{aligned}$$

the density function of the ruin time in the actuarial model is the derivative with respect t of the former expression and using (5.1)

$$\begin{aligned}&P\{X \leq x, Y < y, z < \tau < z + dz \mid u\} \\ &= \int_0^x \int_0^y f(z, a, b \mid u) db da \\ &= \frac{\partial \Psi_{z,u,x,y}(\lambda)}{\partial z} = \Psi'_{z,u,x,y}(\lambda) = \sum_{k=1}^{\infty} \lambda^k \frac{\partial \Lambda_{z,u,x,y}^k}{\partial z} \\ &= \sum_{k=1}^{\infty} \lambda^k \Lambda'_{z,u,x,y}^k\end{aligned}$$

Defining

$$h(u, z) = u(v(z) - 1) + (\bar{s}_z - z)$$

as the pure financial income up to z , we can obtain the following relation for the infinitesimal interval $(z, z + dz)$

$$\begin{aligned}&P\{X^i \leq x, Y^i \leq y, z < \tau^i < z + dz \mid u\} \\ &= P\{0 < Z_z^i \leq x; 0 > Z_{z+dz}^i \geq y \mid u\}\end{aligned}$$

$$\begin{aligned}
&= P\{0 < Z_z + h(u, z) \leq x; 0 > Z_{z+dz} + h(u, z + dz) \geq y \mid u\} \\
&= P\{0 < Z_z \leq x; 0 > Z_{z+dz} \geq y \mid u + h(u, z)\} \\
&= P\{X \leq x, Y \leq y, z < \tau < z + dz \mid u + h(u, z)\} \\
&= \Psi'_{z, u+h(u, z), x, y}(\lambda) dz = \sum_{k=1}^{\infty} \lambda^k \Lambda'_{z, u+h(u, z), x, y}{}^k dz
\end{aligned}$$

because

$$\begin{aligned}
Z_z^i &= uv(z) + c \bar{g}_z - \sum_{k=1}^{N_z} U_k = Z_z + h(u, z) \\
&= (u + h(u, z)) + cz - \sum_{k=1}^{N_z} U_k
\end{aligned}$$

integrating over z we finally obtain (6.3)

$$\begin{aligned}
&\Psi_{i, u, x, y}^i(\lambda) \\
&= \int_0^t P\{X \leq x, Y \leq y, z < \tau < z + dz \mid u + h(u, z)\} \\
&= \int_0^t \left(\sum_{k=1}^{\infty} \lambda^k \Lambda'_{z, u+h(u, z), x, y}{}^k \right) dz = \sum_{k=1}^{\infty} \lambda^k \left(\int_0^t \Lambda'_{z, u+h(u, z), x, y}{}^k dz \right)
\end{aligned}$$

It is easy to prove with (5.1) that

$$\begin{aligned}
&P\{X^i \leq x, Y^i \leq y, z < \tau = \tau^i < z + dz \mid u\} \\
&= P\{X \leq x - h(u, z), y + h(u, z) > Y > h(u, z), z < \tau < z + dz \mid u\} \\
&= (\Psi'_{z, u, x-h(u, z), h(u, z)+y}(\lambda) - \Psi'_{z, u, x-h(u, z), h(u, z)}(\lambda)) dz \\
&= \sum_{k=1}^{\infty} \lambda^k (\Lambda'_{z, u, x-h(u, z), h(u, z)+y}{}^k - \Lambda'_{z, u, x-h(u, z), h(u, z)}{}^k) dz
\end{aligned}$$

because when actuarial ruin occurs in the infinitesimal interval $(z, z + dz)$,

a) In order to provoke also ruin and a deficit less than y in the financial-actuarial model ($Y^i < y$), the deficit in the actuarial model Y should lie inside the following interval

$$h(u, z) < Y < h(u, z) + y$$

b) In order to keep the surplus prior to ruin in the financial-actuarial model less x ($X^i < x$), the surplus in the actuarial model X should be less than $x - h(u, z)$, because in the financial-actuarial model, the pure financial income up to z will be also added.

Finally, integrating over the time of ruin in the actuarial model

$$\begin{aligned}
 & \Psi_{t,u,x,y}^{si}(\lambda) \\
 &= \int_0^t P \{ X \leq x - h(u, z), h(u, z) > Y > y + h(u, z), z < \tau < z + dz \mid u \} \\
 &= \int_0^t \left(\sum_{k=1}^{\infty} \lambda^k (\Lambda'_{z,u,x-h(u,z),h(u,z)+y} - \Lambda'_{z,u,x-h(u,z),h(u,z)}) \right) dz \\
 &= \sum_{k=1}^{\infty} \lambda^k \int_0^t (\Lambda'_{z,u,x-h(u,z),h(u,z)+y} - \Lambda'_{z,u,x-h(u,z),h(u,z)}) dz
 \end{aligned}$$

■

7. PRICING THE RISK OF A GENERAL INSURANCE

The risk of a general insurance portfolio, in our opinion, is best measured using ruin probabilities, specially in the multivariate case, where we can also state the maximum loss at ruin and the surplus just before. Introducing the rate of interest in the model can be considered one more step in the course to approach theoretical models to actuarial practice.

It is clear from (6.2) that $\Psi_{t,u,x,y}^i(\lambda)$ is the sum of two components and we consider most interesting identifying both because it will reveal what are the true options for a company to continue working under actuarial ruin and only supported by the investments. The conditional probability or ratio $R_{t,u,x,y}^i(\lambda)$ could be considered as a measure of the financial independence or autonomy of a portfolio from its actuarial foundations; in other words, a measure of the strength of the investments yields to prevent ruin,

$$R_{t,u,x,y}^i(\lambda) = 1 - \frac{\Psi_{t,u,x,y}^{si}(\lambda)}{\Psi_{t,u,x,y}^i(\lambda)} \in (0, 1) \quad (7.1)$$

Let us name the former ratio as *Financial Autonomy Ratio*. It is clear that the larger the value the smaller the correlation between the time of actuarial and financial-actuarial ruin and, subsequently, the greater the financial autonomy of the portfolio.

However, defining the concept of "price of risk" it is not an easy task. We consider, following our approach based on ruin multivariate ruin probabilities, that the "price of a risk" should be the amount paid to guarantee a certain multivariate ruin probability, in other words, the premiums earned.

In the models contemplated in sections 1 and 6, the premium considered for time unit (before the change of scale in the time units to fit the subsequent theoretical developments) is

$$c = P = \lambda_0 \mu^{-1} (1 + \theta) \quad (7.2)$$

it is clear that the average number of claims, λ_0 and the expected value of the claims size, μ^{-1} are part of the data concerning the model.

Nevertheless, θ , the security loading can be considered as a true decision variable and, using (2.2), define the following one-to-one correspondence

$$P = \lambda_0 \mu^{-1} (1 + \theta) \longleftrightarrow \Psi_{\lambda_0 t_0 (1 + \theta), u, x, y}^i \left(\frac{1}{(1 + \theta) \mu^{-1}} \right)$$

or for simultaneous multivariate ruin probability

$$P = \lambda_0 \mu^{-1} (1 + \theta) \longleftrightarrow \Psi_{\lambda_0 t_0 (1 + \theta), u, x, y}^{si} \left(\frac{1}{(1 + \theta) \mu^{-1}} \right)$$

Following the former reasoning, we will consider P , premium earned per time unit as the price of the risk underwritten.

8. -NUMERICAL ILLUSTRATION

In figures 1 ($t_0 = 2$) and 3 ($t_0 = 3$), multivariate ruin probabilities in the financial-actuarial model ($\Psi_{t_0, u, x, y}^{si}(\lambda_0)$, $\Psi_{t_0, u, x, y}^i(\lambda_0)$) are drawn for exponential claims size and the parameter values specified.

Results were obtained using a 25 points interpolation polynomial considering the values for $\theta = -0.9, -0.8, \dots, 1.5$ (step 0.1) and the former parameters in (6.3) and (6.4), Theorem 4. The McLaurin series was truncated in the seventh term and at least two digits accuracy is guaranteed. Calculations were performed using Maple V Release 4 on a Pentium 120.

Notice that negative values of θ , security loading, means that we are underwriting the considered portfolio with an actual premium, P , less than the actuarial net premium, $\lambda_0 \mu^{-1}$, expected value of the total claims during the time units considered (see (7.2)), as it may be the case under severe market conditions.

Using figures 1 and 3, the risk of the underwritten portfolio with the parameters considered (ruin probabilities) can be easily priced in terms of the annual premium, P .

We can also state that, in the examples considered, when financial-actuarial ruin takes place is mainly because, at the same time, actuarial ruin does so; in other words, it is very unlikely in these examples, specially for small P , that after an actuarial ruin, the firm will keep on working supported by financial yields and be ruined later. The Financial Autonomy ratio, $R_{t_0, u, x, y}^i(\lambda_0)$, shows this last fact in figures 2 and 4.

9. CONCLUDING COMMENTS

In the present work, using of a pure actuarial model (1.1) for the risk process, we first introduced the concept of t -years deferred ultimate multivariate ruin probability ${}_{t_0} \Psi_{u, x, y}(\lambda_0)$ (1.2) presented as a series expansion based on

a recursive scheme, Theorem 1 (2.3). The former result can be used for any waiting times distribution between claims(d.f. $w(t)$).

Later, in the context of the Classical case, we obtained the McLaurin series expansion for ${}_{t_0}|\Psi_{u,x,y}(\lambda_0)$, Theorem 3 (4.1), using a recursive scheme. As a corollary of Theorem 3, it is presented the same expansion for finite time multivariate ruin probability (5.1) $\Psi_{t_0,u,x,y}(\lambda_0)$.

In Theorem 4 (6.4 and 6.3), based on the former results, McLaurin expansion is presented for finite time multivariate ruin probability, $\Psi_{t_0,u,x,y}^i(\lambda_0)$, and the *simultaneous* multivariate ruin probability, $\Psi_{t_0,u,x,y}^{si}(\lambda_0)$ considering a financial-actuarial model for the risk process (6.1) and again using a recursive scheme.

Finally, we applied the results of former sections to price a General Insurance Risk in a Financial-Actuarial model using the couple *Multivariate Ruin Probability-Annual Premium* (figures 1 and 3) and using the *Financial Autonomy ratio*, $R_{t_0,u,x,y}^i(\lambda_0)$ (7.1, figures 2 and 4) the conditional probability was given of the event consisting on the firm will keep on working after an actuarial ruin conditioned to the fact that will be eventually ruined, considering the financial-actuarial model, later as a measure of the strength of the investments' yields to prevent ruin.

Figure 1: Risk Pricing curve $t_0 = 2$

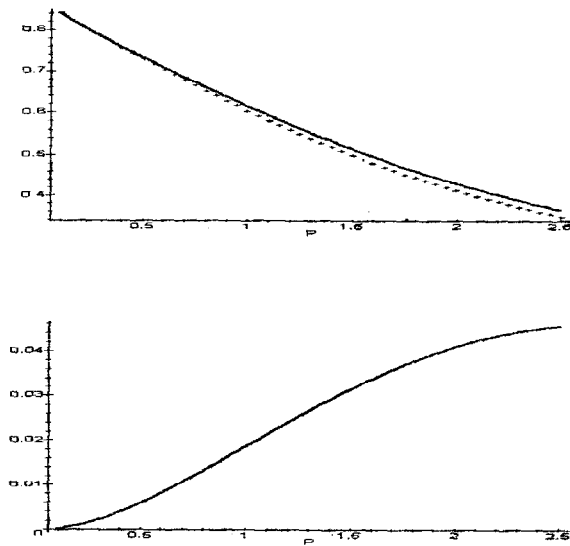


Figure 2: Financial Autonomy Ratio $R_{2,0,10,10}^i(1)$

Actuarial-Financial Model Multivariate Ruin probabilities

Exponential claim size $b(z) = \mu e^{-\mu z}$ $\mu^{-1} = 1$	
$\lambda_0 = 1, u = 0, x = 10, y = 10, t_0 = 2$	
$\delta(s) = s \ln(1 + 0.06) \quad s > 0$	$P = \lambda_0 \mu^{-1} (1 + \theta) \in [0.1, 2.5]$
$\Psi_{2,0,10,10}^i(1)$	$\Psi_{2,0,10,10}^{st}(1) \quad + + +$

Figure 3: Risk Pricing curve $t_0 = 3$

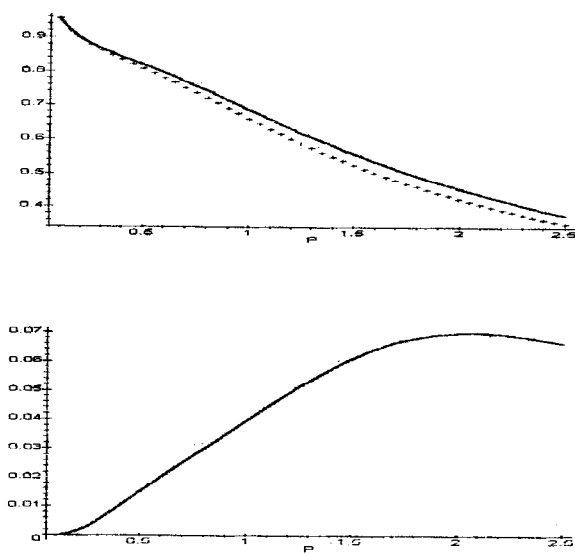


Figure 4: Financial Autonomy Ratio $R^i_{3,0,10,10}(1)$

Actuarial-Financial Model Multivariate Ruin probabilities

Exponential claim size	$b(z) = \mu e^{-\mu z}$	$\mu^{-1} = 1$
$\lambda_0 = 1, \quad u = 0, \quad x = 10, \quad y = 10, \quad t_0 = 3$		
$\delta(s) = s \ln(1 + 0.06) \quad s > 0$	$P = \lambda_0 \mu^{-1} (1 + \theta) \in [0.1, 2.5]$	
$\Psi^i_{3,0,10,10}(1)$	—	$\Psi^{st}_{3,0,10,10}(1) \quad + + +$

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