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# THE RELATIONSHIP BETWEEN GROSS AND NET YIELDS TO REDEMPTION

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1. With reference to the Actuarial Note by Messrs Hathaway, Rickard & Woods (HR&W) *J.I.A.* **113,** 509–520 there is a simpler way, which is much easier to calculate, of approximating from the gross yield to redemption to the net yield, with taxation on the basis assumed by HR&W.

2. From expression (2) in that note by rearranging we get

$$r_G = \frac{I}{P} + \frac{Q}{P} \cdot \frac{r_G}{(1+r_G)^n - 1} = \frac{I}{P} + \frac{Q}{P} \cdot \frac{1}{s_{\overline{n}|}^{r_G}} \tag{1}$$

and from expression (4) in that note by rearranging we get

$$r_N = \frac{I}{P}(1-t) + \frac{Q}{P}(1-t) \cdot \frac{r_N}{(1+r_N)^n - 1} = \frac{I}{P}(1-t) + \frac{Q}{P}(1-t)\frac{1}{s_{\overline{n}|}^{r_N}} \tag{2}$$

These two formulae for the gross and net yields can be very useful because if $Q$ is fairly small the term $(Q/P)\, s_{\overline{n}|}$ often contributes only a small part of $r$, so that a reasonable approximation to $s_{\overline{n}|}$ used instead of the true value will not introduce serious error in the value of $r$.

3. However, it is more usual and simpler to obtain $r$ by the use of a table of Bond Values. These conveniently provide $r_G$ but their use to find $r_N$ is seldom convenient, so the use of a table of Bond Values leaves us with the problem of finding $r_N$ from $r_G$.

From (1) $$r_G - (I/P) = (Q/P)/s_{\overline{n}|}^{r_G} \tag{3}$$

Hence $$s_{\overline{n}|}^{r_G} = (Q/P)/\{r_G - (I/P)\} \tag{4}$$

Quite a reasonable approximation to $s_{\overline{n}|}^{r_N}$ is given by

$$s_{\overline{n}|}^{r_N} \doteqdot n + (1-t)\{s_{\overline{n}|}^{r_N} - n\}$$

$$= t\,n + (1-t)s_{\overline{n}|}^{r_G} = t\,n + (Q/P)(1-t)/\{r_G - (I/P)\} \tag{5}$$

using this in (2) we get

$$r_N \doteqdot (I/P)(1-t) + (Q/P)(1-t)/[t\,n + (Q/P)(1-t)/\{r_G-(I/P)\}]$$

$$= (I/P)(1-t) + \{r_G-(I/P)\} \times \left[\frac{(Q/P)(1-t)}{(Q/P)(1-t) + t\,n\{r_G-(I/P)\}}\right] \tag{6}$$

The corresponding formula for getting $r_G$ from $r_N$ is

$$r_G \doteqdot (I/P) + \{r_N-(I/P)(1-t)\} \times \left[\frac{(Q/P)(1-t)}{(Q/P)(1-t)-t\,n\{r_N-(I/P)(1-t)\}}\right] \tag{7}$$

4. These formulae are suitable for easy calculation as the following example shows, taking HR&W's illustration on page 511 for a $ bond with a coupon payable annually at 16% redeemable at par after 3 years. According to HR&W the gross yield to redemption is 18·311% and the net yield is 12·508% with tax at 32% on both the coupon and the excess of redemption value over the purchase price. The calculations using (6) and (7) are as follows:

| | | | | | |
|---|---|---|---|---|---|
| (i) | $r_G$% | 18·311 | (i) | $r_N$% | 12·508 |
| (ii) | $(I/P)$% | 16·842 | (ii) | $(I/P)(1-t)$% | 11·453 |
| (iii) | (i) − (ii) | 1·469 | (iii) | (i) − (ii) | 1·055 |
| (iv) | $(Q/P)(1-t)$% | 3·5789 | (iv) | $(Q/P)(1-t)$% | 3·5789 |
| (v) | $t\,n$ × (iii) | 1·4102 | (v) | $t\,n$ × (iii) | 1·0128 |
| (vi) | (iv) + (v) | 4·9891 | (vi) | (iv) − (v) | 2·5661 |
| (vii) | {(iii) × (iv)}/(vi) | 1·054 | (vii) | {(iii) × (iv)}/(vi) | 1·471 |
| (viii) | $(I/P)(i-t)$% | 11·453 | (viii) | $(I/P)$% | 16·842 |
| (ix) | (vii) + (viii) | 12·507 | (ix) | (vii) + (viii) | 18·313 |

5. The error in $r_N$ is only ·001 in the value % which shows that the approximation is acceptable at least in this case which might not be so if either $r$ or $n$ or $Q/P$ or any of them were much larger. However, if you have to get $r_N$ from $r_G$ the above is a perfectly practicable approximation for calculation purposes.

6. The reverse process of finding $r_G$ from $r_N$ is not quite so satisfactory (though probably acceptable) since the reverse process tends to exaggerate the effect of even slight errors in $r_N$, even the error involved in rounding off. In this example if $r_N$ were taken as 12·507% (the approximate value obtained from $r_G$) this would have reproduced $r_G$ exactly to 5 significant figures. So an error of ·001 is doubled by the reverse process. However, it is extremely unlikely in practice that anyone will want to find $r_G$ from $r_N$.

7. The approximate Net Yield to Redemption per cent for examples (i) to (viii) of Table 1 of HR&W's note have been recalculated from the Gross Yield to Redemption per cent by the above method and are given in Table 1 to this note, with, for comparison, the true rate from HR&W and their second approximation.

8. However, attention is invited to the method of calculating yields set out in the Addendum (*J.I.A.* **93,** 292–295)—the 1967 formula. That method is very

much simpler, easier and quicker than the one proposed by HR&W, it also copes effortlessly with the very frequent cases where the tax on capital gains is not at the same rate as that on the interest. The values given by this method are also given in Table 1. It will be seen that there is little to choose, but that the values by the 1967 formula are on the whole better.

9. The gross yields calculated using the 1967 formula together with the true gross yields given by HR&W are given in Table 2. It is seen that in some cases where the rate of interest is very high (over 25%) the error in the values by the 1967 formula tend to be unacceptable.

10. The 1967 formula was published when interest rates were still sufficiently low for the error due to the remainder terms to be small enough to be neglected. The general rise in interest rates due to inflation has meant that this may sometimes no longer be the case. There is therefore a need for a method by which the accuracy of a rate may be conveniently checked when that accuracy may be in doubt.

11. One way to do this is to go back to the thoretical basis of the 1967 formula. Reverting to the Text Book notation

$$(g-i)=k/a_{\overline{n}|}$$

if the reciprocal of $a_{\overline{n}|}$ is expanded in terms of $i$ and terms in $i^2$ and above are neglected we get the Text Book formula. By taking the term involving $i^2$ into account the 1967 formula was evolved. Though greater accuracy might be obtained by taking $n/a_{\overline{n}|}$ to more terms this would still leave uncertain the point at which the resultant rate of interest became unreliable, besides complicating the calculation. If instead of expanding the reciprocal of $a_{\overline{n}|}$ in terms of $i$ a different manipulation is tried the formula for the rate of interest can, as shown later, be expressed as

$$i=\left(g-\frac{k}{n}\right)\bigg/\left\{1+k\left(1+\frac{1}{(1+i)^n-1}-\frac{1}{ni}\right)\right\} \tag{8}$$

12. Now with the aid of a pocket electronic calculator using the $y^x$ key, the value of $1/\{(1+i)^n-1\}$ may be obtained in a few seconds. If therefore it is desired to verify a value $i_1$ the R.H.S. of (8) is easily calculated using $i_1$ and if $i_1$ is indeed the correct rate it will be reproduced. If, however, it is not, this calculation will produce a rate $i_2$ which is a better approximation to the true rate than $i_1$. In the cases in Table 2 where the value by the 1967 formula to 2 decimal places % differed from the HR&W values, the first application of (8) produced their rates to 2 decimal places.

13. If to start with only a rough approximation $i_0$ is available a better value $i_1$ may be obtained by the use of (8) and $i_1$ may be used to find a better value $i_2$ and so on. A bad starting value merely adds one or two repetitions to the iteration process which is reasonably quick. The improvement in $i$ by using (8) is first order so that if three successive values of $i$ are available exponential extrapolation may be used to find a very close approximation to the true rate. For the benefit of

Table 1. *Approximations to NYR given GYR and various bond parameters (price, coupon rate and duration) for various marginal tax rates on the following bases:*

(a) *Correct NYR as given by HR&W*
(b) *Approximate NYR by (6)*
(c) *Approximate NYR by 1967 formula*
(d) *HR&W's 2nd approximation to NYR*

(i) $P=80$, $Q=20$, $I=10$, $n=4$, $r_G=17{\cdot}34\%$

| Tax rate | (a) | (b) | (c) | (d) |
|---|---|---|---|---|
| ·32 | 12·05 | 12·05 | 12·06 | 12·05 |
| ·46 | 9·67 | 9·67 | 9·67 | 9·67 |
| ·60 | 7·24 | 7·24 | 7·24 | 7·24 |

(ii) $P=80$ $Q=20$ $I=10$, $n=10$, $r_G=13{\cdot}81\%$

| Tax rate | (a) | (b) | (c) | (d) |
|---|---|---|---|---|
| ·32 | 9·59 | 9·55 | 9·59 | 9·58 |
| ·46 | 7·70 | 7·66 | 7·70 | 7·69 |
| ·60 | 5·77 | 5·73 | 5·77 | 5·76 |

(iii) $P=80$, $Q=20$, $I=20$, $n=4$ $r_G=29{\cdot}09\%$

| Tax rate | (a) | (b) | (c) | (d) |
|---|---|---|---|---|
| ·32 | 20·16 | 20·13 | 20·16 | 20·13 |
| ·46 | 16·16 | 16·13 | 16·16 | 16·13 |
| ·60 | 12·09 | 12·06 | 12·09 | 12·07 |

(iv) $P=80$, $Q=20$, $I=20$, $n=10$ $r_G=25{\cdot}73\%$

| Tax rate | (a) | (b) | (c) | (d) |
|---|---|---|---|---|
| ·32 | 17·73 | 17·65 | 17·77 | 17·64 |
| ·46 | 14·19 | 14·10 | 14·21 | 14·11 |
| ·60 | 10·61 | 10·53 | 10·61 | 10·55 |

(v) $P=120$, $Q=-20$, $I=10$, $n=4$, $r_G=4{\cdot}43\%$

| Tax rate | (a) | (b) | (c) | (d) |
|---|---|---|---|---|
| ·32 | 2·96 | 2·96 | 2·96 | 2·96 |
| ·46 | 2·33 | 2·33 | 2·33 | 2·33 |
| ·60 | 1·71 | 1·71 | 1·71 | 1·71 |

(vi) $P=120$, $Q=-20$, $I=10$, $n=10$, $r_G=7{\cdot}13\%$

| Tax rate | (a) | (b) | (c) | (d) |
|---|---|---|---|---|
| ·32 | 4·76 | 4·77 | 4·76 | 4·76 |
| ·46 | 3·74 | 3·76 | 3·74 | 3·74 |
| ·60 | 2·74 | 2·75 | 2·74 | 2·75 |

(vii) $P=120$, $Q=-20$, $I=20$ $n=4$ $r_G=13{\cdot}24\%$

| Tax rate | (a) | (b) | (c) | (d) |
|---|---|---|---|---|
| ·32 | 8·85 | 8·86 | 8·85 | 8·85 |
| ·46 | 6·97 | 6·98 | 6·97 | 6·98 |
| ·60 | 5·12 | 5·13 | 5·12 | 5·13 |

(viii) $P=120$ $Q=-20$, $I=20$, $n=10$ $r_G=15{\cdot}88\%$

| Tax rate | (a) | (b) | (c) | (d) |
|---|---|---|---|---|
| ·32 | 10·64 | 10·68 | 10·63 | 10·67 |
| ·46 | 8·39 | 8·44 | 8·39 | 8·41 |
| ·60 | 6·16 | 6·21 | 6·16 | 6·19 |

Table 2. *Comparison between the values of* $r_G\%$:

(a) *as given by HR&W and*
(b) *by the 1967 formula*

| | (a) | (b) | | (a) | (b) |
|---|---|---|---|---|---|
| (i) | 17·34 | 17·36 | (v) | 4·43 | 4·43 |
| (ii) | 13·81 | 13·83 | (vi) | 7·13 | 7·13 |
| (iii) | 29·09 | 29·17 | (vii) | 13·24 | 13·24 |
| (iv) | 25·73 | 29·91 | (viii) | 15·88 | 15·83 |

those not familiar with this very useful device the process is as follows: if $x_1$, $x_2$ and $x_3$ are successive approximations to $x$, $x_{r+1}$ being found from $x_r$ then

$$\sum_{r=3}^{r=\infty} \Delta x_r \doteqdot (\Delta x_2)^2 \div (\Delta^2 x_1).$$

The true value of $x$ is found by adjusting $x_3$ by the true value of

$$\sum_{r=3}^{r=\infty} \Delta x_r,$$

so a close approximation to this will yield a close approximation to $x$.

14. Table 3 gives the results of calculating the yields (convertible half-yearly) by the 1967 formula and two successive applications of (8) and by exponential extrapolation on these three values for a bond with a coupon rate of 5% per annum payable half-yearly for unexpired terms of 10, 20, 30 and 40 years where the true value of $i^{(2)}$ is 10%. These results are given in the form of the error in the rate %, this being the most convenient way to show how far the formulae can be relied upon. These examples were chosen because the large difference between the coupon rate and the yield and hence the large value of $k$ imposes a severe test on the formulae. With a smaller difference and hence a smaller value of $k$ the error would be correspondingly reduced.

15. The different manipulation of $(g-i)=k/a_{\overline{n}|}$ referred to in § 11, remembering that $k$ is the excess of the purchase price over the redemption price, yields,

$$(g-i) = \frac{k}{a_{\overline{n}|}} = k\left(\frac{1}{n} + \frac{1}{a_{\overline{n}|}} - \frac{1}{n}\right) = \frac{k}{n} + k\left(\frac{1}{a_{\overline{n}|}} - \frac{1}{n}\right)$$

$$= \frac{k}{n} + k\left(i + \frac{1}{s_{\overline{n}|}} - \frac{1}{n}\right) = \frac{k}{n} + i\,k\left(1 + \frac{1}{i\,s_{\overline{n}|}} - \frac{1}{ni}\right)$$

Hence

$$g-\frac{k}{n}=i+ik\left\{1+\frac{1}{(1+i)^n-1}-\frac{1}{ni}\right\}$$

Hence

$$=\left(g-\frac{k}{n}\right)\Big/\left[1+k\left\{1+\frac{1}{(1+i)^n-1}-\frac{1}{ni}\right\}\right] \qquad (8)$$

This is more convenient for calculation purposes than its algebraic equivalent

$$i=\left(g-\frac{k}{n}\right)\Big/\left[1+k\left\{\frac{1}{1-v^n}-\frac{1}{ni}\right\}\right]$$

16. Using Text Book notation $g$ and $k$ are both per unit of the net redemption value. If, however, a bond is not redeemed at par or if capital gains tax is levied at redemption, the value of these, particularly of $g$, will not be a convenient figure for easy calculation. Further bond interest is almost invariably paid half-yearly, not annually, and what is required is a rate per annum convertible half-yearly. Expressing the net Redemption value as $R$ and multiplying both top and bottom of (8) by $2nR$, as for the 1967 formula, with $g^{(2)}$ as the nominal net coupon rate per annum payable half-yearly we get

$$i_2^{(2)}=(2n\,g^{(2)}-2k)\Big/\left[2nR+k\left\{2nR\left(1+\frac{1}{(1+\frac{1}{2}i_1^{(2)})^{2n}-1}\right)-\frac{2R}{i_1^{(2)}}\right]\right. \qquad (9)$$

Table 3. *The errors in the yield % p.a. convertible half-yearly of a bond with a coupon rate of 5% p.a. payable half-yearly where the price is such that the true yield is 10% p.a. for yields calculated by the following methods:*

(a) *The 1967 formula (1967,* $i_1^{(2)}$*)*
(b) *Formula (8) using 1967,* $i_1^{(2)}$ *as the trial rate (1988(8),* $i_2^{(2)}$*)*
(c) *Formula (8) using 1988(8),* $i_2^{(2)}$ *as the trial note (1988(8)* $i_3^{(2)}$*)*
(d) *By Exponential Extrapolation on* $i_1^{(2)}$, $i_2^{(2)}$ *and* $i_3^{(2)}$ *(Exp. Ext(8))*
(e) *Formula (i) using 1967,* $i_1^{(2)}$ *as the trial rate (1988(1)* $i_2^{(2)}$*)*
(f) *Formula (i) using 1988(i)*$i_2^{(2)}$ *as the trial rate (1988 (1)* $i_3^{(2)}$*)*
(g) *By Exponential Extrapolation on* $i_1^{(2)}$, $i_2^{(2)}$ *and* $i_3^{(2)}$ *(Exp. Ext. (1))*

| Term→ | 10 years | 20 years | 30 years | 40 years |
|---|---|---|---|---|
| (a) (1967, $i_1^{(2)}$) | $3{\cdot}072, 0\times10^{-3}$ | $-3{\cdot}117, 4\times10^{-3}$ | $11{\cdot}276, 0\times10^{-3}$ | $91{\cdot}692, 4\times10^{-3}$ |
| (b) (1987(8) $i_2^{(2)}$) | $\cdot802, 2\times10^{-3}$ | $-\cdot247, 8\times10^{-3}$ | $1{\cdot}318, 6\times10^{-3}$ | $12{\cdot}219, 4\times10^{-3}$ |
| (c) (1987(8)$i_3^{(2)}$) | $\cdot209, 5\times10^{-3}$ | $-\cdot019, 8\times10^{-3}$ | $\cdot155, 4\times10^{-3}$ | $1{\cdot}663, 6\times10^{-3}$ |
| (d) Exp. Ext. (8) | $\cdot1\times10^{-6}$ | $\cdot1\times10^{-6}$ | $\cdot6\times10^{-6}$ | $6{\cdot}9\times10^{-6}$ |
| (e) (1987(1)$i_2^{(2)}$) | $-\cdot444, 4\times10^{-3}$ | $\cdot473, 2\times10^{-3}$ | $-1{\cdot}155, 4\times10^{-3}$ | $-5{\cdot}172, 6\times10^{-3}$ |
| (f) (1987)(1)$i_3^{(2)}$) | $\cdot064, 6\times10^{-3}$ | $-\cdot071, 8\times10^{-3}$ | $\cdot118, 6\times10^{-3}$ | $\cdot295, 6\times10^{-3}$ |
| (g) Exp. Ext. (1) | $\cdot1\times10^{-6}$ | $\cdot0\times10^{-6}$ | $\cdot2\times10^{-6}$ | $3{\cdot}4\times10^{-6}$ |

This form of (8) may often be more convenient for practical calculation. Note that for finding a yield convertible half-yearly the term $\{(1+i)^n-1\}$ becomes $\{(1+\frac{1}{2}\,i^{(2)})^{2n}-1\}$.

17. An alternative to the above is to use formula (1) starting with $s_{\overline{n}|}$ calculated using the result of applying the 1967 formula and then, if desired, recalculating $s_{\overline{n}|}$ at the improved value so obtained to get a still better value. Using an electronic calculator with a key for $y^x$ the calculation is a little simpler than by using (8). Table 3 also includes for the same examples the results of two such successive applications of (1) and of exponential extrapolation on the three values involved.

18. It would appear from the above that if a programmable electronic calculator is readily available, it may be programmed to calculate a yield by iteration using (1) starting with an initial value of $s^{(2)}_{\overline{n}|}$ equal to $n$ and therafter calculating $s^{(2)}_{\overline{n}|}$ by $s^{(2)}_{\overline{n}|}=\{(1+\frac{1}{2}i^{(2)}_r)^{2n}-1\}/i^{(2)}_r$. The use of $s^{(2)}_{\overline{n}|}=n$ for the initial values ensures the use always of a non-zero value of $i^{(2)}$ for calculating subsequent values of $s^{(2)}_{\overline{n}|}$. The use of (1) in this way gets rid of any requirement to express anything per unit (or 100 units) of net redemption value since the calculation constants are $Q/P$ and $I/P$ also $n$ may be a fraction plus an integer to allow for transactions occurring between interest payment dates and perhaps final repayment not being on an interest date.

19. If a programmable calculator is not readily available but an electronic calculator with a $y^x$ key is available the starting value for the use of (1) could be the value obtained by the 1967 formula or the procedure of § 18 could be followed for a few iterative values and exponential extrapolation then used. For example in the 10-year case used in Table 3 if $i^{(2)}_3$ is found using $s^{(2)}_{\overline{n}|}=n$, then if $i^{(2)}_3$ is expressed as a rate % the error in this rate is $-5{\cdot}226\times10^{-3}$ and the rate obtained by exponential extrapolation on $i^{(2)}_1$, $i^{(2)}_2$ and $i^{(2)}_3$ has an error of only $14\times10^{-6}$. The corresponding values of the errors in the 40-year case are $\cdot3072\times10^{-3}$ and $4{\cdot}8\times10^{-6}$. With yields calculated using (1) iteratively it is as well to verify the final result by using (8) or its practical modification (9). Since the latter uses a very different route to the final result it will show up any errors occurring in the use of (1) such as an error made in $Q/P$ or $I/P$.

20. If no electronic calculator is available which readily calculates $y^x$ recourse must be made to the 1967 formula. From Table 3 it will be seen that this gives usable results up to 30 years even where such very large capital gains at maturity are involved. If such capital gains are smaller the error is correspondingly smaller. In this case verification of the resulting yield by the use of (9) or of (1) would require $(1+\frac{1}{2}\,i^{(2)})^{2n}$ to be calculated by logarithms.

21. Some pocket calculators, such as the one on which much of the calculations for this note were done, have a built-in program for calculating yields to maturity, but these require the coupon rate (payable half-yearly) and the net purchase price to be expressed per 100 units of net redemption value, which is inconvenient where tax is payable on the half-yearly coupon payments and a capital gains tax probably at some other rate on the capital gain at maturity.

## APPENDIX

The approximation $n+(1-t)(s_{\overline{n}|}^{i_c}-n)$ for $s_{\overline{n}|}^{i_N}$

I learnt the fairly rough and ready but frequently adequate approximation in class studying in 1924 for the Institute of Actuaries Compound Interest exam though it was not in the syllabus. It forms the basis of the method which at some time later was in the syllabus for finding the grossed up net yield, there then being no capital gains tax, by reducing the unexpired term by the rate of tax and finding the yield using this reduced term from a table of Bond Values. This enabled use to be made of the gross nominal rate of interest for which Bond Values would normally be tabulated instead of the nominal net rate for which they would not.

The validity of the approximation can be justified in two ways (i) by actual calculation which shows whether for the use envisaged the error is small enough and (ii) by investigating the theoretical basis of the approximation.

If the yield is $(I/P)+(Q/s_{\overline{m}|})$, what is relevant is whether the error in $(1/s_{\overline{n}|})$ using the approximate value is sufficiently large to render its use inadmissible. Using a gross yield of 5% and tax at 30%, i.e. a net yield of $3\frac{1}{2}$%, the error in $(1/s_{\overline{n}|})$ comes to ·000189 for 5 years, ·000528 for 10 years, ·000818 for 15 years, ·00105 for 20 years and ·00123 for 25 years which would seem to indicate that provided the rate of interest is not too high nor the term too long the approximation is usable.

Having been a pensioner for 22 years the only tables of compound interest I have available are those in the Institute's Short Collection published in 1912 which I used as a student in which, in those pre-inflation days, the highest rate of interest is 5%.

If $s_{\overline{n}|}$ is expanded in terms of $i$ if

$$T_r=\frac{(n)(n-1)(n-2)(\ldots\ldots)(n-r)}{2\quad 3\quad \ldots\ldots(r+1)}$$

$s_{\overline{n}|}=n+T_1i+T_2i^2+T_3i^3+\ldots\ldots$ The approximate value of $s_{\overline{n}|}$ therefore is

$$s_{\overline{n}|}^{i_N}+T_2i_Ni_G\{1-(1-t)\}+T_3i_Ni_G{}^2\{1-(1-t)^2\}+T_3i_Ni_G{}^3\{1-(1-t)^3\}+\ \ldots$$

This approximation consists of interpolating between the values of $s_{\overline{n}|}$ for $i=0$ and $i=i_G$ for $s_{\overline{n}|}$ at $i=i_N$. This is a fairly rough and ready interpolation but usable if $n$ and $i$ are not so large as to make the error unacceptable *for the purpose for which the value is required.* The results of its use given above show that it is often acceptable.