Institute and Faculty of Actuaries

PhD studentship output

Funded by the Institute and Faculty of Actuaries

Model Independent Price Bounds for the Swiss Re Mortality Bond 2003

Raj Kumari Bahl

University of Edinburgh

(Ph.D. funded by IFoA & UoE)

20th International Congress on Insurance: Mathematics and Economics
IME 2016

July 25, 2016

Biography

- Ph.D. researcher at the School of Mathematics, University of Edinburgh
- Working on the research project **Mortality Linked Derivatives and their Pricing**
- Project funded by the **Actuarial Research Centre (ARC)** of **IFOA** and **UoE**
- Assistant Professor at University of Delhi for the last 13 years
- Teaching Statistics, Probability, Financial Mathematics and Actuaries
- M.Sc. and M.Phil. in Statistics from University of Delhi
- Qualified CT series Actuarial exams with teaching
- Part of the Universitas 21 project
- M.Sc. in Financial Mathematics - from University of Edinburgh & Heriot-Watt University

Quotation

"Nothing is certain in life except death and taxes."

— Benjamin Franklin

Agenda

- Introduction
- Historical Facts
- The Problem
- Available Methodologies
- Case Study: Swiss Re Mortality Bond 2003
- A Model-independent Approach
- Lower Bounds for the Swiss Re Bond
- Upper Bounds for the Swiss Re Bond
- Numerical Results
- What Lies Ahead?
- Further Research
- The Modeling Aspect

Introduction(1)

Motivation

- In the present day world, financial institutions face the risk of unexpected fluctuations in human mortality
- This Risk has two aspects
  - *Mortality Risk*: Actual rates of mortality are in excess of those expected
  - *Longevity Risk*: People outlive their expected lifetimes

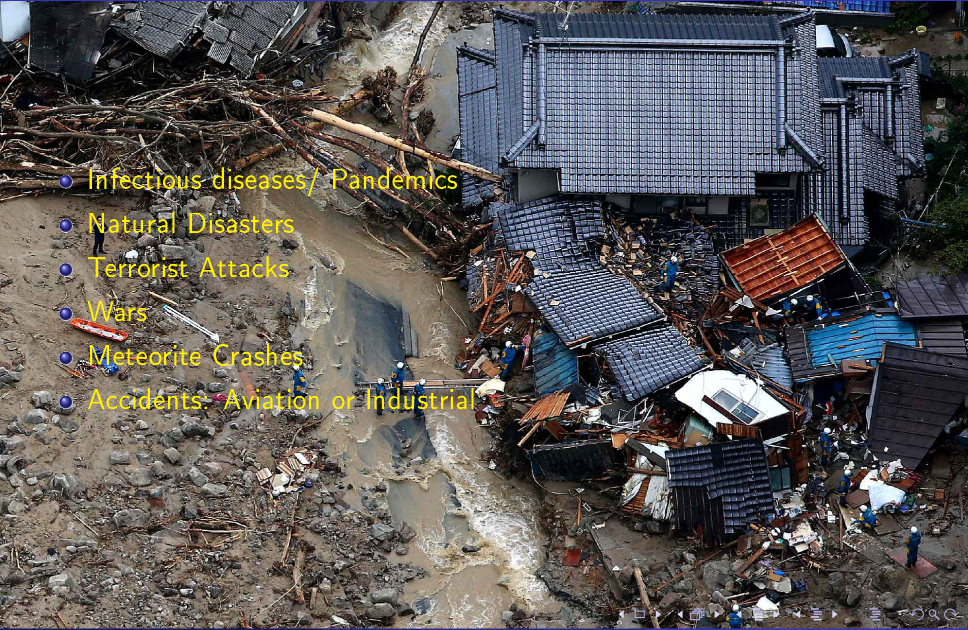
Introduction(2)

- Life insurers interested in *mortality risk*
- Annuity providers, defined benefit plans & social insurance programs interested in *longevity risk*
- A quick note on *longevity risk*
  - Life Expectancy in developed world has been increasing by approx 1.2 months every year
  - Global Life Expectancy has increased by 4.5 months per year
  - Substantial improvements in Longevity at older ages during 20th century
  - Difficulties in Longevity Risk Management in Pension Funds due to wrong estimation of mortality rate
- What are the implications?
  - Underestimation of expected lifetimes leads to aggregate deficit in pension reserves
  - Equitable Life closed to new business in 2000 because GAO's in money
  - In 2010 alone improved life expectancy added 5 billion pounds to corporate pension obligations in UK

Introduction(3)

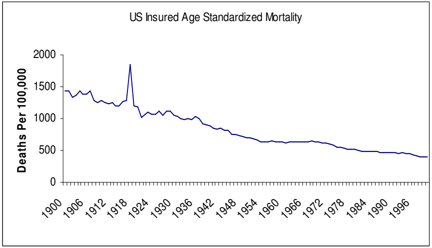
- A quick note on *mortality risk*
  - Life being shorter than expected is referred to as premature death or mortality risk
  - Factors that trigger mass premature deaths are CATASTROPHES!
- Catastrophes can be natural or man-made
- What is a catastrophe?
  - An event in which insured claims, total economic losses, or the number of casualties exceed a certain threshold
  - Lost or missing lives 20, injured 50, homeless 2000
- Number of catastrophes has risen sharply in the last four decades
- In the 1970's roughly 100 catastrophic events per year
- Number has more than tripled in the last decade
- Between 1994 and 2013, EM-DAT recorded 6,873 natural disasters
- Claimed 1.35 million lives or almost 68,000 lives on average each year
- 218 million people affected by natural disasters on average per annum

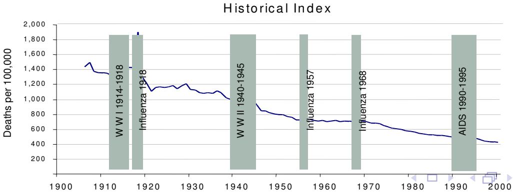
Introduction(4) : Possible Mortality Catastrophes



- Infectious diseases/ Pandemics
- Natural Disasters
- Terrorist Attacks
- Wars
- Meteorite Crashes
- Accidents: Aviation or Industrial

Historical Facts(1): Catastrophes lead to Mortality Spikes





Historical Facts(2): The 1918 Influenza Pandemic

Table 2: The change of death rates per 100,000 for each age group, from 1917 to 1919

Age groups	1917	1919	Ratio	Age groups	1917	1919	Ratio
All	1397.1	1810	1.296	35-44	900.8	1339.3	1.487
<1	10457.2	11167.2	1.068	45-54	1385.6	1524.1	1.100
1-4	1066.0	1573.5	1.476	55-64	2678.6	2648.1	0.989
5-14	256.0	412.8	1.613	65-74	5728.4	5505.0	0.961
15-24	468.9	1070.6	2.283	75-84	12386.2	11295.7	0.912
25-34	649.1	1643.5	2.532	>=85	24593.6	22213.5	0.903

- The 1918 influenza pandemic: Increase in mortality rate by 30% overall.
- Most affected age groups: 15-24 and 25-34
- For individuals aged 55 and over a little decrease in the death rate

Historical Facts(3): The 1918 Influenza Pandemic

"The great flu pandemic of 1918 and 1919 is estimated to have killed between 30 million and 50 million people worldwide. Among them were 675,000 Americans. (source: CNN)"

Historical Facts(4): H1N1 flu

"The global H1N1 flu pandemic may have killed as many as 575,000 people, though only 18,500 deaths were confirmed. The H1N1 virus is a type of swine flu, which is a respiratory disease of pigs caused by the type A influenza virus. (source: CNN)"

Historical Facts(5): Pandemics in general

- 13 or more influenza pandemics since 1500
- 4 Influenza Pandemics in 20th Century
  - Spanish Flu (1918)
    - most severe influenza pandemic
    - more than 675,000 excess deaths b/w Sep 1918 & Apr 1919 in US
  - Asian Flu (1957)
  - Hong Kong Flu (1968)
  - Russian Flu (1977)

Risk of an outbreak in any given year: 1-in-30

Risk of a 1957-caliber outbreak: 1-in-40

Risk of a 1918-caliber outbreak: 1-in-475

Source: RMS pandemic model

- H5N1 Avian Influenza in Hong Kong in 1997
- Swine Flu in 2009
- Could a flu happen again?
- Virologists and Epidemiologists say YES!
- Zika and Ebola: A taste of things to come?

Facts on Pandemic

- **Frequency**: 3 per century
- **Attack Rate**: 10-60%
- **Severity**: 1x to 6x mortality

Historical Facts(6): Pandemics in general

- A (flu) pandemic may occur if three conditions are met:
  - a new influenza virus emerges
  - the virus infects humans
  - the virus spreads efficiently and in a sustained manner
- WHO –The World Health Report 2007: “Scientists agree that the threat of a pandemic from H5N1 continues and that the question of a pandemic of influenza from this virus or another avian influenza virus is still a matter of when, not if.”
- We don’t know how infectious and deadly the new virus will be
  - Unlimited reservoir of influenza sub-types
  - Interspecies transmission, intraspecies variation and altered virulence

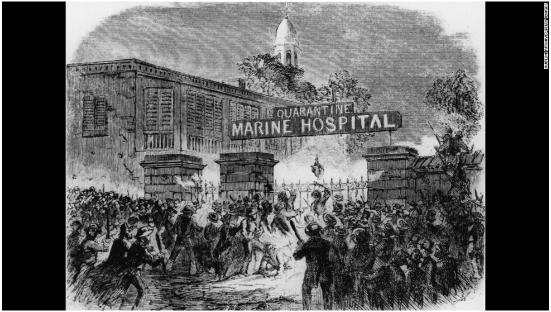
Factors attenuating virulence

- Improvement in medical care
- Establishment of global surveillance
- Crisis/emergency plans

Factors supporting virulence

- Population Growth
- Urbanization
- Increased Global Mobility

Historical Facts(7): Current Pandemic?



"Philadelphia was struck with a yellow fever epidemic in 1793 that killed a 10th of the city's 45,000-person population. (source: CNN)"

"The Ministry of Health in Angola has reported an ongoing outbreak of yellow fever. At least 3,552 suspected & confirmed cases have been reported, including 355 deaths. (source: CDC, 14th July, 2016)"

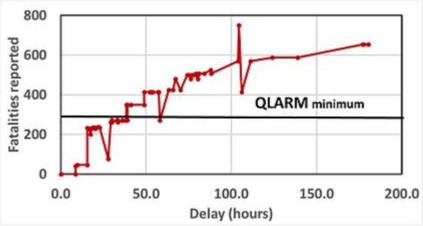
Historical Facts(9): SARS

"Severe Acute Respiratory Syndrome, better known as SARS, was first identified in 2003 in China, though the first case is believed to have occurred in November 2002. By July more than 8,000 cases and 774 deaths had been reported. Diseases like AIDS bring PERSISTENT changes in mortality curve. (source: CNN)"

Historical Facts(9): Earthquakes in 2016

- THE EDUCADOR EARTHQUAKE 16th APRIL 2016

- Atleast 660 people killed
- More than 27,732 injured
- Nearly 7,000 buildings destroyed
- More than 26,000 people in shelters
- Worst natural disaster since 1949



- A DAY EARLIER: KUMAMOTO CITY, JAPAN
  - 39 people killed
  - More than 1,000 injured
  - 8,700 buildings damaged
  - A bridge collapsed in Aso
- ALARMING FIGURES!!!!!

Historical Facts(10): The China Floods in 2016

- THE BLOOMBERG REPORTS ON JULY 11 2016

  “Weeks of torrential rain across central and southern China have caused the country’s worst flooding since 1998, killing 173 people, ruining farms and cutting major transportation arteries – and creating potential headwinds to economy growth.

  A swollen Yangtze and other rivers spilled over their banks. That was compounded by the arrival of Typhoon Nepartak, as it made landfall on Saturday in Fujian province.

  The Ministry of Civil Affairs said flooding and rain associated with the typhoon affected more than 31 million people in 12 provinces, submerged more than 2.7 million hectares (6.7 million acres) of cropland and caused 67.1 billion yuan ($10 billion) in damages.

  Flooding is linked to El Nino, which originates from warm waters in the Pacific Ocean near the equator and disrupts global weather patterns.

  While forecasters said the worst weather has passed, analysts said the economic impact from farm damage and transport disruptions would be tallied for months to come."

Types of Terrorism Attacks

AON
Empower Results®

4	7	7	3	3
Nuclear	**Conventional**	**Radiological**	**Biological**	**Chemical**
100 kiloton	Cruise missile	Cruise missile	Large event	Large event
20 kiloton	Multiple aircraft	Multiple aircraft	Medium event	Medium event
10 kiloton	Single aircraft	Single aircraft	Small event	Small event
1 kiloton	Large truck bomb	Large truck bomb		
	Small truck bomb	Small truck bomb		
	Car bomb	Car bomb		
	Human bomb	Human bomb		

Total attack types = 24

The Problem (1) - Extreme Mortality Risk

- Life insurance companies provide protection to their policyholders in the form of a payout made in the event of a policyholder's death, in exchange for a premium
- Extreme mortality events, such as a severe pandemic or a large terrorist attack, could result in a life insurance company needing to make sudden payouts to many policyholders
- This large payout would be exacerbated in that the investment portfolio would not yet have delivered sufficient returns – the payouts to policyholders are made sooner than expected
- Therefore it is crucial for life insurers, and life reinsurers, to manage their exposure to extreme mortality risks where insurance portfolio diversification by itself is insufficient

The Problem (2)

- Mortality jumps are infrequent but when they occur they
  - Trigger a large number of unexpected death claims
  - Affect the financial strength of the life insurance industry
- [Stracke and Heinen(2006)]estimated that the worst pandemic would result in
  - Approximately €45 billion of additional claim expenses in Germany
  - Amount equivalent to five times the total annual gross profit
  - Or 100% of the policyholder bonus reserves in the German Life Insurance market
- [Toole(2007)] found that in a severe pandemic scenario
  - Additional claim expenses would consume 25% of the Risk Based Capital (RBC) of the entire US life insurance industry
  - Companies with less than 100 % of RBC are at the risk of being insolvent

Available Methodologies for Hedging Extreme Mortality Risk

- *Natural Hedging*: compensating longevity risk by mortality risk
  - Drawback: Cost prohibitive
- *Mortality-linked Securities (MLS's)* or *Catastrophe (CAT) Mortality (CATM) Bonds* or *Extreme Mortality Bonds (EMB's)*: Cash flows linked to a mortality index such that the bonds get triggered by a catastrophic evolution of death rates of a certain population
  - Swiss Re Bond 2003 (VITA I): The first mortality bond
  - Swiss re Bond 2015 (VITA VI): The latest mortality bond

Valuation approaches on MLS's

- *Risk-adjusted process/ No-arbitrage Pricing:*
  - Estimate the distribution of future mortality rates in the real world probability measure
  - Transform the real-world distribution to its risk-neutral counterpart
  - Calculate the price of MLS by discounting the expected payoff under the risk-neutral probability measure at the risk-free rate
- *The Wang Transform:*
  - Employs a distortion operator that transforms the underlying distribution into a risk-adjusted distribution
  - MLS price is the expected value under the risk-adjusted probability discounted by risk-free rate
- *Instantaneous Sharpe Ratio:* Expected return on the MLS equals the risk-free rate plus the Sharp ratio times its standard deviation
- *The utility-based valuation:* Maximisation of the agent's expected utility subject to wealth constraints to obtain the MLS equilibrium

History of Mortality Linked Securities

- Tontines: 17th and 18th century in France
- Annuities in Geneva: Payoffs directly linked to the survival of Genevan "mademoiselles"
- Speculations came to an end during French Revolution
- Detailed overview in [Bauer(2008)]

Recent Developments(1)

Blake & Burrows (2001)	derived the concept of longevity bond
Swiss Re. (2003)	issued the first mortality bond
European investment bank (2004)	issued the first longevity bond
Cowley & Cummins(2005)	show that securitization may increase a firm's value
Lin & Cox (2005)	study and price the mortality bonds and swaps
Cairns, Blake & Dowd(2006)	show how to price mortality-linked financial instruments such as the EIB bond
Blake et al. (2006)	Introduce five types of longevity bonds

Recent Developments(2)

Pessler (2000)	Criticism of Wang Transform
Chen and Cox (2009)	Modelling mortality with Jumps
Cox et al (2010)	Mortality Risk Modelling
Milidonis et al (2011)	A regime switching mortality model with two states
Shang et al (2011)	Recursive Approach to MLS
Deng et al (2012)	Double-exponential jump diffusion model for mortality jumps & cohort effects
Cox et al (2013)	Mortality portfolio Risk Management
Lin et al (2013)	Pricing mortality securities with correlated indexes
Huang et al (2014)	Price jumps of MLS in incomplete markets
Hunt & Blake (2015)	Analysing the Swiss Re Kortis Bond

Prime Focus(1)

- Catastrophe Mortality Bonds or CATM Bonds
- What are these?
    - Bonds designed to transfer the risk of extreme mortality from a sponsor to investors
    - Coupon & Principal payments depend on the non-occurence of a pre-defined catastrophic event
- Transaction involves three parties
    - The Ceding company or Sponsor
    - Special Purpose Vehicle (SPV) or issuer
    - Investors generally large institutional buyers
- Transaction begins with formation of a SPV
- Investment Period: 3 to 5 years
- Can be purchased as OTC products
- High yield debt instruments

Prime Focus(2)

- SPV issues bonds to investors
- SPV invests the received capital in high quality securities such as government or corporate AAA bonds
- Generally held in a trust account
- Coupon Payment
  - Investment returns from trust account &
  - Risk premium from ceding company
- Embedded in the bonds is a call option
- This call option gets triggered by a defined catastrophic event
- Well defined *Attachment or Trigger* and *Exhaustion Points*
- Principal is fully at risk
- Our choice: Swiss Re Bond 2003

Prime Focus(3)

Specifications	VITA I	VITA II	TARTAN
Sponsor	Swiss Re	Swiss Re	Scottish Re
Arranger	Swiss Re	Swiss Re	Goldman Sachs
Modelling Firm	Milliman	Milliman	Milliman
SPV domicile	Cayman Islands	Cayman Islands	Cayman Islands
Size	$ 400M	$ 362M	$ 155M
No.of Tranches	1	3	2
Issue date	December 2003	April 2005	May 2006
Maturity	3 years	5 years	3 years
Index	US, UK, France, Italy, Switzerland	US, UK, Germany, Japan, Canada	US

Table 1: The Initial CAT Mortality Bonds

Prime Focus(4)

Specifications	OSIRIS	VITA III	NATHAN
Sponsor	AXA	Swiss Re	Munich Re
Arranger	Swiss Re	Swiss Re	Munich Re
Modelling Firm	Milliman	Milliman	Milliman
SPV domicile	Ireland	Cayman Islands	Cayman Islands
Size	€ 345M	$ 705M	$ 100M
No.of Tranches	3	2	1
Issue date	November 2006	January 2007	February 2008
Maturity	4 years	4 & 5 years	5 years
Index	France, Japan, US	US, UK, Germany, Japan, Canada	US, UK, Canada, Germany

Table 2: The Middle Stage CAT Mortality Bonds

Prime Focus(5)

Specifications	Vita IV	Vita IV	Vita V
Sponsor	Swiss Re	Swiss Re	Swiss Re
Arranger	Swiss Re	Swiss Re	Swiss Re
Modelling Firm	RMS	RMS	RMS
SPV domicile	Cayman Islands	Cayman Islands	Cayman Islands
Size	\$ 300M	\$ 180M	\$ 275M
No.of Tranches	4	2	2
Issue date	I: Nov'09; II: May'10 III & IV: Oct 2010	July 2011	July 2012
Maturity	4 & 5 years	5 years	5 years
Index	I:US, UK; II:US/UK III: US/Japan, IV: Germany/ Canada	IV:Canada/ Germany(Ger.), V:Canada/Ger./ UK/US	D-1:Australia, Canada E-1:Australia, Canada, US

Table 3: The Middle Stage CAT Mortality Bonds (Contd...)

Prime Focus(6)

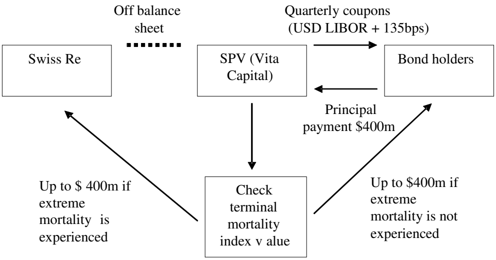
Specifications	Mythen Re	Atlas IX	VITA VI
Sponsor	Swiss Re	SCOR Re	Swiss Re
Arranger	Swiss Re	Aon, BNP Paribas, Natixis	Swiss Re
Modelling Firm	AIR/RMS	RMS	RMS
SPV domicile	Cayman Islands	Ireland	Cayman Islands
Size	$ 200M	$ 180M	$ 100M
No.of Tranches	2	2	1
Issue date	November 2012	September 2013	December 2015
Maturity	4 & 5 years	5 years	5 years
Index	U.S. hurricane, UK mortality	US	Australia, Canada, UK

Table 4: The Latest CAT Mortality Bonds

Prime Focus(7)

- Why Swiss Re Bond...?
  - An Innovative Security...one of its kind
  - A kind of pioneer and path setter
  - Shifted the risk exposure from the balance sheet to the capital markets
- Attracted lot of attention and was fully subscribed (Euroweek, 19 December 2003)
- Investors included a large number of pension funds
- Established a Special Purpose Vehicle (SPV) called VITA I for the securitization
- A 3-year bond issued in December 2003 with maturity on Jan 1, 2007
- Principal s.t. mortality risk defined in terms of an index q_i in yr t_i
- Quarterly coupons of three-month US-dollar LIBOR + 135 basis points
- Strength: Extreme Transparency

The Bond Mechanism



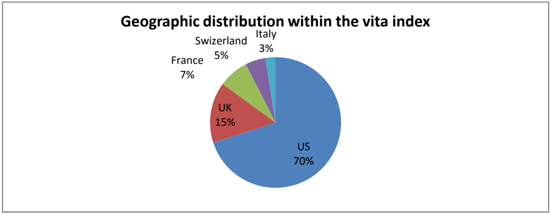
The Mortality Index

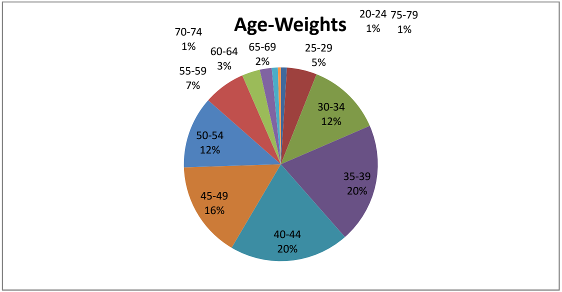
- Mortality index constructed as a weighted average of mortality rates (deaths per 100,000) over age, sex (male 65%, female 35%) and nationality (US 70%, UK 15%, France 7.5%, Italy 5%, Switzerland 2.5%)

$$q_i = \sum_j C_j \sum_k A_k \left(G^m q^m_{k,j,t_i} + G^f q^f_{k,j,t_i} \right)$$

- q^m_{k,j,t_i} and q^f_{k,j,t_i} = mortality rates (deaths per 100,000) for males and females respectively in the age group k for country j at time t_i
- C_j = weight attached to country j
- A_k = weight attributed to age group k (same for males and females)
- G^m and G^f = gender weights applied to males and females respectively
- q_0 = base index

Index Distribution





Design of the Swiss Re Bond(1)

Principal Loss Percentage

$$L_i = \begin{cases} 0 & \text{if } q_i \leq K_1 q_0 \\ \frac{(q_i - K_1 q_0)}{(K_2 - K_1) q_0} & \text{if } K_1 q_0 < q_i \leq K_2 q_0 \\ 1 & \text{if } q_i > K_2 q_0 \end{cases} \quad (1)$$

- For Swiss Re Bond: Trigger Point $K_1 = 1.3$ and Exhaustion Point $K_2 = 1.5$

Coupons

$$C0_j = \begin{cases} \left(\frac{SP + LI_j}{4}\right).C & \text{if } j = \frac{1}{4}, \frac{2}{4}, \ldots, \frac{11}{4}, \\ \left(\frac{SP + LI_j}{4}.C + X\right) & \text{if } j = 3, \end{cases} \quad (2)$$

- SP: Spread value (1.35%), LI_j: LIBOR rates, C: Face Value, X: a random variable

Design of the Swiss Re Bond(2)

- Proportion of the principal returned to the bondholders on the maturity date:

$$X = C\left(1 - \sum_{i=1}^{3} L_i\right)^{+}, \tag{3}$$

- $C = \$400$ million
- Risk-neutral price of the random pay-off at time 0 with Q as the EMM

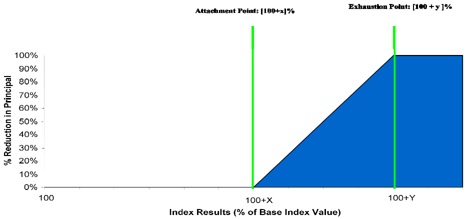
$$P = e^{-rT}\mathrm{E}_Q[X] \tag{4}$$

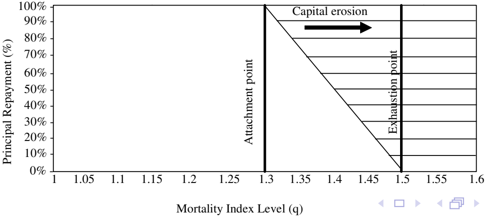
- r is nominal annual interest rate

Discounted Cashflow of Payments

$$DC\left(r\right) = \sum_{i=1}^{12} \frac{CO_{\frac{i}{4}}}{\left(1+\frac{r}{4}\right)^{i}} \tag{5}$$

Design of the Swiss Re Bond(3)





Pricing?

What is the main Problem?

Pricing the Swiss Re Bond with no closed form solution

What can be done?

An incomplete mortality market that has no arbitrage guarantees the existence of at least one risk-neutral measure termed the equivalent martingale measure Q that can be used for calculating fair prices of mortality securities

Our Approach for Bond Evaluation

Steps

- Adapt the payoff of the bond in terms of the payoff of an Asian put option
- Assume the existence of an Equivalent Martingale Measure (EMM)
- Present model-independent bounds
- Exploit comonotonic theory as illustrated in [Albrecher et al.(2008)Albrecher, Mayer, and Schoutens] for the pricing of Asian options
- Carry out Monte Carlo simulations to estimate the bond price under a variety of models
- Draw graphs of the bounds by varying the interest rate r and mortality rate q_0

Payoff as that of an Asian Put Option

Alternative form of writing Payoff

$$P = De^{-rT}\mathrm{E}\left[(q_0 - S)^+\right] \quad (6)$$

- $D = \frac{C}{q_0}$
- $S_i = 5\left(q_i - 1.3q_0\right)^+$
- $S = \sum_{i=1}^{3} S_i$

Call counterpart of the payoff

$$P_1 = De^{-rT}\mathrm{E}\left[(S - q_0)^+\right] \quad (7)$$

Put-call parity for the Swiss Re Bond

The relation

$$P_1 - P = De^{-rT}\left[5\sum_{i=1}^{3} e^{rt_i} C\left(1.3q_0, t_i\right) - q_0\right] \tag{8}$$

- Define

$$G = De^{-rT}\left[5\sum_{i=1}^{3} e^{rt_i} C\left(1.3q_0, t_i\right) - q_0\right] \tag{9}$$

- Bounding P_1 by bounds l_1 and u_1
- Corresponding bounds for the Swiss Re Mortality Bond:

$$(l_1 - G)^+ \leq P \leq (u_1 - G)^+ \tag{10}$$

Some Basic Concepts

Definition

Stop-loss Premium: The stop-loss premium with retention d of a random variable X is defined as $\mathbf{E}\left[(X-d)^+\right]$.

Definition

Stop-loss Order: Consider two random variables X and Y. Then X is said to precede Y in the stop-loss order sense, written as $X \leq_{sl} Y$, if and only if X has lower stop-loss premiums than Y:

$$\mathbf{E}\left[(X-d)^+\right] \leq \mathbf{E}\left[(Y-d)^+\right] \qquad -\infty < d < \infty \tag{11}$$

Definition

Convex Order: X is said to precede Y in terms of convex order, written as $X \leq_{cx} Y$, if and only if $X \leq_{sl} Y$ and $\mathbf{E}[X] = \mathbf{E}[Y]$.

Lower Bound for the Call Counterpart

Lower Bound using Jensen's Inequality

$$P_1 \geq De^{-rT}\mathbf{E}\left[\left(\sum_{i=1}^{n} 5\left(\mathbf{E}\left(q_i|\Lambda\right) - 1.3q_0\right)^+ - q_0\right)^+\right] \qquad (12)$$

- We define: $Z_i = 5\left(\mathbf{E}\left(q_i|\Lambda\right) - 1.3q_0\right)^+ ; i = 1, 2, ..., n$ & $Z = \sum_{i=1}^{n} Z_i$
- $S \geq_{sl} Z$ or $\mathrm{E}\left[(S - q_0)^+\right] \geq \mathrm{E}\left[(Z - q_0)^+\right]$
- The conditioning variable Λ is chosen in such a way that $\mathbf{E}\left[q_i|\Lambda\right]$ is either increasing or decreasing for every i
- This implies the vector: $\mathbf{Z^l} = (Z_1, \ldots, Z_n)$ is comonotonic & yields

Stop-loss lower bound for the call-counterpart

$$P_1 \geq De^{-rT}\sum_{i=1}^{n}\mathbf{E}\left[\left(5\left(\mathbf{E}\left(q_i|\Lambda\right) - 1.3q_0\right)^+ - F_{Z_i}^{-1}\left(F_Z\left(q_0\right)\right)\right)^+\right] \qquad (13)$$

The Trivial Lower Bound

- if the random variable Λ is independent of the mortality evolution $\{q_t\}_{t\geq 0}$ we get

The Trivial Lower Bound

$$P_1 \geq Ce^{-rT}\left(\sum_{i=1}^{n} 5\left(\exp\left(rt_i\right) - 1.3\right)^+ - 1\right)^+ =: \mathrm{lb}_0 \tag{14}$$

- Using

$$G = De^{-rT}\left[5\sum_{i=1}^{3} e^{rt_i} C\left(1.3q_0, t_i\right) - q_0\right] \tag{15}$$

- Corresponding bound for the Swiss Re Mortality Bond:

$$P \geq \left(\mathrm{lb}_0 - G\right)^+ =: \mathrm{SWLB}_0 \tag{16}$$

The Lower Bound $SWLB_1$

- We choose $\Lambda = q_1$ in (13)
- Use the martingale argument for the discounted mortality process

$$\mathbf{E}\left[q_i | q_1\right] = \mathbf{E}\left[e^{rt_i} e^{-rt_i} q_i | q_1\right] = e^{r\left(t_i - t_1\right)} q_1 .$$

The Lower Bound $SWLB_1$

$$P_1 \geq 5D \sum_{i=1}^{n} e^{-r(T-t_i)} C\left(q_0 . \max\left(x, \frac{1.3}{e^{r(t_i-t_1)}}\right),\ t_1\right) =: \mathrm{lb}_1 . \tag{17}$$

- where x is the solution of $\sum_{i=1}^{n}\left(e^{r\left(t_j - t_1\right)} x - 1.3\right)^{+} = 0.2$
- $C\left(K, t_1\right)$ is the price of a European call on the mortality index with strike K, maturity t_1 and current mortality index q_0

A Model-independent Lower Bound(1)

- Additional assumption that holds good for stationary exponential Lèvy models

$$\sum_{i=1}^{n} q_i \geq_{sl} \left(\sum_{i=1}^{j-1} q_0^{(1-t_i/t)} q_t^{t_i/t} + \sum_{i=j}^{n} e^{r(t_i-t)} q_t \right) \tag{18}$$

- for $0 \leq t \leq T$ and $j = \min\{i \,:\, t_i \geq t\}$
- We then use the following two results

Proposition

Let $(X,\ Y) \sim BVN\left(\mu_X, \mu_Y, \sigma_X^2, \sigma_Y^2, \rho\right)$, where BVN stands for bivariate normal distribution. The conditional distribution function of X, given the event $Y = y$, is given as

$$F_{X|Y=y}(x) = \Phi\left(\frac{x - \left(\mu_X + \rho\frac{\sigma_X}{\sigma_Y}(y - \mu_Y)\right)}{\sigma_X\sqrt{1-\rho^2}}\right) \tag{19}$$

A Model-independent Lower Bound(2)

Proposition

Let $W = (W_t), t \geq 0$ be a standard Brownian motion. Then the conditional expectation of W_{t_i} given W_t is given as

$$\boldsymbol{E}\left[W_{t_i} | W_t\right] = \frac{t_i}{t} W_t \quad \textit{for any } t_i < t$$

- The above proposition then leads to the following proposition

Proposition

The additional assumption (18) *holds for stationary exponential Lèvy models with mortality evolution $q_t = q_0 \exp\left(U_t\right)$, where $\left(U_t\right)_{t \geq 0}$ is a Lèvy process*

A Model-independent Lower Bound(3)

- We use this result to achieve the lower bound for the Asian-type call option

$$\sum_{i=1}^{n} 5\left(\mathbf{E}\left(q_i|q_t\right) - 1.3q_0\right)^+ = \sum_{i=1}^{j-1} 5q_0\left(\left(\frac{q_t}{q_0}\right)^{t_i/t} - 1.3\right)^+ + \sum_{i=j}^{n} 5q_0\left(\frac{q_t}{q_0}e^{r(t_i-t)} - 1.3\right)^+ =: S^{l_2}. \quad (20)$$

- S^{l_2} is the same as Z with Λ being replaced by q_t
- So we have $S \geq_{sl} S^{l_2}$

A Model-independent Lower Bound(4)

- Define $\mathbf{Y} = (Y_1, \ldots, Y_n)$ with

$$Y_i = \begin{cases} 5q_0\left(\left(\frac{q_t}{q_0}\right)^{t_i/t} - 1.3\right)^+ & i < j \\ 5q_0\left(\left(\frac{q_t}{q_0}\right)e^{r(t_i - t)} - 1.3\right)^+ & i \geq j \end{cases}$$

- $i = 1, 2, ..., n$
- $\mathbf{Y}$ is comonotonic:-components are strictly increasing functions of q_t
- By the comonotonic theory

$$\mathbf{E}\left[\left(S^{l_2} - q_0\right)^+\right] = \sum_{i=1}^{n} \mathbf{E}\left[\left(Y_i - F_{Y_i}^{-1}\left(F_{S^{l_2}}\left(q_0\right)\right)\right)^+\right] \qquad (21)$$

- where $F_{S^{l_2}}(q_0)$ is the distribution function of S^{l_2} evaluated at q_0

A Model-independent Lower Bound(5)

- such that for an arbitrary t, we have:

$$
\begin{aligned}
F_{S^{l_2}}(q_0) &= \mathbf{P}\left[S^{l_2} \leq q_0\right] \\
&= \mathbf{P}\left(\sum_{i=1}^{j-1}\left(\left(\frac{q_t}{q_0}\right)^{t_i/t} - 1.3\right)^+ \right. \\
&\qquad \left. + \sum_{i=j}^{n}\left(\left(\frac{q_t}{q_0}\right)e^{r(t_i-t)} - 1.3\right)^+ \leq 0.2\right) \quad (22)
\end{aligned}
$$

- Substitute x for q_t/q_0 in (22)
- where x solves

$$
\sum_{i=1}^{j-1}\left(x^{t_i/t} - 1.3\right)^+ + \sum_{i=j}^{n}\left(xe^{r(t_i-t)} - 1.3\right)^+ = 0.2 \qquad (23)
$$

- Then $S^{l_2} \leq q_0$ if and only if $q_t \leq xq_0$

A Model-independent Lower Bound(6)

- This yields

$$F_{S^{l_2}}(q_0) = F_{q_t}(xq_0) = \begin{cases} F_{Y_i}\left(5q_0\left(x^{t_i/t} - 1.3\right)^+\right) & i < j \\ F_{Y_i}\left(5q_0\left(xe^{r(t_i-t)} - 1.3\right)^+\right) & i \geq j \end{cases}$$

The Lower Bound $\mathrm{lb}_t^{(2)}$

$$\begin{aligned} P_1 \quad \geq \quad & 5De^{-rT}\left(\sum_{i=1}^{j-1} q_0^{1-t_i/t}\mathbf{E}\left[\left(q_t^{t_i/t} - q_0^{t_i/t}.\max\left(x^{t_i/t}, 1.3\right)\right)^+\right]\right. \\ & \left. + \sum_{i=j}^{n} e^{rt_i} C\left(q_0.\max\left(x, \frac{1.3}{e^{r(t_i-t)}}\right),\ t\right)\right) \\ =: \quad & \mathrm{lb}_t^{(2)} \end{aligned} \tag{24}$$

A Model-independent Lower Bound(7)

- $\mathrm{lb}_t^{(2)}$ is a lower bound for all t and can be maximized w.r.t. t to yield the optimal lower bound:

$$P_1 \geq \max_{0 \leq t \leq T} \mathrm{lb}_t^{(2)} \tag{25}$$

- As before, we have on using the put-call parity

$$P \geq \left(\mathrm{lb}_t^{(2)} - G\right)^+ =: \mathrm{SWLB}_t^{(2)} \tag{26}$$

Proposition

The payoff of the call option is a convex function[a] *of the strike price, i.e.,* $\boldsymbol{E}\left[(X-x)^{+}\right]$ *is convex in x.*

---

[a]A function $f : I \rightarrow \mathbb{R}$, where I is an interval in $\mathbb{R}$, is convex if and only if $f\left(ax+(1-a)\,y\right) \leq af\left(x\right)+(1-a)\,f\left(y\right) \quad \forall a \in [0,1]$ and any pair of elements $x, y \in I$.

- Define a vector $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_n)$ such that $\lambda_i \in \mathbb{R}$ and $\sum_{i=1}^{n} \lambda_i = 1$
- With the help of $\boldsymbol{\lambda}$ we can write the payoff of the Asian-type call option as

$$P_1 = Ce^{-rT}\mathrm{E}\left[\left(\sum_{i=1}^{n}\left(5\left(\frac{q_i}{q_0}-1.3\right)^{+}-\lambda_i\right)\right)^{+}\right]. \qquad (27)$$

- The above result for the call option implies

$$P_1 \leq 5De^{-rT}\sum_{i=1}^{n} e^{rt_i}C\left(q_0\left(1.3+\frac{\lambda_i}{5}\right), t_i\right) \qquad (28)$$

First Upper Bound for the Swiss Re Bond(2)

- Employing the Lagrangian with ϕ as the Lagrange's multiplier, we have

$$L(\boldsymbol{\lambda}, \phi) = \frac{5}{q_0}\sum_{i=1}^{n} e^{rt_i} C\left(q_0\left(1.3 + \frac{\lambda_i}{5}\right), t_i\right) + \phi\left(\sum_{i=1}^{n}\lambda_i - 1\right) \quad (29)$$

The Upper Bound ub_1

$$P_1 \leq 5De^{-rT}\sum_{i=1}^{n} e^{rt_i} C\left(F_{q_i}^{-1}(x), t_i\right) =: \mathrm{ub}_1 \quad (30)$$

- where $x \in (0,1)$ solves $\sum_{i=1}^{n} F_{q_i}^{-1}(x) = \frac{q_0}{5}(1 + 6.5n)$
- Put-Call parity yields: $P \leq (\mathrm{ub}_1 - G)^+ =: \mathrm{SWUB}_1$

First Upper Bound for the Swiss Re Bond (3)(Aliter)

- The same upper bound by using comonotonicity theory
- Define the comonotonic counterpart of $\mathbf{q} = (q_1, ..., q_n)$ as
- $\mathbf{q^u} = \left(F_{S_1}^{-1}(U), ..., F_{S_n}^{-1}(U) \right)$, $U \sim U(0,1)$
- Let

$$S^c = \sum_{i=1}^{n} F_{S_i}^{-1}(U) = \sum_{i=1}^{n} S_i^c. \tag{31}$$

- Clearly,

$$S \leq_{cx} S^c \tag{32}$$

- cx denotes convex ordering
- So

$$\mathbf{E}\left[\left(\sum_{i=1}^{n} S_i - q_0 \right)^+ \right] \leq \sum_{i=1}^{n} \mathbf{E}\left[\left(S_i - F_{S_i}^{-1}(F_{S^c}(q_0)) \right)^+ \right]. \tag{33}$$

- As a result, an upper bound for the call counterpart of the Swiss Re bond is given as

$$P_1 \leq 5De^{-rT} \sum_{i=1}^{n} e^{rt_i} C\left(1.3q_0 + \frac{F_{S_i}^{-1}\left(F_{S^c}\left(q_0\right)\right)}{5}, t_i\right) \tag{34}$$

- So the upper bound becomes

$$P_1 \leq 5De^{-rT} \sum_{i=1}^{n} e^{rt_i} C\left(1.3q_0 + \frac{F_{S_i}^{-1}\left(x\right)}{5}, t_i\right) \tag{35}$$

- $x \in (0, 1)$ is the solution of the equation

$$\sum_{i=1}^{n} F_{S_i}^{-1}\left(x\right) = q_0 \tag{36}$$

- In fact, this yields the same upper bound

$$P_1 \leq 5De^{-rT} \sum_{i=1}^{n} e^{rt_i} C\left(F_{q_i}^{-1}\left(x\right), t_i\right) =: \mathrm{ub}_1 \tag{37}$$

Improved Upper Bound for the Swiss Re Bond(1)

- A sharper upper bound is possible
- if we assume that some additional information concerning the stochastic nature of $(q_1, q_2, ..., q_n)$ is available
- That is, if we can find a random variable Λ, with a known distribution
- s.t. the individual conditional distributions of q_i given the event $\Lambda = \lambda$
- are known for all i and all possible values of λ
- Define

$$S^u = \sum_{i=1}^{n} F^{-1}_{S_i|\Lambda}(U) = \sum_{i=1}^{n} S_i^u \tag{38}$$

- Then

$$S \leq_{cx} S^u \leq_{cx} S^c \tag{39}$$

- Let $\mathbf{q^u} = (S_1^u, ..., S_n^u)$
- $\left(F^{-1}_{S_1|\Lambda=\lambda}, ..., F^{-1}_{S_n|\Lambda=\lambda}\right)$ is comonotonic, so that

$$F^{-1}_{S^u|\Lambda=\lambda}(p) = \sum_{i=1}^{n} F^{-1}_{S_i|\Lambda=\lambda}(p), \; p \in (0,1). \tag{40}$$

Improved Upper Bound for the Swiss Re Bond(2)

- It follows that, in this case

$$\sum_{i=1}^{n} F^{-1}_{S_i|\Lambda=\lambda}\left(F_{S^u|\Lambda=\lambda}(q_0)\right) = q_0. \tag{41}$$

- The tower property & the convex order relationship given by (39) yield

The upper bound $\mathrm{ub}_t^{(1)}$

$$5De^{-rT}\sum_{i=1}^{n}\int_{-\infty}^{\infty}\mathbf{E}\left[\left(q_i - F^{-1}_{q_i|\Lambda=\lambda}(x)\right)^{+}\middle|\Lambda=\lambda\right] dF_{\Lambda}(\lambda) =: \mathrm{ub}_t^{(1)} \tag{42}$$

- where $x \in (0,1)$ solves the equation

$$\sum_{i=1}^{n} F^{-1}_{q_i|\Lambda=\lambda}(x) = \frac{q_0}{5}(1+6.5n). \tag{43}$$

- This is is an upper bound for all t and minimise (42) over $t \in [0, T]$

Examples

Bounds for Black Scholes Case

- A Tight Lower Bound on lines of $\mathrm{SWLB}_t^{(2)}$
- Improved Upper Bound assuming dependence of Mortality index q_i on Brownian Motion

Bound for Transformed Gamma Distribution

A compact expression for $\mathrm{SWLB}_t^{(2)}$

A Lower Bound under Black-Scholes Model(1)

- Assume that the mortality evolution process $\{q_t\}_{t\geq 0}$ follows the Black-Scholes model written as $q_t = e^{U_t}$
- where
$$U_t = \log_e(q_0) + \left(r - \frac{\sigma^2}{2}\right)t + \sigma W_t^* \tag{44}$$
and $\{W_t^*\}_{t\geq 0}$ denotes a standard Brownian motion
- $$U_t \sim N\left(\log_e q_0 + \left(r - \frac{\sigma^2}{2}\right)t,\ \sigma^2 t\right) \tag{45}$$

Proposition

If $(X, Y) \sim BVN\left(\mu_X, \mu_Y, \sigma_X^2, \sigma_Y^2, \rho\right)$, the conditional distribution of the lognormal random variable e^X, given the event $e^Y = y$ is

$$F_{e^X|e^Y=y}(x) = \Phi\left(\frac{\log_e x - \left(\mu_X + \rho\frac{\sigma_X}{\sigma_Y}\left(\log_e y - \mu_Y\right)\right)}{\sigma_X\sqrt{1-\rho^2}}\right) \tag{46}$$

A Lower Bound under Black-Scholes Model(2)

- Given the time points t_i, t for each i
- let ρ be the correlation between U_{t_i} and U_t
- Then, $(U_{t_i}, U_t) \sim \text{BVN}\left(\mu_{U_{t_i}}, \mu_{U_t}, \sigma^2_{U_{t_i}}, \sigma^2_{U_t}, \rho\right)$
- where $\mu_{U_{t_i}}, \mu_{U_t}, \sigma^2_{U_{t_i}}$ and $\sigma^2_{U_t}$ are given by (46)
- Now $q_t = e^{U_t}$
- The distribution function of q_i conditional on the event $q_t = s_t$ is given as

$$F_{q_i|q_t=s_t}(x) = \Phi\left(a(x)\right)$$

- where $a(x)$ is given by

$$a(x) = \frac{\log_e x - \left(\log\left(q_0\left(\frac{s_t}{q_0}\right)^{\rho\sqrt{\frac{t_i}{t}}}\right) + \left(r - \frac{\sigma^2}{2}\right)\left(t_i - \rho\sqrt{t_i t}\right)\right)}{\sigma\sqrt{t_i\left(1-\rho^2\right)}}. \tag{47}$$

A Lower Bound under Black-Scholes Model(3)

- For the mortality evolution process $\{q_t\}_{t\geq 0}$ defined as $q_t = e^{U_t}$

$$\mathbf{E}\left(q_i|q_t\right) = \begin{cases} q_0\left(\frac{q_t}{q_0}\right)^{\frac{t_i}{t}} e^{\frac{\sigma^2 t_i}{2t}(t-t_i)} & t_i < t, \\ q_t e^{r(t_i-t)} & t_i \geq t. \end{cases} \tag{48}$$

- Use this result to achieve the lower bound for the Asian-type call option
-
- Define $\mathbf{Y} = (Y_1, \ldots, Y_n)$
- where

$$Y_i = \begin{cases} 5q_0\left(\left(\frac{q_t}{q_0}\right)^{t_i/t} e^{\frac{\sigma^2 t_i}{2t}(t-t_i)} - 1.3\right)^+ & i < j \\ 5q_0\left(\left(\frac{q_t}{q_0}\right) e^{r(t_i-t)} - 1.3\right)^+ & i \geq j \end{cases}$$

- $i = 1, 2, ..., n$
- $\mathbf{Y}$ is comonotonic

A Lower Bound under Black-Scholes Model(4)

- Define $S^{l_3} = \sum_{i=1}^{n} Y_i$
- By the comonotonic theory

$$\mathbf{E}\left[\left(S^{l_3} - q_0\right)^+\right] = \sum_{i=1}^{n} \mathbf{E}\left[\left(Y_i - F_{Y_i}^{-1}\left(F_{S^{l_3}}\left(q_0\right)\right)\right)^+\right] \tag{49}$$

- where $F_{S^{l_3}}\left(q_0\right)$ is the distribution function of S^{l_3} evaluated at q_0
- such that for an arbitrary t, we have:

$$\begin{aligned} F_{S^{l_3}}\left(q_0\right) &= \mathbf{P}\left[S^{l_3} \leq q_0\right] \\ &= \mathbf{P}\left(\sum_{i=1}^{j-1}\left(\left(\frac{q_t}{q_0}\right)^{t_i/t} e^{\frac{\sigma^2 t_i}{2t}(t-t_i)} - 1.3\right)^+ \right. \\ &\qquad \left. + \sum_{i=j}^{n}\left(\left(\frac{q_t}{q_0}\right) e^{r(t_i-t)} - 1.3\right)^+ \leq 0.2\right) \end{aligned} \tag{50}$$

A Lower Bound under Black-Scholes Model(5)

- Substitute x for q_t/q_0 in (50)
- where x solves

$$\sum_{i=1}^{j-1}\left(x^{t_i/t}e^{\frac{\sigma^2 t_i}{2t}(t-t_i)}-1.3\right)^+ + \sum_{i=j}^{n}\left(xe^{r(t_i-t)}-1.3\right)^+ = 0.2 \quad (51)$$

- Then $S^{l_3} \leq q_0$ if and only if $q_t \leq xq_0$
- This yields

$$F_{S^{l_3}}(q_0) = F_{q_t}(xq_0) = \begin{cases} F_{Y_i}\left(5q_0\left(x^{t_i/t}e^{\frac{\sigma^2 t_i}{2t}(t-t_i)}-1.3\right)^+\right) & i<j, \\ F_{Y_i}\left(5q_0\left(xe^{r(t_i-t)}-1.3\right)^+\right) & i \geq j \end{cases}$$

A Lower Bound under Black-Scholes Model(6)

- As a result we have:

$$
\begin{aligned}
P_1 \quad \geq \quad & 5De^{-rT}\left(\sum_{i=1}^{j-1} q_0^{1-t_i/t}\mathbf{E}\left(\left(q_t^{t_i/t} e^{\frac{\sigma^2 t_i}{2t}(t-t_i)}\right.\right.\right. \\
& \left.\left.\left. - q_0^{t_i/t}\left(1.3+\left(x^{t_i/t} e^{\frac{\sigma^2 t_i}{2t}(t-t_i)} - 1.3\right)^{+}\right)\right)^{+}\right) \\
& + \sum_{i=j}^{n} e^{rt_i} C\left(q_0\left(\frac{1.3}{e^{r(t_i-t)}} + \left(x - \frac{1.3}{e^{r(t_i-t)}}\right)^{+}\right), t\right)\Bigg)
\end{aligned}
$$

A Lower Bound under Black-Scholes Model(7)

- Denote the term within the first summation as E_1 and its value is given below.

$$\mathbf{E}_1 = 5q_0 \left(e^{rt_i} \Phi\left(d_{1ai}\right) - \left(1.3 + \left(x^{t_i/t} e^{\frac{\sigma^2 t_i}{2t}(t-t_i)} - 1.3\right)^{+}\right) \Phi\left(d_{2ai}\right)\right) \tag{52}$$

- where d_{2ai} and d_{1ai} are given respectively as

$$d_{2ai} = \frac{-\log_e\left(\frac{da_i}{q_0}\right) + \left(r - \frac{\sigma^2}{2}\right)t}{\sigma\sqrt{t}} \tag{53}$$

$$d_{1ai} = d_{2ai} + \sigma\frac{t_i}{\sqrt{t}} \tag{54}$$

- and da_i is given as

$$da_i = q_0 \left(\frac{1.3}{e^{\frac{\sigma^2 t_i}{2t}(t-t_i)}} + \left(x^{t_i/t} - \frac{1.3}{e^{\frac{\sigma^2 t_i}{2t}(t-t_i)}}\right)^{+}\right)^{t/t_i} \tag{55}$$

A Lower Bound under Black-Scholes Model(8)

- As a result we have

The Lower Bound $\mathrm{lb}_t^{(BS)}$

$$5De^{-rT}\left(\sum_{i=1}^{j-1} q_0\left(e^{rt_i}\Phi\left(d_{1ai}\right) - \max\left(1.3, x^{t_i/t} e^{\frac{\sigma^2 t_i}{2t}(t-t_i)}\right).\Phi\left(d_{2ai}\right)\right)\right.$$
$$\left.+\sum_{i=j}^{n} e^{rt_i} C\left(q_0 \max\left(\frac{1.3}{e^{r(t_i-t)}}, x\right), t\right)\right) =: \mathrm{lb}_t^{(BS)} \quad (56)$$

- The bound $\mathrm{lb}_t^{(BS)}$ can undergo treatment similar to $\mathrm{lb}_t^{(2)}$ in sense of maximization with respect to t yielding

$$P_1 \geq \max_{0\leq t\leq T} \mathrm{lb}_t^{(BS)} \quad (57)$$

The Upper Bound $\text{SWUB}_t^{(BS)}$ (1)

- SWUB_1 is improved if there exists Λ s.t. $\text{Cov}\left(X_i, \Lambda\right) \neq 0\ \forall i$.
- Suppose the mortality index $\{q_t\}_{t\geq 0}$ depends on an underlying standard Brownian motion $\{W_t\}_{t\in[0,T]}$
- Then

$$P_1 \leq 5De^{-rT}\sum_{i=1}^{n}\int_{-\infty}^{\infty}\mathbf{E}\left[\left(q_i - F^{-1}_{q_i|W_t=w}\left(x\right)\right)^+ \middle| W_t = w\right] d\Phi\left(\frac{w}{\sqrt{t}}\right) \tag{58}$$

- where x solves

$$\sum_{i=1}^{n} F^{-1}_{q_i|W_t=w}\left(x\right) = \frac{q_0}{5}\left(1 + 6.5n\right). \tag{59}$$

- An explicit formula for the conditional inverse distribution function of q_i given the event $W_t = w$, is provided by the following result

The Upper Bound $SWUB_t^{(BS)}$ (2)

Proposition

Under the assumptions of the Black-Scholes model, conditional on the event $W_t = w$, the conditional distribution function of q_i is given by

$$F^{-1}_{q_i|W_t=w} = \begin{cases} q_0 e^{\left(r-\frac{\sigma^2}{2}\right)t_i+\sigma\frac{t_i}{t}w+\sigma\sqrt{\frac{t_i}{t}(t-t_i)}\Phi^{-1}(x)} & i < j, \\ q_0 e^{\left(r-\frac{\sigma^2}{2}\right)t_i+\sigma w+\sigma\sqrt{(t_i-t)}\Phi^{-1}(x)} & i \geq j. \end{cases} \tag{60}$$

where $j = min\{i : t_i \geq t\}$.

- From equation (59), we then solve the following for x.

$$0.2 + 1.3n = \sum_{i=1}^{j-1} e^{\left(r-\frac{\sigma^2}{2}\right)t_i+\sigma\frac{t_i}{t}w+\sigma\sqrt{\frac{t_i}{t}(t-t_i)}\Phi^{-1}(x)} + \sum_{i=j}^{n} e^{\left(r-\frac{\sigma^2}{2}\right)t_i+\sigma w+\sigma\sqrt{(t_i-t)}\Phi^{-1}(x)} \tag{61}$$

- The improved upper bound for the call counterpart of the Swiss Re bond in the Black-Scholes case

The Upper bound $\text{ub}_t^{(BS)}$

$$P_1 \leq 5Ce^{-rT} \int_{-\infty}^{\infty} \left(\sum_{i=1}^{n} e^{\left(r - \frac{\sigma^2 (t_i \wedge t)^2}{2 t_i t}\right) t_i + \sigma \frac{t_i \wedge t}{t} w} \Phi\left(c_1^{(i)}\right) - (0.2 + 1.3n)(1 - x) \right) d\Phi\left(\frac{w}{\sqrt{t}}\right) =: \text{ub}_t^{(BS)} \quad (62)$$

- with

$$c_1^{(i)} = \begin{cases} \sigma \sqrt{\frac{t_i}{t}(t - t_i)} - \Phi^{-1}(x) & i < j, \\ \sigma \sqrt{(t_i - t)} - \Phi^{-1}(x) & i \geq j \end{cases} \quad (63)$$

- and $x \in (0, 1)$ solves equation (61)
- For optimal upper bound minimise (62) over $t \in [0, T]$

Log Gamma Distribution (1)

- Log Gamma distribution: a particular type of transformed Gamma distribution
- The mortality index 'q' follows log Gamma distribution if

$$\frac{\log_e q - \mu}{\sigma} = x \sim Gamma\,(p, a)\,, \tag{64}$$

- where μ, σ, p and a are parameters (> 0) and *log* is the natural logarithm
- Useful references for transformed gamma distribution are
  - [Johnson et al.(1994)Johnson, Kotz, and Balakrishnan]
  - [Vitiello and Poon(2010)]
  - [Cheng et al.(2014)Cheng, Tzeng, Hsieh, and Tsai]

The Lower Bound $\mathrm{lb}_t^{(LG)}$

$$5Ce^{-rT}\left(\sum_{i=1}^{j-1} q_0^{-t_i/t}\left(\frac{e^{\frac{t_i}{t}\mu}}{(\sigma^{''})^p}\left[1-G\left(d_2^{'},\ p,\sigma^{''}\right)\right]-K_1\left[1-G\left(d_2^{'},\ p\right)\right]\right)\right.$$
$$\left.+\sum_{i=j}^{n}\frac{e^{r(t_i-t)}}{q_0}\left(q_0e^{rt}\left[1-G\left(d_1,\ p\right)\right]-K_2\left[1-G\left(d_2,\ p\right)\right]\right)\right) \qquad (65$$

- s.t. $\sigma^{''}=1-\sigma^{'}\frac{t_i}{t}$, $\sigma^{'}=1-\left(q_0e^{rt-\mu}\right)^{1/p}$, $d_2^{'}=\frac{lnd_1^{'}-\mu}{\sigma}$,

  $d_1^{'}=q_0.\max\left(x^{t_i/t},1.3\right)^{t/t_i}$, $K_1=\left(d_1^{'}\right)^{t_i/t}$, $K_2=q_0.\max\left(x,\frac{1.3}{e^{r(t_i-t)}}\right.$

- $d_1=\frac{lnK_2-\mu}{q_0e^{rt-\mu}-1}$, $d_2=d_1+lnK_2-\mu$,

  $G\left(x,p\right)=\int_0^x\frac{1}{\Gamma\left(p\right)}x^{p-1}e^{-x}dx$, $G\left(x,p,\sigma^{''}\right)=\int_0^x\frac{\left(\sigma^{''}\right)^p}{\Gamma\left(p\right)}x^{p-1}e^{-\left(\sigma^{''}x\right)}d$

- Assume that the mortality evolution process $\{q_t\}_{t\geq 0}$ obeys the Black-Scholes model, specified by the following stochastic differential equation (SDE)

$$dq_t = rq_t dt + \sigma q_t dW_t.$$

- In order to simulate a path, we will consider the price of the asset on a finite set of $n = 3$ evenly spaced dates $t_1, ..., t_n$.

The Brownian Simulation

$$q_{t_j} = q_{t_{j-1}} \exp\left[\left(r - \frac{1}{2}\sigma^2\right)\delta t + \sigma\sqrt{\delta t}\, U_j\right] \quad U_j \sim N(0,1), \quad j = 1, 2, \ldots, n \tag{66}$$

Parameter choices in accordance with [Lin and Cox(2008)]

$$q_0 = 0.008453,\ r = 0.0,\ T = 3,\ t_0 = 0,\ n = 3,\ \sigma = 0.0388$$

Numerical Results(2)

Table 5: Table showing the various bounds and the Monte Carlo estimate for the B-S Model for varying values of r

r	SWLB0	SWLB1	SWLBt_(BS)	MC	SWUBt_(BS)	SWUB_(1)
0.035	0.899130889131400	0.899130889153152	0.899131577418890	0.899130939228525	0.899131588499602	0.899131637780299
0.03	0.913324024542464	0.913324024546338	0.913324256505855	0.913324120543246	0.913324317265175	0.913324320930395
0.025	0.927447505802074	0.927447505802722	0.927447580428344	0.927447582073642	0.927447605312234	0.927447619324390
0.02	0.941626342686440	0.941626342686542	0.941626365599735	0.941626356704134	0.941626369726985	0.941626384748977
0.015	0.955935721003105	0.955935721003120	0.955935727716106	0.955935715488521	0.955935732229503	0.955935736078305
0.01	0.970419124545862	0.970419124545864	0.970419126422140	0.970419112046475	0.970419126801821	0.970419129771609
0.005	0.985101139986133	0.985101139986134	0.985101140486345	0.985101142704466	0.985101140839740	0.985101141738075
0	0.999995778015617	0.999995778015617	0.999995778142797	0.999995730678518	0.999995778174612	0.999995778583618

Table 6: Table showing the various bounds and the Monte Carlo estimate for B-S Model for varying values of q0 when r=0.0

q0	SWLB0	SWLB1	SWLBt_(BS)	MC	SWUBt_(BS)	SWUB_(1)
0.007	0.999999999999517	0.999999999999517	0.999999999999517	1.000000000000000	0.999999999999517	0.999999999999517
0.008	0.999999915251651	0.999999915251651	0.999999915252175	0.999999935586330	0.999999915252765	0.999999915253115
0.008453	0.999995778015617	0.999995778015617	0.999995778142797	0.999995730678518	0.999995778174612	0.999995778583618
0.009	0.999821987943444	0.999821987949893	0.999822025862818	0.999816103328680	0.999822374801022	0.999822875816246
0.01	0.978292691034648	0.978310383929407	0.978503560221499	0.978738658827918	0.978292691184203	0.986262918346612
0.011	0.572750782003669	0.610962124257773	0.610962123857400	0.652440509314875	0.572755594265253	0.877336305501968
0.012	0.000000000000000	0.040209774144029	0.040209770810359	0.094615386163640	0.000000000000000	0.395672911251278
0.013	0.000000000000000	0.000000000000000	0.000000000000000	0.001662471990070	0.000000000000000	0.083466184427206
0.014	0.000000000000000	0.000000000000000	0.000000000000000	0.000003376858132	0.000000000000000	0.008942985848261

Figure1: Rel. Diff. of LBt(2), LBt(3) and UB1 w.r.t. MC estimate under Black-Scholes model

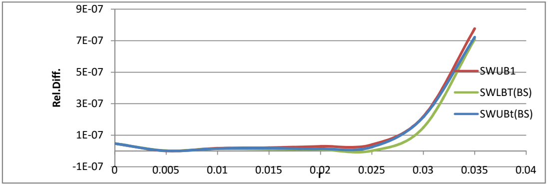
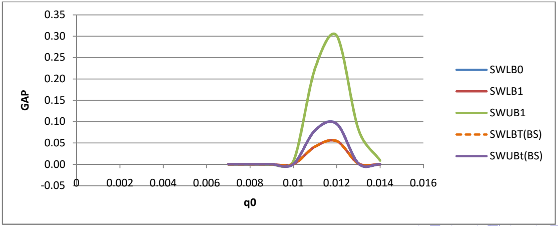
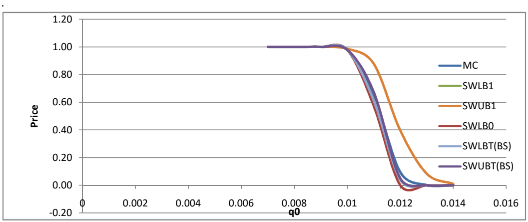


Figure2: Comparison of different bounds under B-S model in terms of difference from MC estimate for r=0



Numerical Results(4)

Figure3: Price Bounds under Black-Scholes model for the parameter choice of Lin and Cox(2008) Model



Numerical Results(5)

- Assume that the mortality rate 'q' obeys the four-parameter transformed Normal (S_u) distribution ([Johnson(1949)] and [Johnson et al.(1994)Johnson, Kotz, and Balakrishnan]) which is defined as follows

$$sinh^{-1}\left(\frac{q-\alpha}{\beta}\right) = x \sim N\left(\mu, \sigma^2\right), \tag{67}$$

- α, β, μ and σ are parameters ($\beta, \sigma > 0$) and $sinh^{-1}$ is the inverse hyperbolic sine function
- Let $q_0 = 0.008453$.

Parameter choices in accordance with [Tsai and Tzeng(2013)]

$$\alpha = [0.008399, 0.008169, 0.007905],\ \beta = [0.000298, 0.000613, 0.000904],$$

$$\mu = [0.70780, 0.58728, 0.58743] \text{ and } \sigma = [0.67281, 0.50654, 0.42218].$$

Numerical Results(6)

Table 7: Table showing the various bounds and the Monte Carlo estimate for the Su distn. for varying values of r

r	SWLB0	SWLB1	SWLBt_(2)	MC	SWUB_(1)
0.035	0.883255461690070	0.884321427701533	0.885548150428771	0.884689900254432	0.886806565750194
0.03	0.903403981322902	0.904010021303490	0.904693957669362	0.904223406591320	0.905481788284534
0.025	0.921607066867317	0.921935518850858	0.922291170234705	0.922030679117868	0.922759498340311
0.02	0.938407830148741	0.938576980453810	0.938747560828014	0.938598989786277	0.939010425491579
0.015	0.954287129640998	0.954369722665066	0.954444088119093	0.954415686472720	0.954582647473048
0.01	0.969639544072264	0.969677756802278	0.969706604342752	0.969683647401862	0.969774875755017
0.005	0.984762743262391	0.984779521693468	0.984789115794819	0.984784143645972	0.984820459036106
0	0.999861354235404	0.999868375732131	0.999870879263060	0.999871208429012	0.999884274666239

Note: LBt2 obtained by Numerical Integration in MATLAB

Numerical Results(7)

- Assume that the mortality index $\{q_t\}_{t\geq 0}$ follows log gamma distribution, which is defined as

$$\frac{\log_e q - \mu}{\sigma} = x \sim Gamma\,(p, a)\,, \tag{68}$$

- μ, σ, p and a are parameters (> 0) and *log* is the natural logarithm.

Parameter choices in accordance with [Cheng et al.(2014)Cheng, Tzeng, Hsieh, and Tsai]

$$q_0 = 0.0088,\;\; p = [61.6326, 64.2902, 71.8574],\;\; a = [0.0103, 0.0098, 0.0080],$$

$$\mu = [-5.2452, -5.4600, -5.7238] \;\&\; \sigma = [7.4\times10^{-5}, 9.5\times10^{-5}, 9.4\times10^{-5}].$$

Table 8: Table showing the various bounds and the Monte Carlo estimate for the TGD for varying values of r

r	SWLB0	SWLB1	SWLBt_(LG)	MC	SWUB_(1)
0.035	0.848032774815386	0.848424044789595	0.855969730838120	0.854167495146694	0.866104360048102
0.03	0.873577023530120	0.873813448730075	0.879110918002518	0.878026709161428	0.887240130128182
0.025	0.897102805167311	0.897242672828637	0.900881660116024	0.900486935407607	0.907283088296566
0.02	0.918896959516680	0.918977921696450	0.921421185492669	0.921030195923945	0.926366403382851
0.015	0.939240965473512	0.939286791778834	0.940888331577441	0.941092453291025	0.944633306794068
0.01	0.958403723325991	0.958429070673721	0.959452704642603	0.959485386731500	0.962230654369936
0.005	0.976635430514097	0.976649121750369	0.977286229664468	0.977322136744823	0.979302971604630
0	0.994162849651329	0.994170066410978	0.994555652671267	0.994698510160850	0.995987334249625

Table 9: Table showing the various bounds and the Monte Carlo estimate for the TGD for varying values of q0 when r=0.0

q0	SWLB0	SWLB1	SWLBt_(LG)	MC	SWUB_(1)
0.008	0.999766066714250	0.999766066846378	0.999772840361840	0.999793281501976	0.999779562416927
0.0088	0.994162849651329	0.994170066410978	0.994555652671267	0.994686720834666	0.995987334249625
0.009	0.989104987070782	0.989146149900171	0.989952105692831	0.990012775482666	0.993383346707654
0.01	0.876692543049394	0.888049181229988	0.896376305638172	0.891609413787780	0.958189590378894
0.011	0.410971060715423	0.596089667856852	0.596089667856850	0.568675584083477	0.837207974723077
0.012	0.000000000000000	0.271045973759684	0.271045973759680	0.207081909248152	0.613838720959082
0.013	0.000000000000000	0.082740708460284	0.082740708460278	0.045779872978350	0.381822437530697
0.014	0.000000000000000	0.012702023135424	0.012702023135418	0.006694089213835	0.212229375394606
0.015	0.000000000000000	0.000000000000000	0.000000000000000	0.000883157235603	0.110420349200491
0.016	0.000000000000000	0.000000000000000	0.000000000000000	0.000084710725625	0.055539272590864
0.017	0.000000000000000	0.000000000000000	0.000000000000000	0.000004497045497	0.027576845294053
0.018	0.000000000000000	0.000000000000000	0.000000000000000	0.000000019842250	0.013697961782757

Numerical Results(9)

Figure 4: Rel. Diff. of Lower Bounds and UB1 w.r.t. MC estimate under Transformed Gamma Distribution

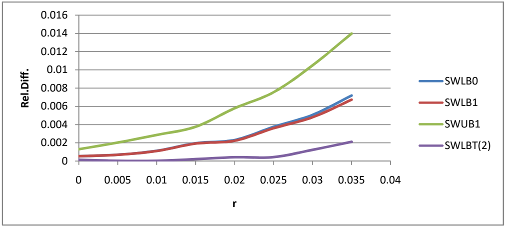
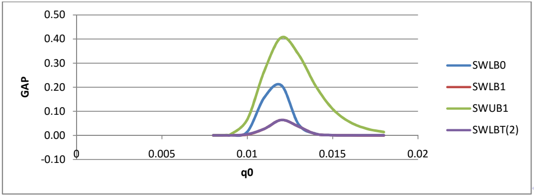
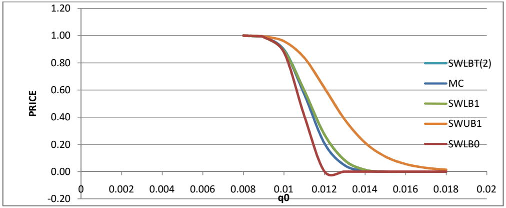


Figure 5: Comparison of different bounds under Transformed Gamma distn in terms of difference from MC estimate for r=0



Numerical Results(10)

Figure 6: Price Bounds under Transformed Gamma Distn. for the parameter choice of Lin and Cox(2008) Model



What did Swiss Re achieve?

- Swiss Re thrives from Life Insurance Business
  - It achieved Mortality Risk Transfer
  - Protection against extreme mortality events
  - Got counter parties to offload mortality risk
  - No dependence on retrocessionaire
- Profitability negatively correlated to mortality rates
- Methodology: Catastrophic bond with loss measurement based on a parametric index
- Investors in the bond took opposite position
- Received an enhanced return if an extreme mortality event doesn't occur

What Lies Ahead...?

- Mortality risk transfer expected to become more of a concern for life insurers and reinsurers
- Under Solvency II access to fully collateralized ILS capacity beneficial on a capital efficiency basis
- More such transactions predicted in the future
- ILS investors pleased to see VITA VI
  - A new extreme (or excess) mortality catastrophe bond deal
  - Keen to access the diversification it can offer
  - The fact that it is Swiss Re again welcomed
- The giant has transferred over $ 2.2 billion of mortality risk to the capital market
- A lot of variations being tried
- Swiss Re has experimented with
  - Longevity Trend Bond - KORTIS (2010)
  - Multiple Peril Bond - MYTHEN RE (2012)
- A more transparent and liquid Longevity and mortality market is emerging (since the formation of LLMA (2010))

Further Research

- This research is a crucial breakthrough in the pricing of catastrophic mortality bonds
- Model-independent bounds give freedom of choice for selecting mortality models
- Only one earlier publication by [Huang et al.(2014)Huang, Tsai, Yang, and Cheng] in direction of price bounds for the Swiss Re bond
- These authors propose gain-loss bonds that suffer from model risk
- The present scenario poised for further research
- Deriving even more tighter upper bound
- Using these bounds for the *Longevity Trend Bond* - KORTIS
- The success of our research hinges upon the trading of vanilla options written on the mortality index

Benefits

- TO THE BOND ISSUER
  - Securing protection for insurance liabilities when claims are horrendous
  - Multiyear coverage compared to 1-year given by stop-loss reinsurance
  - Gaining access to capital from investors which is used to generate further returns
  - Flexibility to access capital markets when required by using shelf programs
  - A kind of 'ALTERNATIVE RISK TRANSFER'
- TO THE BUYER
  - High yields offered from these bonds
  - Diversification to the portfolio
  - A type of charity for the rich
- A WIN-WIN situation for both
- One phrase to summarize these bonds: 'HIGH RISK HIGH REWARD'

A Few Disadvantages

- Significant up-front transaction costs
  - Legal
  - Risk Modeling
  - Broker
  - Rating agency
  - Bank fees
- that require minimum transaction sizes for the issuance to be economical
- 'BASIS RISK'
  - since the payoff trigger is index based
  - and the actual loss suffered is unlikely to be perfectly matched by the bond payoff
- Capital Credit given by regulators and rating agencies may be reduced for CATM's in comparison to traditional reinsurance
- Terms are fixed throughout the duration of coverage but can be adjusted for traditional reinsurance every year allowing for short term commitment and flexibility

The Modeling Aspect

- Pricing of the CAT mortality bonds depends on the estimation and forecast of mortality rates
- The development of new catastrophic mortality bonds and longevity-linked securities is
- Aided by and in turn encouraged the development of increasingly sophisticated 'Mortality Models'
- Many stochastic models are being proposed
- Experimentation being done with the celebrated
  - Lee-Carter Model ([Lee and Carter(1992)])
  - CBD Model ([Cairns et al.(2006)Cairns, Blake, and K.])
- Mortality modeling with Lévy Processes very popular
- Mortality jumps are being incorporated in these models
- Examples of Mortality Models
  - DEJD: Double Exponential Jump Diffusion ([Deng et al.(2012)Deng, Brockett, and MacMinn])
  - Geometric Brownian Motion with log-normal jump size distribution ([Lin and Cox(2008)])

> "If there will be one day such a severe world-wide pandemic that one of the bonds I bought will be triggered, there will be more important things to look after than an investment portfolio."
>
> — ANONYMOUS CATM INVESTOR

Thanks!

Bibliography I

H. Albrecher, P. A. Mayer, and W. Schoutens.
General Lower Bounds for Arithmetic Asian Option Prices.
Applied Mathematical Finance, 15(2):123–149, 2008.

D. Bauer.
Stochastic Mortality Modelling and Securitizaton of Mortality Risk.
ifa-Verlag, Ulm, Germany, 2008.

A.J.G. Cairns, D. Blake, and Dowd K.
A Two-factor Model for Stochastic Mortality with Parameter Uncertainty: Theory and Calibration.
Journal of Risk and Insurance, 73:687–718, 2006.

H. W. Cheng, C.-F. Tzeng, M.-H. Hsieh, and J. T. Tsai.
Pricing of Mortality-Linked Securities with Transformed Gamma Distribution.
Academia Economic Papers, 42(2):271–303, 2014.

Y. Deng, P.L. Brockett, and R.D. MacMinn.
Longevity/Mortality Risk Modeling and Securities Pricing.
The Journal of Risk and Insurance, 79(3):697–721, 2012.

Y.L. Huang, J.T. Tsai, S.S. Yang, and H.W. Cheng.
Price Bounds of Mortality-linked Security in Incomplete Insurance Market.
Insurance: Mathematics and Economics, 55:30–39, 2014.

N.L. Johnson.
Systems of Frequency Curves Generated by Methods of Translation.
Biometrika, 36:149–176, 1949.

N.L. Johnson, S. Kotz, and N. Balakrishnan.
Continuous Univariate Distributions, Vol. 1.
Wiley, New York, 1994.

R.D. Lee and L.R. Carter.
Modeling and Forecasting U.s. Mortality.
Journalof the American Statistical Association, 87:659–675, 1992.

Y. Lin and S.H. Cox.
Securitization of Mortality Risks.
Insurance: Mathematics and Economics, 42:628–637, 2008.

A. Stracke and W. Heinen.
Influenza Pandemic: The impact on an Insured Lives Life Insurance Portfolio.
The Actuary June, 2006.

J. Toole.
Potential Impact of Pandemic Influenza on the Us Life Insurance Industry.
Research Report, Society of Actuaries, 2007.

Bibliography IV

J.T. Tsai and L.Y. Tzeng.
Pricing of Mortality-linked Contigent Claims: an Equilibrium Approach.
Astin Bulletin, 43(2):97–121, 2013.

L. Vitiello and S.-H. Poon.
General Equilibrium and Preference Free Model for Pricing Options under Transformed Gamma Distribution.
Journal of Future Markets, 30(5):409–431, 2010.