RESERVING, PRICING AND HEDGING FOR POLICIES WITH GUARANTEED ANNUITY OPTIONS

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ABSTRACT

In this paper we consider reserving and pricing methodologies for a pensions-type contract with a simple form of guaranteed annuity option. We consider only unit-linked contracts, but our methodologies and, to some extent, our numerical results would apply also to with profits contracts.

The Report of the Annuity Guarantees Working Party, Bolton *et al.* (1997), presented the results of a very interesting survey as at the end of 1996 of life assurance companies offering guaranteed annuity options. There was no consensus at that time among the companies on how to reserve for such options. The Report discussed several approaches to reserving but concluded that it was unable to recommend a single approach. This paper is an attempt to fill that gap.

We investigate two approaches to reserving and pricing. In the first sections of the paper we consider quantile, and conditional tail expectation, reserves. The methodology we adopt here is very close to that proposed by the Maturity Guarantees Working Party in its Report to the profession, Ford *et al.* (1980). We show how these policies could have been reserved for in 1985, and what would have been the outcome of using the proposed method.

In a later section we consider the feasibility of using option pricing methodology to dynamically hedge a guaranteed annuity option. It is shown that this is possible within the context of the model we propose, but we submit that, in practical terms, dynamic hedging is not a complete solution to the problem since suitable tradeable assets do not in practice exist.

Finally, we describe several enhancements to our models and methodology, which would make them even more realistic, though generally they would have the effect of increasing the required contingency reserves

KEYWORDS

Guaranteed Annuity Options; Contingency Reserves; Quantile Reserves; Conditional Tail Expectations; Charging for Contingency Reserves; Mortality Improvements; Quanto Options; Option Prices; Hedging Proportions; Dynamic Hedging; Empirical Hedging; Hedging Errors; Transaction Costs; Practicability of Hedging; Fat-tailed Innovations; Stochastic Mortality Models; Stochastic Hypermodels; Stochastic Bridges; Brownian Bridges; Ornstein-Uhlenbeck Bridges

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1. Introduction

1.1 This paper describes methods for the calculation of reserves and charges for policies with guaranteed annuity options (GAOs). It is substantially based on the work carried out for a doctoral thesis by the last-named author (Yang, 2001).

1.2 We start in Section 2 by going back to early 1985, by which date the up-to-date actuary, familiar with the then current U.K. actuarial literature, should have had sufficient knowledge to have been able to attack the problem of reserving and pricing for policies with GAOs, on the basis of work that had been published by that date, mainly the Report of the Maturity Guarantees Working Party (MGWP, Ford *et al.*, 1980), and the Wilkie investment model, whose first version had been presented to the Faculty of Actuaries in late 1984, though not published until later (Wilkie, 1986a). In Section 3 we widen the discussion, to consider certain problems that arise with the MGWP methodology.

1.3 In Section 4 we continue the historical approach by showing how this methodology would have fared for new business in the years since then (up to the end of 2001). In Section 5 we introduce the effect of new mortality tables published since 1985, and in Section 6 the use of the 1995 version of the 'Wilkie stochastic investment model'. In Section 7 we consider how the reserves on an initial tranche of new business would have altered since 1985, allowing both for changes in interest rates and changes in the basis.

1.4 In Section 8 we review the developments in methodology that have happened since then. In Section 9 we discuss conditional tail expectations as an alternative to quantiles for reserving. In Section 10 we discuss the very important topic of option pricing theory and hedging and how far this helps to reduce the contingency reserves that might otherwise be required. In Section 11 we discuss further extensions of our method that are desirable, but which we have not yet implemented. Section 12 provides a summary and conclusion.

1.5 Our overall methodology is quite similar to that of Boyle & Hardy (1997), though they are dealing with maturity guarantees on unit-linked policies.

1.6 The subject of GAOs has proved controversial and has been settled in part in the courts. We do not discuss the legal aspects of these cases, but we do make clear our actuarial approach to this subject.

2. Reserving and Pricing on the Basis of Quantile Reserves in 1985

2.1 *The Position in 1985*

2.1.1 We start by trying to put ourselves in the position of an actuary who, at the beginning of 1985, was considering how to value GAOs issued by a life office, and who was familiar with the U.K. actuarial literature that had been published by that date. Such an actuary would have been aware of the work of the Maturity Guarantees Working Party (Ford *et al.*, 1980), which showed how what we would now call quantile reserves could be calculated, both for single policies and for a portfolio of policies, by stochastic, or 'Monte Carlo', simulation of a large number of possible future outcomes, and then by choosing to reserve on the assumption that the 99% (say) most costly outcome occurred.

2.1.2 The problem faced by the MGWP was that many life offices had written unit-linked life policies, typically endowment assurances, which included a guarantee that the maturity value would be not less than some fixed amount, for example the sum of the premiums paid. As share prices rose during the 1960s and early 1970s it had seemed that such a guarantee would be very unlikely to be of value, and therefore needed neither to be reserved for nor charged for. The sharp drop in stock market values in the mid-1970s showed that such an assumption was mistaken, and after a great deal of discussion and investigation the MGWP set out a methodology for reserving for such policies with guarantees. The methodology was based on a stochastic model, which modelled share prices as the ratio of a dividend index and a dividend yield.

2.1.3 We can now see that the problem faced by the MGWP was essentially the same as the valuation of a put option on shares, but in the 1980s option pricing methodology was only just becoming known, and the MGWP was not convinced of its applicability to their problem. We shall see the same in relation to GAOs. See Ford *et al.* (1980), Section 6.

2.1.4 The MGWP had developed a stochastic model for share prices. This would not have been enough for the consideration of GAOs, whose value depends also on interest rates. However, Wilkie was one of the members of the MGWP and in the following years he considered how a more comprehensive model could be constructed. This resulted in the first version of 'the Wilkie model', presented to the Faculty of Actuaries in November 1984 (Wilkie, 1986a). This included stochastic models both for shares, on the same lines as the MGWP, and for long-term interest rates, 'consols', linked by a model for retail price inflation, which was assumed to influence both share dividends and nominal interest rates. It might have been nicer to have had a model for short-term interest rates, and indeed a complete yield curve too, but it would have been possible to value GAOs simply by assuming a flat yield curve with all yields at the consols rate.

2.1.5 The MGWP had been asked to consider only how to reserve for polices with maturity guarantees. It was not necessary to consider how to charge for the option included in the policies, since the existing policies had included the guarantee without any charge. However, in the discussions about guarantees that took place during the late 1970s, Wilkie had written a note (Wilkie, 1978), making suggestions about how a pricing methodology could be devised. We shall explain and use this methodology in the course of this paper.

2.1.6 Wilkie was not the only one to take an interest in maturity guarantees. Besides Wilkie (1976), other authors in the U.K. included Benjamin (1976), Scott (1976) and Corby (1977). In North America, Brennan & Schwarz (1976) had shown how to apply option pricing methodology to the problem

2.2 *The GAO Model*

2.2.1 The model that would almost certainly have been chosen in 1985 for assessing the cost of GAOs would have been on the following lines. We assume pension contracts written as unit-linked savings policies. These could have been either single premium or annual or monthly premium policies. For simplicity in this paper we shall consider only single premium policies. (See Yang, 2001, for a similar treatment of annual premium policies.) At maturity the policyholder would have had the choice of taking the maturity proceeds either as cash which, under the 'open market option', could be applied to purchase a pension annuity with any life office, and thus be at 'market annuity rates' at the time, or as an annuity from the original life office at the guaranteed rate. For simplicity we assume that the annuity would be paid annually in advance. This can be thought of as something of a compromise between the more realistic pension paid monthly and one guaranteed for 5 years. We shall ignore any joint life benefits.

2.2.2 In our specimen calculations we shall consider only males, and assume that all policies mature at age 65. We shall use a guaranteed annuity rate of £111 per annum per £1,000 purchase price. According to Bolton *et al.* (1997) this was the most common GAO rate used by life offices. It presumably derives from the 1 for 9 conventional rate of turning cash into pension, which in turn may come from the assumed equivalence of a pension based on 1/60ths and one based on 1/80ths along with a lump sum of 3/80ths (all of final salary times years service).

2.2.3 To calculate the value of an annuity we need to make some mortality assumptions. In 1985 the latest tables published by the C.M.I. Bureau for pensioners were the PA(90) tables. These are often treated as a single entry table, but the actuary of 1985 should have been aware that the same tables could also be treated as the base tables of 1967-70, forecast for each future calendar year by assuming an age reduction of one year of age per 20 calendar years. Using the C.M.I.'s more recent notation, these double-entry tables could have been described as the '68 series' tables. We shall assume that the actuary of 1985 would have used tables from this series appropriate to each policyholder's year of birth, what might now be called the PMA68B*yyyy* tables, where the annuitant was born in year *yyyy*. If calculations are being carried out as at year *zzzz*, using the mortality table appropriate for the year of birth of each life, this is denoted as PMA68U*zzzz* (where U stands for 'year of use').

2.2.4 We need also to make assumptions about mortality before age 65, though this may be of less significance. For convenience we shall assume the same mortality as after age 65, i.e. PMA68B*yyyy* for a life born in year *yyyy*.

2.2.5 We shall ignore expenses. Of course an actuary dealing with the policies of a real life office would need to take into account expenses, as well as the actual guaranteed annuity rates offered by the office, the method of payment of the annuity, the mortality experience of the office's own insured, and the investment policy of the office.

2.2.6 We shall assume 100% investment in a share portfolio, which performs exactly in line with the shares in the Wilkie model. In practice many life offices attached GAOs to with-profits policies, for which the investment portfolio might be less than 100% shares, and for which the maturity proceeds might not be precisely the 'asset share' derived from the office's portfolio, but some smoothed version of this. With-profit policies are discussed further by Yang (2001), but we shall consider in this paper only unit-linked policies, linked to the total return (gross of tax) on shares. In practice, as we shall demonstrate, the initial cost of the guarantee and the subsequent contingency reserves do not depend on the investment portfolio at all, though the outcome of dynamic hedging does.

2.2.7 There are other possible types of guaranteed annuity benefit that might have been issued. In particular a with-profit policy might have been written with a guarantee that the minimum annuity obtained might be not less than some annual amount. This might be equivalent to applying some guaranteed minimum rate to the original sum assured, applying the current market rate to the full policy proceeds, and taking the larger annuity. In another type of policy a guaranteed rate might have applied to the sum assured and reversionary bonus (strictly the better of the guaranteed rate and the current market rate), with the market rate being applied to the terminal bonus. In order to attack such types of policy one would need to make some assumptions about the size of the basic sum assured in relation to the initial premium and about the office's bonus declaration strategy. However, apart from this, the principles to be used would be the same as we describe in this paper.

2.3 *The Formal Model*

2.3.1 We now introduce some notation and give a formal description of our methodology. Assume a male born in year y, who is therefore aged $x = 1985 - y$ in 1985. He will reach age 65 in year $y + 65$, so the term of his policy is $\mathrm{T} = 65 - x = y - 1920$ years. He pays a single premium of £100, which is invested in share units, whose unit value at time t is $\mathrm{S}(t)$. $\mathrm{S}(t)$ is derived from the value of shares, with dividends reinvested, gross of tax, according to the Wilkie model, in which:

$D(t)$ is the dividend received at time t,
$Y(t)$ is the dividend yield at time t
and $P(t) = D(t)/Y(t)$ is the share price at time t

so that:

$$S(t) = S(t-1).(P(t) + D(t))/P(t-1)$$

At this stage we can assume that no tax is payable on dividend income.

2.3.2 We can put $t = 0$ in 1985, and choose, arbitrarily, $P(0) = 1.0$, and $S(0) = 100$, so the value of the policy at time t is $S(t)$. Although we can choose $P(0)$ and $S(0)$ arbitrarily, the Wilkie model requires initial conditions for the other features. We discuss all these together in Section 2.4.

2.3.3 We assume that deaths may occur between 1985 and maturity in $1985 + T$. We assume that on death the only benefit is a return of the value of the fund; or at least that any other death benefits are covered elsewhere. Thus a proportion of only ${}_Tp_x$ policies, where ${}_Tp_x$ is calculated using PMA68B*yyyy* mortality, survive to maturity, to be potential claimants of any GAO.

2.3.4 We assume that at maturity current interest rates, for all terms, are the same as the value of the 'consols' yield in the Wilkie model, C(T). The policy proceeds at maturity, when $t = T$, are S(T). The guaranteed annuity rate is $g = 0.111$ per unit, or £111 per £1,000 cash. The value of this, using forecast mortality and market interest rates, is $g.a(T)$, where $a(T)$ is the value of an annuity of 1 per annum, payable in advance, to a life aged 65 in year $1985 + T$, using interest rate C(T). The value of the policy proceeds applied to the guaranteed annuity is therefore $S(T).g.a(T)$.

2.3.5 The value of the policy proceeds at maturity is S(T). If this is applied to purchase an annuity at market rates, the annuity that could be purchased is $S(T)/a(T)$, and the value of this annuity is just $S(T)/a(T).a(T) = S(T)$. The GAO has value if $g > 1/a(T)$, i.e. if $S(T).g.a(T) > S(T)$, and in that case the value is $S(T).(g.a(T) - 1)$.

2.3.6 Since only those who survive to age 65 can receive any value from their GAO, we can reduce the value of the GAO by multiplying by ${}_Tp_x$ to give $S(T).(g.a(T) - 1).{}_Tp_x$ per initial life. We also need to discount this to the start of the policy. We consider that how we discount depends on what we assume that any reserves to meet the cost of the GAO are invested in. It would have seemed correct in 1985 for these reserves to have been invested in shares, because this would most tidily match the amount of the liability. The better shares perform, the higher the value of S(T), and the higher the liability. Shares are likely to be more volatile than interest rates, so it would seem appropriate to match the share element rather than the interest rate element of the cost of the guarantee. To meet the cost of the GAO at time T of $S(T).(g.a(T) - 1).{}_Tp_x$, we would have needed to reserve at time 0 an amount of $S(T).(g.a(T) - 1)._T p_x.S(0)/S(T) = S(0).(g.a(T) - 1)._T p_x$, which we shall denote V0. Calculated in this way, the initial value of the guarantee does not depend on the investments in which the premium is invested, but this does assume that the amount reserved to meet the guarantee is invested in the same assets as the premium.

2.3.7 In the expression $V0 = S(0).(g.a(T) - 1)._T p_x$, S(0) and g are fixed, and we assume (at this stage) deterministic mortality, so ${}_T p_x$ is also fixed. However, the value of a(T) depends on C(T), so it is not known till time T, and at time 0 is unknown. Thus V0 is a random variable, with a complicated distribution that depends on the movement of interest rates between 0 and T. Our method is to simulate say $J = 10,000$ values of the variables in the Wilkie model, calculate the value of a(T) in each simulation, and hence the value of V0 in each simulation. We then sort the values of V0 into an increasing sequence, so that $V0_{j-1} < V0_j$. We then choose a percentage security level, say 99% or 99.9%, so that there is a chance of 1 in 100 or 1 in 1,000 that the reserve will be insufficient, and find $V0_{9901}$ or $V0_{9991}$ as required. (We could have taken e.g. $V0_{9900}$ or the average of $V0_{9900}$ and $V0_{9901}$, but there is a small advantage in using the slightly higher value, $V0_{9901}$, which we explain in Section 9 when we discuss CTEs.)

2.3.8 We denote the security level by α, e.g. $\alpha = 0.99$ or 0.999, equivalent to a chance of 1 in $1/(1 - \alpha)$ of having insufficient reserves at time n (conditional, we should note, on conditions at time $t = 0$). Then we choose $V0_{(J\alpha+1)}$ as the desired initial reserve, denoting it Q_α.

2.3.9 The initial 'quantile (contingency) reserve' required is Q_α. But how should this be financed? It is immediately reasonable that the policyholder should pay, in addition to his invested £100, an extra premium for the benefit of the guarantee, at least equal to the expected value of the extra benefit under the policy. We estimate the expected value of the benefit by the mean of the J simulated values of V0, denoted $A = E[V0]$. Thus we start by assuming that the policyholder pays A and the shareholders (or with-profit policyholders generally, if the office is a mutual one) put up $Q_\alpha - A$ to give an initial contingency reserve of Q_α.

2.3.10 But the shareholders' money is at risk. Their amount $Q_\alpha - A$ is initially invested in shares, or at least in the same way as the policyholder's funds, but they are not certain to receive the return of investment of that amount. They may get back either more or less, depending on the outcome of the policy. Their upside risk is limited, because at the best the GAO costs nothing, and they get back all of the proceeds of the investment of Q_α. Their downside risk is generally much greater, because interest rates may go down to very low levels, and shares might go up to very high ones. Although the interest rate risk is limited by a downside of 0%, so that a(T) is calculated as just the expectation of life, shares might still go to very high values. In that case the invested Q_α would also rise, but there is still a chance of $1 - \alpha$ that the reserve will not be enough, and funds intended for other purposes may have to be called on. There may be a final limit on the liability of the shareholders if the life office becomes insolvent, but one hopes that affairs can be managed so as to avoid that contingency.

2.3.11 The method suggested by Wilkie (1978), and followed by Hare *et al.* (2000), is to assume that the shareholders wish to earn on their invested funds an extra h per annum more than they will earn from the normal investment proceeds. The value of h might be say 0.01 or 0.02, equivalent to an extra 1% or 2% return. One way of providing this is to charge the policyholder in year t an amount equivalent to $h(Q_\alpha - A).S(t)/S(0)$. This could be done by making an extra charge on the units of k per annum, where $k = h(Q_\alpha - A)$. If, for example, for a £1 single premium, $Q_\alpha = 0.22$, $A = 0.02$, and $h = 0.02$, then k would equal $0.02 \times (0.22 - 0.02) = 0.004$, or an extra charge of 0.4% of the unit value.

2.3.12 Calculated as an annual charge the amount paid to service the shareholders' funds varies from simulation to simulation. However, the present value of it is in fact the same for all simulations. This can be shown by the following rationale. If the policyholder is charged an extra k per unit of the value of his funds per annum, then his investment declines, relative to making no charge, by an annual factor $v = 1/(1 + k)$. At the end of n years his investment is reduced by a factor of v^T, and is therefore worth $v^T.S(T)$ instead of S(T). The charge has therefore accumulated to $(1 - v^T).S(T)$, provided that it has been invested in the same assets as the policyholder's fund. The value of this at the start of the policy is $(1 - v^T).S(0)$, which is the same for all simulations.

2.3.13 Now let us assume that, instead of making an annual charge to service the policyholders funds, we charge the policyholder the present value of the charge, calculated as just shown, i.e. $(1 - v^T).S(0)$. Denote this as B. The policyholder therefore contributes an extra premium of $A + B$ initially. But then the shareholders only need to provide $Q_\alpha - (A + B)$ instead of $Q_\alpha - A$. This has an influence on the value of k, which now equals $h(Q_\alpha - (A + B))$, which in turn affects B. We need to use a simple recursive calculation to find the equilibrium value of B, such that $B = (1 - v^T).S(0)$, where $v = 1/(1 + h(Q_\alpha - (A + B)))$.

2.3.14 It can be observed that we have ignored mortality in the discussion in the last two paragraphs. However, this is the correct approach. The value of the GAO has already been reduced by a factor of ${}_Tp_x$, as described in ¶2.3.6. Thus the reserve of Q_α needs to be held for the full duration of an assumed group of identical policies, and is not reduced as policyholders die. No refund is made on this account to policyholders who do die, and who therefore cannot collect the benefit of the GAO. Since the reserve of Q_α is to be held for the full duration, the charge at the rate of $k = h(Q_\alpha - (A + B))$ needs to made for the full duration, and should not be reduced as deaths occur. When we calculate its capitalised value we can therefore ignore deaths. One could make other, more complicated assumptions, which would require the value of k to be increased to allow for deaths, but this should have no effect on the initial capital value.

2.4 *The 1984 Wilkie Model*

2.4.1 For the calculations in this part of the paper we use the Wilkie model as presented in 1984 (Wilkie, 1986a), using the 'full basis' and the corresponding parameters described therein. For the record, we repeat the formulae, without explanation.

2.4.2 The retail price index in year t is denoted $\mathrm{Q}(t)$, the dividend yield $\mathrm{Y}(t)$, the dividend index $\mathrm{D}(t)$, and the 'consols' yield $\mathrm{C}(t)$. The innovations, $\mathrm{QZ}(t)$, $\mathrm{YZ}(t)$, $\mathrm{DZ}(t)$, and $\mathrm{CZ}(t)$ are all independent unit normal random variates. The formulae to move the model from year to year can be expressed as:

$$\mathrm{QE}(t) = \mathrm{QSD.QZ}(t)$$
$$\mathrm{I}(t) = \mathrm{QMU} + \mathrm{QA.}(\mathrm{I}(t-1) - \mathrm{QMU}) + \mathrm{QE}(t)$$
$$\mathrm{Q}(t) = \mathrm{Q}(t-1).\exp(\mathrm{I}(t))$$

$$\mathrm{YE}(t) = \mathrm{YSD.YZ}(t)$$
$$\mathrm{YN}(t) = \mathrm{YA.YN}(t-1) + \mathrm{YE}(t)$$
$$\mathrm{Y}(t) = \exp(\mathrm{YW.I}(t) + \ln \mathrm{YMU} + \mathrm{YN}(t))$$

$$\mathrm{DM}(t) = (1 - \mathrm{DD}).\mathrm{DM}(t-1) + \mathrm{DD.I}(t)$$
$$\mathrm{DE}(t) = \mathrm{DSD.DZ}(t)$$
$$\mathrm{K}(t) = \mathrm{DW.DM}(t) + (1 - \mathrm{DW}).\mathrm{I}(t) + \mathrm{DY.YE}(t-1) + \mathrm{DB.DE}(t-1) + \mathrm{DE}(t)$$
$$\mathrm{D}(t) = \mathrm{D}(t-1).\exp(\mathrm{K}(t))$$

$$\mathrm{CM}(t) = (1 - \mathrm{CD}).\mathrm{CM}(t-1) + \mathrm{CD.I}(t)$$
$$\mathrm{CE}(t) = \mathrm{CSD.CZ}(t)$$
$$\mathrm{CN}(t) = \mathrm{CA1.CN}(t-1) + \mathrm{CA2.CN}(t-2) + \mathrm{CA3.CN}(t-3) + \mathrm{CY.YE}(t) + \mathrm{CE}(t)$$
$$\mathrm{CR}(t) = \mathrm{CN}(t) + \ln \mathrm{CMU}$$
$$\mathrm{C}(t) = \mathrm{CM}(t) + \exp(\mathrm{CR}(t))$$

2.4.3 The parameter values used are taken from Wilkie (1986) and are: QMU = 0.05, QA = 0.6, QSD = 0.05, YW = 1.35, YMU = 0.04, YA = 0.6, YSD = 0.175, DD = 0.2, DW = 0.8, DMU = 0.0, DY = −0.2, DB = 0.375, DSD = 0.075, CD = 0.045, CMU = 0.035, CA1 = 1.2, CA2 = −0.48, CA3 = 0.2, CY = 0.06, CSD = 0.14.

Table 2.4.1. Values of initial conditions for the 1984 Wilkie model as at 31 December 1984

Initial condition	Value	Initial condition	Value
I(0)	0.055344	DM(0)	0.080497
I(−1)	0.044781	DE(−1)	0.005634
I(−2)	0.051785	CM(0)	0.066988
Y(0)	4.33%	C(0)	9.80%
Y(−1)	4.42%	C(−1)	9.90%
Y(−2)	4.62%	C(−2)	9.71%

2.4.4 The simulations require initial conditions at $t = 0$, and also for $t = -1$, and $t = -2$. Some are arbitrary, such as Q(0), taken as 1.0. The important ones depend on market conditions at 31 December 1984, and previous years. Some can be observed directly from market indices; others depend on the model parameters, and are derived by calculations with the model for many past years. They are as in Table A1 in Appendix A, but the values for 31 December 1984 are shown in Table 2.4.1.

2.4.5 A small feature of the Wilkie model should be noted. If the initial 'carried forward' value of inflation, CM(0) is too large in comparison with the value of the 'consols' yield C(0), the value of CR(0) can be negative or small. In practice CR(0) is set at a minimum value of 0.005, i.e. 0.5%, and the value of CM(0) is adjusted downwards. This occurs in both the 1984 model and the 1995 model (see Section 6) in some years after 1993. In the 1984 model it may be necessary to make this adjustment for years −1 and −2 from the start. A further constraint may be necessary: if inflation is negative for some years, the value of CM(t) may be negative, so the calculated value of C(t) may also be negative. If this happens the value of C(t) is also set to a minimum value, also 0.005 or 0.5%. This happens very rarely, at the most for one year each in four simulations out of the 10,000 carried out.

2.5 *Numerical Results*

2.5.1 In Table 2.5.1 we show some of our first results. The economic basis is the 1984 Wilkie model, with initial conditions as at the end of December 1984. The mortality basis is denoted as PA68U1985, that is the mortality for each policyholder is based on his own year of birth, assuming that the policy commences in 1985 (i.e. 'year of use' 1985). We show for single policies of terms 10, 15, 20, 25, 30, 35 and 40, and also for our standard portfolio of policies with one each of terms 10, 11, 12, ... 40, all per £100 single premium (for the portfolio a premium of £100/31 per policy, or £100 overall), the following statistics:

— the percentage of simulations in which the discounted present value (DPV) of the cost of the GAO was non-zero, denoted NZ%;

Table 2.5.1. Present value of cost of GAO per £100 single premium: 1984 Wilkie model, initial conditions of 31 December 1984, mortality PMA68U1985.

Term	NZ%	Mean	Q_{90}	Q_{95}	$Q_{97.5}$	Q_{99}	$Q_{99.5}$	$Q_{99.9}$
10	6.57	0.26	0.00	1.00	3.91	7.84	10.16	13.69
15	11.94	0.59	0.90	4.47	7.82	11.96	15.02	21.16
20	16.51	0.99	3.36	7.52	11.03	15.36	19.26	26.87
25	20.72	1.34	4.94	9.50	13.36	18.15	22.79	29.80
30	24.05	1.74	6.69	11.37	16.17	21.08	26.09	35.39
35	27.01	2.02	7.70	12.44	16.92	23.57	28.06	37.15
40	29.74	2.30	8.69	13.70	18.05	23.69	28.88	41.29
Portfolio	59.65	1.33	4.25	6.72	9.27	12.43	15.44	20.95
Sum	29.74	1.33	4.67	8.74	12.66	17.78	21.72	29.84

— the mean cost of the DPV;
— the quantile reserves for 90%, 95%, 97.5%, 99%, 99.5% and 99.9%, denoted Q_{90}, Q_{95}, etc. Q_{90} is calculated, for example, as the 9001*th* largest result out of the 10,000 simulations.

2.5.2 In the last line we show the sum of the quantile reserves for each policy (divided by 31). These are calculated by adding the sorted values of the results for each simulation. They indicate by how much the portfolio quantiles are less than the sums of the individual quantiles.

2.5.3 We can see at once that although the mean values of the DPV of the GAO are not enormous, they are not negligible; further, the quantiles are considerable, sometimes very much so. The costs increase with the term to go (a result which depends on the initial conditions; it is not always true). The DPVs for the portfolio are intermediate between the results for different terms. Even for the shortest term, there is a chance of 6.57% that the GAO will 'bite', a 99% reserve would be 7.84 per £100 single premium and a 99.9% reserve would be 13.69. For the longest term these figures rise to 29.74%, and 23.69 and 41.29 per £100. There is a 59.65% chance that, for the whole portfolio, the GAO will cost something, even if not for all policies in the same simulation.

2.5.4 If the mortality basis of PA(90) males, which was perhaps the standard table in use in 1985, had been used, then the results would have been as shown in Table 2.5.2.

2.5.5 The costs are lower on this heavier mortality basis, especially for longer terms where the mortality is not projected to improve. The quantile reserves, however, remain quite significant.

2.5.6 Following the rationale of our charging methodology, as set out in ¶¶2.3.12-2.3.14, we show in Table 2.5.3 the values of the policyholder charges 'A', 'B' and C = A + B, using the 99% and 99.9% quantiles, with a 1% and 2% extra return to shareholders. We revert to the mortality basis PMA68U1985.

Table 2.5.2. Present value of cost of GAO per £100 single premium: 1984 Wilkie model, initial conditions of 31 December 1984, mortality PA(90) Males.

Term	NZ%	Mean	Q_{90}	Q_{95}	$Q_{97.5}$	Q_{99}	$Q_{99.5}$	$Q_{99.9}$
10	4.55	0.17	0.00	0.00	2.37	6.12	8.34	11.70
15	8.29	0.37	0.00	2.39	5.54	9.42	12.28	18.01
20	11.38	0.60	0.86	4.70	7.93	11.90	15.47	22.40
25	13.52	0.76	1.81	5.93	9.41	13.71	17.86	24.10
30	15.36	0.94	2.86	7.01	11.25	15.56	19.93	28.01
35	16.72	1.01	3.23	7.35	11.23	16.94	20.78	28.49
40	17.49	1.06	3.57	7.84	11.52	16.27	20.62	30.90
Portfolio	44.80	0.71	2.27	4.18	6.13	8.90	11.25	16.08
Sum	17.49	0.71	1.77	5.16	8.67	13.23	16.74	23.89

Table 2.5.3. Charge to policyholders per £100 single premium: different combinations of α and h; 1984 Wilkie model, initial conditions of 31 December 1984, mortality PMA68U1985.

Term		99%, 1%		99%, 2%		99.9%, 1%		99.9%, 2%	
	A = Mean	B	C	B	C	B	C	B	C
10	0.26	0.69	0.94	1.26	1.51	1.21	1.47	2.22	2.47
15	0.59	1.47	2.06	2.60	3.19	2.65	3.24	4.66	5.25
20	0.99	2.37	3.36	4.04	5.03	4.23	5.22	7.19	8.18
25	1.34	3.32	4.65	5.50	6.83	5.56	6.90	9.18	10.51
30	1.74	4.38	6.12	7.08	8.82	7.53	9.27	12.11	13.84
35	2.02	5.47	7.48	8.63	10.65	8.79	10.81	13.83	15.85
40	2.30	5.97	8.28	9.25	11.55	10.69	12.99	16.46	18.76
Portfolio	1.33	2.20	3.52	3.65	4.98	3.85	5.18	6.38	7.70
Sum	1.33	3.42	4.75	5.57	6.89	5.82	7.14	9.43	10.76

2.5.7 For the portfolio we have calculated A and B assuming that an equal charge is made on policies of each duration, and in the calculation of B that an equal amount of reserve is released at the end of each duration as the polices for that duration expire. This does not take into account the different risks contributed by policies of different durations. We discuss the distribution of charges to different policyholders further in Section 3.3 and Section 9.

2.5.8 The row marked 'Sum' shows the sum of the charges for the individual durations, assuming 1/31 of a unit effected at each duration. The mean is the same as for the portfolio, but the values of A, B and C are the sums of the values for the individual terms, each divided by 31.

2.5.9 The charges to the policyholder do not seem to us to be excessive, except perhaps for the longer terms. It is interesting to speculate, however, on how many proposers for insurance would have chosen to purchase the option at these prices, if they had had the choice of whether or not to add the option to the contract.

3. Problems with the Methodology

3.1 *Unanswered Questions*

3.1.1 The methodology that we have described in Section 2 leaves some questions unanswered. What should the value of α be? What should the value of h be? Are these connected? Next, how should one actually charge the policyholder? Should the charge to the policyholder depend on the total portfolio of the office, or depend only on the circumstances of the one policy? How should the initial contingency reserve be updated from year to year? Is the method for this that was suggested by the MGWP satisfactory, or not? Many of these points were not considered by the MGWP. We discuss them in turn.

3.2 *The Values of α and h*

3.2.1 We have indicated that the value of α might be for example 0.99 (99% security or 1 in 100 chance of failure) or 0.999 (99.9% security or 1 in 1,000 chance of failure), and we have quoted results in Section 2.5 on these two bases. The MGWP indicated that 99% seemed a reasonable security level. A 1% chance of failure seems reasonably small. We should note that the probabilities of sufficiency or failure are over the whole policy term, starting at the issue of the policy, or for a whole portfolio until the last policy matures (or is converted to an annuity) and are not, like a banking 'Value at Risk', the probabilities of success or failure over some short period. Yet they are analogous to the Value at Risk concept.

3.2.2 The level of security required is a matter for judgement, rather than calculation. Yet we have seen little discussion in the literature of the 'right' probability level. We would be disturbed by the idea that either life assurance companies or banks were regulated so that as many 1% of them failed during the course of a year, and we would expect others to agree with us. Yet we would not be so concerned by the idea that 1% of existing life assurance companies might fail at some time in the future. Indeed as few as that might be a rather good outcome.

3.2.3 The event that the contingency reserve for a single policy or for a portfolio proves to be insufficient is very different from the insolvency of the office. If the reserve falls to too low a level during the currency of the policy, it may mean that the shareholders (or the with-profit policyholders) have to allocate more capital to that reserve. We discuss this further in Section 3.5. But the initial security level is more a matter of the probability that more capital will be required during the currency of the policy. It may therefore reasonable for it to be at a rather weaker level, such as 95%.

3.2.4 What then about the value of the excess return, or premium for risk, h, expected by the shareholders? We have indicated possible value of $h = 0.01$ (1%) or 0.02 (2%), and Hare *et al.* (2000) used these same values. Yet the appropriateness of these values has not, we believe, been discussed. Perhaps the value of h should be higher if the value of α is lower. Thus if there is a 5% chance of the contingency reserves being insufficient, the shareholders may expect a higher premium for risk, because the risk may appear greater. If the chance of failure is much lower, say 0.1%, the premium for risk could be lower. One could suggest a sort of scale, where the probability of failure and the premium for risk increase together, as in the table below, but we do not know whether these numbers would be acceptable to shareholders, or to 'the market'.

Probability of failure	Premium for risk
0.1%	1%
1%	2%
5%	3%

3.2.5 It might be argued by some that the premium for risk should depend on the '*beta*', the regression coefficient of returns on the portfolio versus returns on 'the market portfolio', as in the Capital Asset Pricing Model. While this concept may have some uses in the context of a share portfolio, we do not think that it is of general applicability, because there is no agreement among investors as to what assets form 'the market portfolio', nor in what currency returns should be considered, nor whether real or nominal returns should be considered, nor over what time period one should look, nor what effect the different liabilities of investors have.

3.2.6 A further argument that might be put is that certain types of risk are independent of others, so are 'diversifiable', and so should require no premium for risk. This does not apply to the risks of GAOs, because they are dependent on both investment returns and interest rates, so are not wholly independent of other investments, whereas it could be argued that mortality risks are independent of investment markets. But even if the risk were independent we believe that shareholders in practice are not willing to invest in a more risky undertaking without the hope of some extra reward, whether or not the risks are independent of others.

3.2.7 All this could be thought of as a matter for 'the market' to decide. It would be an interesting experiment for a portfolio of policies with GAOs to be securitised, with a known level of contingency reserves, in such a way that they could be sold on the market independently. One might then discover what price investors were willing to pay for a given level of risk. Yet we doubt whether this would be really practicable.

3.2.8 We conclude that the levels of α and h remain a matter for the judgement of the management or the directors of the particular company writing the business. We shall continue to quote values on the basis of 99% and 99.9% for the one, 1% and 2% for the other.

3.3 *Charging the Policyholder*

3.3.1 We have suggested in Section 2 that the policyholder could be charged for the guarantee in the first place by an initial premium of A, followed by an annual management charge of so much per cent on the policyholder's funds, or by capitalising the latter into an further initial single premium, which we denoted as B. A further way of charging those policyholders with with-profits polices could be through a reduction in the annual bonus. For policies already written where the benefit has been granted free, it is of course too late to make an initial charge to each policyholder, and the office may have to consider how any losses should be met from other sources. However, our comments about charging apply to any benefit that would require contingency reserves to be set up.

3.3.2 However, the charge for each policyholder should depend on the circumstances at the time the policy is issued, in particular on market interest rates at that time, as well as on the term to maturity of the policy (and also on the level of the guaranteed annuity rate and the age of the policyholder at maturity if these can vary). It would be administratively very inconvenient to make the annual management charge vary for each particular policy, and equally difficult to make any annual bonus depend on the circumstances of each particular policy. We therefore suggest that the only practicable and fair way of making a charge to policyholders is for an additional premium at the inception of a single premium policy, which can take into account the circumstances of the policy at issue. For an annual or monthly premium policy the charge could be an addition to the periodic premium, the amount of which is determined at issue.

3.3.3 The question can then be raised as to whether the charge for a particular policy should depend only on the circumstances of that policy, or also on the total portfolio issued by the life office. We have seen in Section 2.5 that the contingency reserve for a portfolio of policies with different durations is less than the sum of the contingency reserves for the single policies (with the same values of α and h), and this might be more pronounced if the portfolio of policies already in force were to be considered. Note that there is no 'averaging out' for policies with the same dates of starting and maturing, but there is some averaging across maturity dates, even though the outcomes are correlated. It may be appropriate for the office to hold contingency reserves on a portfolio basis, yet for policyholders to be charged on an individual basis.

3.3.4 We suggest that one way of achieving this result is to use different levels of α, one, α_1, for portfolio reserving and other, α_2, for policyholder charging, choosing two values that are compatible. The value of α_2 should be less than the value of α_1. If, for example the office wished to reserve for the portfolio at a 99.5% level, and its portfolio of new business were to be the same as we have used in Section 2.5 (which is only a specimen and not intended to be realistic), then the quantile reserve for the portfolio (see Table 2.5.1) would be 15.44%. The sum of the quantile reserves for the individual policies on a 99.5% basis is larger, at 21.72%. At a 97.5% level it is 12.66%, and at a 99% level it is 17.78%. Given the results for all simulations, one can calculate that the sum of the individual quantile reserves at a 98.5% level is 15.45, close to the portfolio reserve. Alternatively, one could choose say a 95% charging level, for which the sum of the individual quantile reserves is 8.74% and calculate that this corresponds to a portfolio reserving level of about 97.2%. Note that all these results are based on simulations, and so are subject to simulation sampling error.

3.3.5 The exact balance between the two reserving levels depends on the composition of the portfolio, and also on all the other elements of the calculation basis, market interest rates, guarantee level, and so on. However, we believe the method described might have been a suitable *ad hoc* way of charging policyholders individually on a fair basis, and having an appropriate level of portfolio reserve. We consider this further in Section 9, where we describe another, much sounder, method, which has only recently been proposed.

3.4 *Reserving from Year to Year*

3.4.1 The calculations discussed in Section 2 indicate how an initial reserve at the start of the policy could be calculated. We now need to consider how we move forward during the currency of the policy. A year after the policy has been written market conditions, including interest rates, may well have changed (indeed they may have changed after only a day). If interest rates have fallen, then the probability of a future claim on the guarantee will generally have increased. The amount of the initial reserve for a comparable policy, a year of age older and a year shorter duration, may well be different from the amount of the initial reserve on the one-year-old policy, even allowing for the investment return on the initial reserve.

3.4.2 The Maturity Guarantees Working Party considered this problem. They felt (Ford *et al.*, 1980, p 110) that it was desirable that the initial reserve for a policy should be 'coherent' (or 'robust'), in the sense that there should preferably be no requirement for additional reserves during the currency of the policy. If the initial quantile reserve that had been set up had a sufficiently small probability of being insufficient, then that initial probability remained unchanged, although as events unfolded the conditional probability of the reserve being insufficient might change. On this argument the initial reserve should remain unchanged.

3.4.3 An alternative might be to adjust the reserves to the same probability level each year, taking account of the current circumstances, in particular current interest rates, doing what would now be called 'marking to market'. However, it was felt that this would involve an uncomfortable instability of the reserves, with movements to and from free capital rather frequently. Further, as the policy approached the option date, it was surely unnecessary to keep a reserve at all times, even in the last few days, so that the probability of having insufficient capital remained small, say at 1 in 100 or 1 in 1,000.

3.4.4 The compromise suggested by the MGWP was to establish a band of permissible probability levels, with defined lower and upper limits. Assume that the initial reserve was at the 99%, or 1 in 100 level. If the reserve at any year-end was below the lower probability level, say 98%, or 1 in 50 level, then it should be strengthened to the original 1 in 100 level. If the reserve at any year-end was above the upper probability level, say 99.9% or 1 in 1,000 level, then it could be reduced to the 1 in 100 level. This method would make movements to and from free capital less frequent, though of course they might be larger when they occurred than if the adjustments had been made more frequently.

3.4.5 The two methods, 'marking to market', and keeping within a band, have different consequences. If it were required, for example by the regulatory authority, that the office should at all times (or at least on each valuation date) have sufficient reserves that the probability of their being insufficient was always less than 1 in 100, then a prudent actuary might wish to see initial reserves set up so that this regulatory barrier was never breached, or rather that the probability of its being breached was suitably small. It is clear that, normally, the level of initial reserves then required would be much higher, probably very much higher, than the initial 1 in 100. It is difficult to see how the initial reserves corresponding to the actuary's chosen probability level could be calculated without simulations within simulations, for each possible course of future events as at time 0, and for each possible future course as at each future time, conditional on where we have reached by that time. Such calculations are not yet easily practicable.

3.4.6 Even if this problem cannot be solved easily, it is clear that, if the marking to market approach is used, the initial reserve could reasonably be on a weaker basis than if initial reserves had to be set up large enough so that an increase was unlikely to be needed in future. Thus if the office was comfortable with a 1 in 1,000 probability overall, then an initial reserve of 1 in 100 or even 1 in 50, with marking to market, might give a similar level of comfort, even though the numerical effect was unknown.

3.4.7 Where would the extra capital come from each year (or go to, if it were released), if marking to market were used, or if a band were used and extra capital were required? This could in the first place be a charge on the annual profits, simply altering the annual profits or losses, each year if marking to market were used, or in occasional years if a band were used. This in turn would come from the 'free reserves'; if absolutely necessary it could come from an increase in the share capital. Unless circumstances had moved very unfavourably, the portfolio of business would still appear likely to be profitable, and it would normally be worth while the shareholders continuing to finance it. Only if the reserves had fallen below the expected cost of the guarantees, or below a level so that there was the chance of some profit from the reserves, albeit a small one, would it be worth considering the consequences of closing the business.

4. Progress Since 1985 of Our 1985 model

4.1 *Introduction*

4.1.1 We now turn to considering the progress of our 1985 model since that date. In the first place we consider what the outcome would have been if there had been no change of valuation basis. In later sections we introduce changes that might have taken place in the basis, and in the methodology.

4.1.2 In Appendix A, Tables A.1 and A.2 we show the market conditions as at 31 December for each year from 1984 to 2001. The initial conditions for the Wilkie model include some values from previous years, which are also shown for 1984.

4.1.3 It is obvious that interest rates have reduced over the 17 years since the end of 1984. The required reserves must therefore almost certainly have risen. We can approach this analysis in two ways. First we can see what the results would have been for new policies entering in each of the years from 1985 onwards; this we consider in Section 4.2. Then we can look at how the original portfolio of policies, written in 1985, would have fared during the years since then; we consider this in Section 7 after we have introduced possible changes in the basis since 1985.

4.2 *New Policies since 1985*

4.2.1 We start by repeating the calculations shown in Tables 2.5.1 and 2.5.3, in which we used the market investment conditions as at the end of 1984, and assumed mortality on the basis of PA68U1985. For comparison we show the results for 1984 in Table 4.2a (for fewer terms than shown in Table 2.5.1). Then in Tables 4.2b to 4.2r we show similar figures for policies entering in years 1986 to 2002. For policies entering in year *zzzz* we use mortality PA68U*zzzz* and market investment conditions as at 31 December of year $zzzz - 1$. We show the quantile reserves (Q) for 95%, 97.5%, 99% and 99.9%, and we show the proposed policyholder charge (C) for combinations $\alpha = 99\%$ and 99.9%, $h = 1\%$ and 2%.

Table 4.2a. Statistics for GAO per £100 single premium: 1984 Wilkie model, initial conditions of 31 December 1984, mortality PMA68U1985.

Term	NZ%	Mean	Q_{95}	$Q_{97.5}$	Q_{99}	$Q_{99.9}$	$C_{99,1}$	$C_{99,2}$	$C_{99.9,1}$	$C_{99.9,2}$
10	6.57	0.26	1.00	3.91	7.84	13.69	0.94	1.51	1.47	2.47
20	16.51	0.99	7.52	11.03	15.36	26.87	3.36	5.03	5.22	8.18
30	24.05	1.74	11.37	16.17	21.08	35.39	6.12	8.82	9.27	13.84
40	29.74	2.30	13.70	18.05	23.69	41.29	8.28	11.55	12.99	18.76
Portfolio	59.65	1.33	6.72	9.27	12.43	20.95	3.52	4.98	5.18	7.70
Sum	29.74	1.33	8.74	12.66	17.78	29.84	4.75	6.89	7.14	10.76

Table 4.2b. Statistics for GAO per £100 single premium: 1984 Wilkie model, initial conditions of 31 December 1985, mortality PMA68U1986.

Term	NZ%	Mean	Q_{95}	$Q_{97.5}$	Q_{99}	$Q_{99.9}$	$C_{99,1}$	$C_{99,2}$	$C_{99.9,1}$	$C_{99.9,2}$
10	7.42	0.29	1.47	4.41	8.23	14.19	1.01	1.60	1.54	2.58
20	17.12	1.03	7.78	11.30	15.68	27.21	3.45	5.15	5.31	8.30
30	24.52	1.78	11.55	16.36	21.30	35.67	6.20	8.92	9.36	13.96
40	30.08	2.34	13.85	18.20	23.88	41.55	8.36	11.65	13.09	18.89
Portfolio	60.39	1.37	6.89	9.45	12.71	21.29	3.61	5.09	5.28	7.84
Sum	30.08	1.37	8.99	12.92	18.07	30.17	4.83	7.00	7.24	10.88

Table 4.2c. Present value of cost of GAO per £100 single premium: 1984 Wilkie model, initial conditions of 31 December 1986, mortality PMA68U1987.

Term	NZ%	Mean	Q_{95}	$Q_{97.5}$	Q_{99}	$Q_{99.9}$	$C_{99,1}$	$C_{99,2}$	$C_{99.9,1}$	$C_{99.9,2}$
10	11.06	0.46	3.26	6.24	10.25	16.33	1.34	2.08	1.89	3.07
20	19.84	1.23	8.87	12.52	17.00	28.80	3.83	5.66	5.73	8.88
30	26.18	1.95	12.30	17.14	22.21	36.77	6.54	9.36	9.73	14.46
40	31.14	2.48	14.36	18.73	24.51	42.36	8.63	12.00	13.40	19.29
Portfolio	63.82	1.55	7.59	10.20	13.70	22.62	3.95	5.53	5.68	8.38
Sum	31.14	1.55	9.98	13.98	19.24	31.54	5.18	7.46	7.62	11.39

Table 4.2d. Present value of cost of GAO per £100 single premium: 1984 Wilkie model, initial conditions of 31 December 1987, mortality PMA68U1988.

Term	NZ%	Mean	Q_{95}	$Q_{97.5}$	Q_{99}	$Q_{99.9}$	$C_{99,1}$	$C_{99,2}$	$C_{99.9,1}$	$C_{99.9,2}$
10	12.34	0.53	3.85	6.87	10.94	17.12	1.46	2.25	2.02	3.26
20	20.91	1.32	9.33	13.02	17.56	29.49	4.00	5.88	5.92	9.13
30	26.91	2.04	12.66	17.55	22.66	37.36	6.71	9.58	9.93	14.71
40	31.83	2.56	14.65	19.05	24.87	42.86	8.79	12.19	13.59	19.54
Portfolio	65.09	1.63	7.89	10.50	14.16	23.19	4.11	5.74	5.86	8.62
Sum	31.83	1.63	10.40	14.44	19.75	32.17	5.34	7.67	7.80	11.64

Table 4.2e. Present value of cost of GAO per £100 single premium: 1984 Wilkie model, initial conditions of 31 December 1988, mortality PMA68U1989.

Term	NZ%	Mean	Q_{95}	$Q_{97.5}$	Q_{99}	$Q_{99.9}$	$C_{99,1}$	$C_{99,2}$	$C_{99.9,1}$	$C_{99.9,2}$
10	10.68	0.44	3.11	6.10	10.08	16.20	1.31	2.04	1.86	3.04
20	19.82	1.23	8.88	12.53	17.04	28.87	3.84	5.67	5.74	8.90
30	26.37	1.98	12.41	17.28	22.37	37.01	6.59	9.43	9.80	14.55
40	31.49	2.53	14.54	18.93	24.75	42.71	8.73	12.12	13.53	19.46
Portfolio	63.86	1.56	7.65	10.27	13.75	22.70	3.97	5.56	5.71	8.41
Sum	31.49	1.56	10.03	14.04	19.32	31.67	5.21	7.51	7.66	11.45

Table 4.2f. Present value of cost of GAO per £100 single premium: 1984 Wilkie model, initial conditions of 31 December 1989, mortality PMA68U1990.

Term	NZ%	Mean	Q_{95}	$Q_{97.5}$	Q_{99}	$Q_{99.9}$	$C_{99,1}$	$C_{99,2}$	$C_{99.9,1}$	$C_{99.9,2}$
10	7.85	0.31	1.70	4.68	8.70	14.68	1.07	1.70	1.61	2.68
20	18.04	1.11	8.27	11.86	16.29	28.08	3.61	5.37	5.51	8.59
30	25.54	1.90	12.14	17.04	22.06	36.72	6.47	9.28	9.68	14.41
40	31.20	2.50	14.47	18.91	24.67	42.67	8.69	12.07	13.50	19.43
Portfolio	62.03	1.47	7.28	9.89	13.28	21.99	3.80	5.34	5.49	8.12
Sum	31.20	1.47	9.49	13.49	18.73	31.08	5.04	7.28	7.49	11.23

Table 4.2g. Present value of cost of GAO per £100 single premium: 1984 Wilkie model, initial conditions of 31 December 1990, mortality PMA68U1991.

Term	NZ%	Mean	Q_{95}	$Q_{97.5}$	Q_{99}	$Q_{99.9}$	$C_{99,1}$	$C_{99,2}$	$C_{99.9,1}$	$C_{99.9,2}$
10	5.62	0.22	0.40	3.27	7.16	13.07	0.85	1.37	1.38	2.34
20	16.29	0.98	7.52	11.06	15.53	27.26	3.38	5.07	5.28	8.28
30	24.61	1.81	11.74	16.65	21.64	36.23	6.31	9.07	9.51	14.18
40	30.78	2.45	14.28	18.73	24.47	42.47	8.60	11.96	13.41	19.32
Portfolio	59.97	1.37	6.83	9.49	12.61	21.17	3.59	5.06	5.26	7.80
Sum	30.78	1.37	8.87	12.85	18.05	30.32	4.85	7.03	7.29	10.96

Table 4.2h. Present value of cost of GAO per £100 single premium: 1984 Wilkie model, initial conditions of 31 December 1991, mortality PMA68U1992.

Term	NZ%	Mean	Q_{95}	$Q_{97.5}$	Q_{99}	$Q_{99.9}$	$C_{99,1}$	$C_{99,2}$	$C_{99.9,1}$	$C_{99.9,2}$
10	10.85	0.45	3.18	6.25	10.25	16.50	1.34	2.07	1.90	3.09
20	20.47	1.29	9.28	12.94	17.50	29.52	3.96	5.84	5.90	9.12
30	27.24	2.08	12.88	17.86	22.97	37.88	6.81	9.72	10.08	14.92
40	32.53	2.66	15.05	19.52	25.38	43.63	9.00	12.47	13.87	19.91
Portfolio	64.95	1.64	7.94	10.59	14.17	23.22	4.12	5.75	5.87	8.64
Sum	32.53	1.64	10.41	14.49	19.82	32.40	5.38	7.72	7.87	11.73

Table 4.2i. Present value of cost of GAO per £100 single premium: 1984 Wilkie model, initial conditions of 31 December 1992, mortality PMA68U1993.

Term	NZ%	Mean	Q_{95}	$Q_{97.5}$	Q_{99}	$Q_{99.9}$	$C_{99,1}$	$C_{99,2}$	$C_{99.9,1}$	$C_{99.9,2}$
10	15.75	0.74	5.34	8.47	12.74	19.04	1.82	2.71	2.38	3.74
20	23.92	1.58	10.56	14.37	19.07	31.38	4.56	6.48	6.44	9.82
30	29.13	2.30	13.76	18.76	24.05	39.16	7.22	10.24	10.52	15.50
40	33.81	2.83	15.62	20.12	26.09	44.54	9.31	12.86	14.23	20.37
Portfolio	69.03	1.89	8.78	11.42	15.25	24.80	4.53	6.27	6.37	9.30
Sum	33.81	1.89	11.58	15.73	21.21	34.00	5.81	8.29	8.34	12.35

Table 4.2j. Present value of cost of GAO per £100 single premium: 1984 Wilkie model, initial conditions of 31 December 1993, mortality PMA68U1994.

Term	NZ%	Mean	Q_{95}	$Q_{97.5}$	Q_{99}	$Q_{99.9}$	$C_{99,1}$	$C_{99,2}$	$C_{99.9,1}$	$C_{99.9,2}$
10	54.37	3.79	15.03	18.06	22.39	29.92	5.46	6.84	6.13	8.06
20	40.77	3.18	15.88	19.90	25.15	37.04	6.78	9.31	8.68	12.50
30	36.90	3.12	16.42	21.62	27.33	42.91	8.59	11.94	11.98	17.32
40	37.61	3.28	17.03	21.73	27.71	46.33	10.08	13.80	15.03	21.35
Portfolio	84.65	3.26	12.42	15.59	19.51	30.41	6.46	8.56	8.55	11.99
Sum	54.37	3.26	15.99	20.32	26.08	39.44	7.75	10.61	10.36	14.79

Table 4.2k. Present value of cost of GAO per £100 single premium: 1984 Wilkie model, initial conditions of 31 December 1994, mortality PMA68U1995.

Term	NZ%	Mean	Q_{95}	$Q_{97.5}$	Q_{99}	$Q_{99.9}$	$C_{99,1}$	$C_{99,2}$	$C_{99.9,1}$	$C_{99.9,2}$
10	12.23	0.55	4.03	7.21	11.59	18.09	1.55	2.37	2.13	3.43
20	22.86	1.54	10.54	14.36	19.29	31.86	4.46	6.51	6.48	9.92
30	29.46	2.39	14.11	19.34	24.59	40.08	7.41	10.49	10.79	15.87
40	34.65	2.96	16.09	20.75	26.70	45.55	9.58	13.19	14.59	20.85
Portfolio	68.67	1.89	8.78	11.54	15.34	24.77	4.55	6.30	6.37	9.29
Sum	34.65	1.89	11.57	15.81	21.36	34.49	5.87	8.37	8.45	12.52

Table 4.2l. Present value of cost of GAO per £100 single premium: 1984 Wilkie model, initial conditions of 31 December 1995, mortality PMA68U1996.

Term	NZ%	Mean	Q_{95}	$Q_{97.5}$	Q_{99}	$Q_{99.9}$	$C_{99,1}$	$C_{99,2}$	$C_{99.9,1}$	$C_{99.9,2}$
10	36.07	2.17	11.27	14.52	18.88	25.95	3.68	4.92	4.31	6.06
20	33.86	2.51	14.04	18.15	23.24	35.63	5.91	8.30	7.90	11.64
30	34.49	2.90	15.86	21.08	26.67	42.13	8.26	11.56	11.63	16.91
40	37.11	3.25	16.96	21.70	27.75	46.64	10.07	13.79	15.09	21.46
Portfolio	78.75	2.71	11.16	14.19	18.13	28.52	5.75	7.74	7.74	11.02
Sum	37.12	2.71	14.61	18.95	24.71	38.04	7.08	9.85	9.69	14.05

Table 4.2m. Present value of cost of GAO per £100 single premium: 1984 Wilkie model, initial conditions of 31 December 1996, mortality PMA68U1997.

Term	NZ%	Mean	Q_{95}	$Q_{97.5}$	Q_{99}	$Q_{99.9}$	$C_{99,1}$	$C_{99,2}$	$C_{99.9,1}$	$C_{99.9,2}$
10	26.95	1.53	9.22	12.50	17.19	23.89	2.94	4.11	3.54	5.19
20	30.88	2.24	13.19	17.27	22.24	35.13	5.52	7.83	7.59	11.31
30	33.58	2.82	15.65	20.86	26.43	42.10	8.15	11.42	11.56	16.85
40	36.99	3.25	17.02	21.69	27.81	47.32	10.09	13.82	15.15	21.55
Portfolio	76.24	2.49	10.61	13.52	17.59	28.00	5.47	7.42	7.47	10.71
Sum	37.00	2.49	13.95	18.30	24.05	37.44	6.80	9.53	9.43	13.75

Table 4.2n. Present value of cost of GAO per £100 single premium: 1984 Wilkie model, initial conditions of 31 December 1997, mortality PMA68U1998.

Term	NZ%	Mean	Q_{95}	$Q_{97.5}$	Q_{99}	$Q_{99.9}$	$C_{99,1}$	$C_{99,2}$	$C_{99.9,1}$	$C_{99.9,2}$
10	52.97	3.75	15.12	18.38	22.67	30.48	5.46	6.86	6.15	8.12
20	40.97	3.27	16.30	20.41	25.83	38.04	6.97	9.56	8.92	12.84
30	37.87	3.31	17.09	22.41	28.19	44.05	8.92	12.36	12.36	17.83
40	38.93	3.50	17.74	22.48	28.58	47.77	10.48	14.29	15.57	22.05
Portfolio	84.90	3.38	12.78	16.11	20.12	30.58	6.68	8.84	8.69	12.13
Sum	52.97	3.38	16.50	20.91	26.84	40.49	8.00	10.93	10.66	15.21

Table 4.2o. Present value of cost of GAO per £100 single premium: 1984 Wilkie model, initial conditions of 31 December 1998, mortality PMA68U1999.

Term	NZ%	Mean	Q_{95}	$Q_{97.5}$	Q_{99}	$Q_{99.9}$	$C_{99,1}$	$C_{99,2}$	$C_{99.9,1}$	$C_{99.9,2}$
10	81.41	8.89	24.23	27.96	33.04	42.09	11.06	12.84	11.86	14.28
20	56.05	5.51	21.82	26.34	32.37	45.85	9.90	12.96	12.04	16.54
30	46.37	4.53	20.58	26.24	32.41	49.44	10.80	14.63	14.47	20.45
40	43.97	4.27	19.97	24.94	31.34	51.28	11.78	15.88	17.04	23.89
Portfolio	94.64	5.43	17.19	20.71	25.03	36.78	9.28	11.80	11.52	15.46
Sum	81.41	5.43	21.31	26.09	32.52	47.39	10.67	14.01	13.53	18.59

Table 4.2p. Present value of cost of GAO per £100 single premium: 1984 Wilkie model, initial conditions of 31 December 1999, mortality PMA68U2000.

Term	NZ%	Mean	Q_{95}	$Q_{97.5}$	Q_{99}	$Q_{99.9}$	$C_{99,1}$	$C_{99,2}$	$C_{99.9,1}$	$C_{99.9,2}$
10	72.88	7.19	22.01	25.93	31.15	39.94	9.35	11.11	10.12	12.51
20	51.84	4.98	20.87	25.48	31.29	45.32	9.28	12.28	11.50	16.01
30	45.01	4.39	20.31	26.03	32.14	49.08	10.64	14.45	14.29	20.24
40	43.89	4.26	19.99	25.00	31.42	51.58	11.80	15.91	17.12	24.01
Portfolio	92.60	4.96	16.33	19.90	24.25	36.10	8.75	11.23	11.01	14.92
Sum	72.88	4.96	20.56	25.39	31.79	46.70	10.16	13.47	13.03	18.07

Table 4.2q. Present value of cost of GAO per £100 single premium: 1984 Wilkie model, initial conditions of 31 December 2000, mortality PMA68U2001.

Term	NZ%	Mean	Q_{95}	$Q_{97.5}$	Q_{99}	$Q_{99.9}$	$C_{99,1}$	$C_{99,2}$	$C_{99.9,1}$	$C_{99.9,2}$
10	88.95	11.36	27.48	31.36	36.55	45.67	13.62	15.48	14.43	16.93
20	65.00	7.03	24.62	29.28	35.36	49.24	11.66	14.88	13.85	18.55
30	54.04	5.68	23.08	28.81	35.10	52.40	12.29	16.32	16.00	22.21
40	50.93	5.28	22.25	27.25	33.76	54.02	13.18	17.48	18.50	25.58
Portfolio	97.40	6.88	19.77	23.33	27.74	39.73	10.97	13.64	13.25	17.37
Sum	88.95	6.88	23.99	28.87	35.40	50.57	12.43	16.04	15.27	20.62

Table 4.2r. Present value of cost of GAO per £100 single premium: 1984 Wilkie model, initial conditions of 31 December 2001, mortality PMA68U2002.

Term	NZ%	Mean	Q_{95}	$Q_{97.5}$	Q_{99}	$Q_{99.9}$	$C_{99,1}$	$C_{99,2}$	$C_{99.9,1}$	$C_{99.9,2}$
10	74.04	7.49	22.54	26.52	31.86	40.69	9.68	11.48	10.46	12.88
20	52.93	5.16	21.35	26.03	31.87	46.10	9.53	12.58	11.79	16.35
30	45.99	4.54	20.74	26.51	32.69	49.80	10.87	14.74	14.56	20.58
40	44.70	4.40	20.38	25.43	31.92	52.27	12.04	16.20	17.40	24.36
Portfolio	93.03	5.14	16.75	20.34	24.77	36.74	8.99	11.52	11.28	15.24
Sum	74.04	5.14	21.02	25.90	32.37	47.45	10.42	13.77	13.32	18.42

4.2.2 A number of general remarks can be made on the basis of the results in the tables above, some of which can be confirmed by separate experiments:

(a) As the years progress the costs drift upwards because of the assumption of steadily improving mortality rates.
(b) The costs are higher when the initial 'consols' yield is lower, because there is then a greater chance that future interest rates will also be lower; this is probably the most important single factor affecting the results from year to year.
(c) The costs are higher when the initial inflation rate is lower, because a low inflation rate in one year implies, according to the Wilkie model, lower inflation and also lower interest rates in future years.
(d) The costs for shorter terms vary much more when the initial conditions change than do the costs for longer terms.
(e) When interest rates are high the costs increase with term; when interest rates are lower the costs may reduce with term, or reduce first and increase later (see for example Table 4.2r), though this effect varies with the different measures that are shown.

(f) The values of $C_{99,2}$ and $C_{99.9,1}$, the calculated policyholder's contributions on two different bases, are often quite close; sometimes one is the greater, sometimes the other.
(g) The sum of the costs for individual policies always exceeds the costs when the portfolio is treated as a unit.
(h) Over the whole period the costs have increased, but the effect has been proportionately different for different measures of cost; thus for term 10 between 1984/85 and 2001/02 (compare Tables 4.2a and 4.2r) the average has multiplied by 28.8, whereas the 99.9% quantile has multiplied by only 2.97; the value of $C_{99,1}$ has multiplied by 10.3, but the value of $C_{99.9,2}$ has multiplied by 5.2.

4.2.3 Some of these results can be seen from Figure 4.1, which shows values of Q_{99} for Term 10, Term 40 and the Portfolio, mortality PMA68, 1984 Wilkie model, along with values of 100/consols yield%.

4.3 *'At-the-money-ness'*

4.3.1 When a traded option is purchased, it is common to consider whether an option is 'in the money' or 'out of the money', which depends on whether the current price of the security on which the option is written is above or below the exercise price. The same would apply to guaranteed annuity options. However, although the current price of the annuity

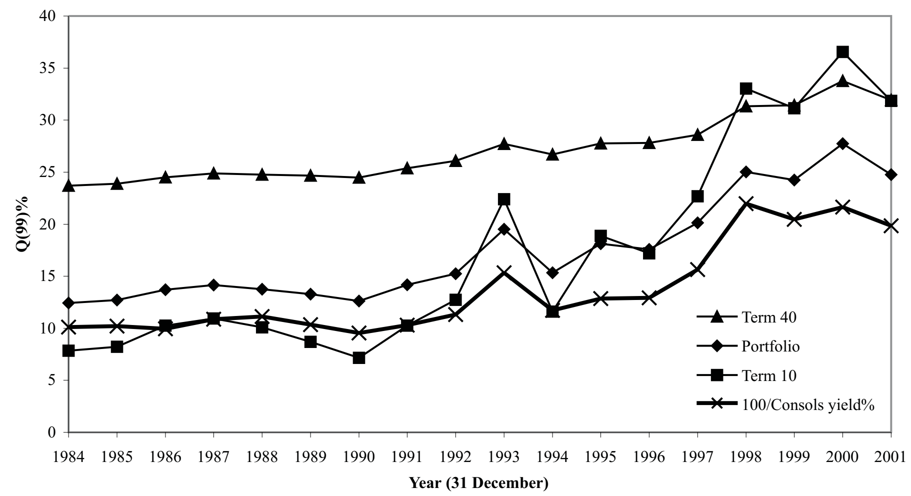

Figure 4.1. Values of Q_{99} for term 10, term 40 and the portfolio, 1984 Wilkie model, mortality PMA68, and of 100/consols yield%.

obviously depends on market interest rates, it also depends on projected mortality rates. We therefore have to use a hypothetical current price for the annuity, based on the projected mortality for the policyholder as at the retirement age (of 65), together with current interest rates. No annuities would be on the market at these rates, because the policyholders now aged 65 would be assumed to have different mortality, but we can assume, in effect, that these are the rates that would apply if the projected mortality were still the same when the policyholder reaches age 65, and if the then current interest rates were the same as they are now.

4.3.2 In Table 4.3.1 we show the annuity rate per mille, using, as at for each year the interest rates at the preceding 31 December, and the projected mortality for contracts with terms to retirement of 10, 20, 30 and 40 years, i.e. for policyholders aged 55, 45, 35 and 25 at the time of purchase. We show the rates per £1,000 purchase price, as usual for annuities paid annually in advance with no guaranteed periods, which should be compared with the rate of £111 per £1,000 which we have assumed for the guaranteed rate. If the rate shown is above £111 the guarantee is out of the money and if it is less than £111 it is in the money.

Table 4.3.1. At-the-money annuity rates per £1,000, calculated using the noted mortality basis (PMA68U*yyyy* or PA(90)M), and the specified consols yield.

Interest rates at 31 December	Policies entering, *yyyy*	Consols yield %	PMA68U*yyyy*				PA(90)M
			Term in years				
			10	20	30	40	
1984	1985	9.90	129.9	128.6	127.3	126.1	131.6
1985	1986	9.80	129.1	127.8	126.5	125.3	130.9
1986	1987	10.06	130.8	129.4	128.2	127.0	132.7
1987	1988	9.21	124.8	123.5	122.2	121.0	126.9
1988	1989	8.99	123.2	121.9	120.6	119.4	125.4
1989	1990	9.66	127.6	126.3	125.1	123.9	130.0
1990	1991	10.48	133.1	131.8	130.6	129.4	135.5
1991	1992	9.71	127.7	126.4	125.2	124.0	130.3
1992	1993	8.83	121.6	120.3	119.0	117.8	124.4
1993	1994	6.52	105.6	104.3	103.0	101.8	108.7
1994	1995	8.53	119.2	117.9	116.7	115.5	122.3
1995	1996	7.78	114.0	112.7	111.4	110.2	117.2
1996	1997	7.74	113.6	112.3	111.0	109.8	117.0
1997	1998	6.39	104.2	102.9	101.6	100.4	107.9
1998	1999	4.55	91.6	90.3	89.0	87.8	95.5
1999	2000	4.89	93.8	92.5	91.2	89.9	97.8
2000	2001	4.62	91.8	90.5	89.2	88.0	96.0
2001	2002	5.04	94.5	93.2	92.0	90.7	98.8

4.3.3 It can be observed that the annuity rates reduce as the term to go until retirement increases; and are lower for all the PA68U rates than for PA(90)M and that, of course, the rates are higher the higher the interest rate. The rates in 1984 are such that £111 per £1,000 is well out of the money, but that they all get into the money by 1993, when the consols yield drops to 6.52%, but go out again for the next few years; that the rates move back below £111 by 1996 for the longer terms, and for all terms from 1997 onwards. The increasing cost of the guarantee annuity options we have shown in Section 4.2 is therefore not surprising.

5. New Mortality Tables Since 1985

5.1 *The C.M.I. Experience*

5.1.1 In Section 2 we assumed that the actuary had used the projected tables from which the PA(90) tables were constructed, for each age using the mortality table appropriate to the year of birth of the policyholder, which we described as the PMA68B*yyyy* tables. We now consider how the contingency reserves would have changed as the actuary took into account the latest experience published by the C.M.I. Bureau and the subsequent sets of mortality tables in the '80 series' and '92 series', denoted the PMA80B*yyyy* and PMA92B*yyyy* tables.

5.1.2 The actuary who followed with care the *C.M.I. Reports* might have observed that, from 1977-79 onwards, the experience rates fell steadily below those forecast on the basis of the 1967-70 experience. Table 3.1.2 of a report in *C.M.I.R.* **8** (1986), page 34, shows the mortality experience of pensioners insured by life offices, on the basis of amounts, as a percentage of the rates expected under the projected tables which we have described as the PMA68 tables. The percentages for all ages, for males and females, are as shown in Table 5.1.1. It can be seen that, by 1982, the rates were around 90%

Table 5.1.1. Experience of life office pensioners who retired at or after the normal retirement age: actual deaths, on the basis of amounts, expressed as a percentage of those expected on the PA(90) projected rates, all ages

Year	Males	Females
1975	100	102
1976	105	103
1977	101	98
1978	101	97
1979	92	96
1980	93	94
1981	91	93
1982	91	90

of those forecast for that date, and in fact were not too far away from the level of PA(90), which it had not been expected would be reached till 1990.

5.1.3 New tables, with a basis for projection, based on the experience of 1979-82, the '80 series' tables were published in *C.M.I.R.* **10** in 1990. Even if our hypothetical actuary had not altered his basis on the evidence shown above, it is reasonable to suppose that he might have revised the calculations described in Section 4.2 to use the new mortality basis. This is discussed further in Section 5.2.

5.1.4 Further *C.M.I. Reports* should have altered our actuary's opinion. The reports in *C.M.I.R.* **14** (1995) and *C.M.I.R.* **16** (1998) show the experience of pensioners in insured group pension schemes as compared with the rates projected on the '80 series' for the corresponding calendar year. The percentages for all ages (ages 61-100 from 1987 onwards), for males and females, are as shown in Table 5.1.2. It can be seen that the mortality of males was improving much faster than expected, though the mortality of females was improving rather more slowly than expected.

5.1.5 Further new tables based on the experience of 1991-94, the '92 series' tables, were published in *C.M.I.R.* **16** (1998) and a new basis for projection was published in *C.M.I.R.* **17** in 1999. Again it is reasonable to suppose that the actuary would have revised his calculations to use this new mortality basis. This is discussed further in Section 5.3.

5.1.6 In *C.M.I.R.* **19** (2000) a further four years' data is made available for pensioners in insured group pension schemes. The percentages for ages 61-100, for males and females, as compared with the rates projected on the

Table 5.1.2. Experience of life office pensioners who retired at or after the normal retirement age: actual deaths, on the basis of amounts, expressed as a percentage of those expected on the PMA80 and PFA80 tables for the corresponding year.

Year	Males	Females
1983	97	108
1984	97	105
1985	96	103
1986	92	88
1987	91	110
1988	90	102
1989	92	107
1990	85	101
1991	78	100
1992	84	115
1993	83	92
1994	79	120

Table 5.1.3. Experience of life office pensioners who retired at or after the normal retirement age: actual deaths, on the basis of amounts, expressed as a percentage of those expected on the PMA92 and PFA92 tables for the corresponding year.

Year	Males	Females
1991	103	109
1992	103	101
1993	100	96
1994	95	93
1995	94	98
1996	89	90
1997	84	85
1998	86	85

'92 series' for the corresponding calendar year are as shown in Table 5.1.3. Yet again the mortality, this time of both sexes, was improving faster than projected. This would perhaps justify reconsidering the basis once again. However, we have not carried out any further calculations on these lines.

5.1.7 Guaranteed annuity options have often been added to self-employed retirement annuities or to personal pensions, and it may be thought that the mortality of such groups of policyholders would be more relevant to them than is the mortality of group pensioners. The first set of standard tables published by the C.M.I. Bureau for retirement annuitants, in the vested section of the investigation, were in the '92 series', the RMV92 and RFV92 tables (*C.M.I.R.* **17**, 1999). The projection factors recommended for use with them are the same as for group pensioners.

5.1.8 The mortality rates of retirement annuitants in the vested section are relatively high at younger pensioner ages (i.e. 50 to 70), presumably because those in ill health start to draw their pensions rather than continue working. The mortality of a healthy retirement annuity policyholder retiring at age 65 might be rather lighter than the published table, just as the mortality of pensioners who retire at or after the normal pension age (on which the Pxx92 tables are based) is rather lighter than that of the combined pensioners (including early retirements) reflected in the PCxx92 tables (see *C.M.I.R.* **19**, 2000). The mortality table based on the combined experience of both deferred and vested retirement annuitants is perhaps more appropriate. We make use of this, denoting it as the RMC92 table, even though it is not a standard C.M.I. table. We discuss this further in Section 5.4.

5.1.9 Figures 5.1 to 5.3 show the values of q_x for various calendar year and year of birth mortality tables, all for males ages 65 to 119, and all expressed as relative to the values of q_x for PMA68Base. Figure 5.1 compares the various base tables, PMA68Base, PMA80Base, PMA92Base and RMC92Base, as well as PA(90)Males. One can observe that the overall level of PMA80base

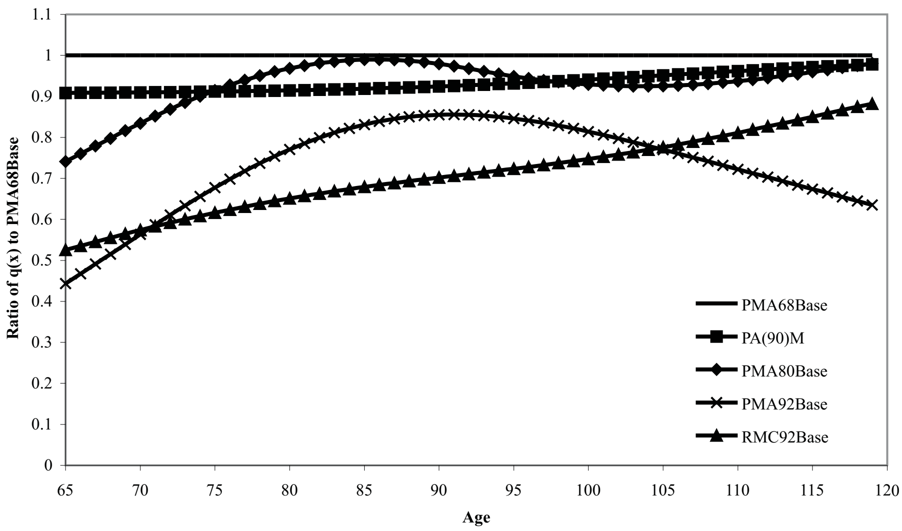

Figure 5.1. Mortality rates of tables PMA68Base, PA(90)Males, PMA80Base, PMA92Base and RMC92Base, all expressed as relative to PMA68Base; ages 65 to 119.

is not very different from that of PA(90)Males, but with a different shape; remember that it is applicable to 1979-82, not to 1990. The levels of PMA92Base and RMC92Base, applicable to 1991-94, are considerably lower than PMA80Base, but have similar levels, and, again, different shapes.

5.1.10 Figure 5.2 compares projected year of birth tables for a male born in 1930 (who was therefore 55 in 1985, and is the oldest life we have considered), again relative to the values of q_x for PMA68Base. One can see how the rates generally reduce as the base year increases. Figure 5.3 shows the same for a male born in 1977 (who was therefore 25 in 2002, and is the youngest life we have considered). The same results can be observed.

5.1.11 One can calculate the 'at-the-money' annuity rates for each of these tables, on the same basis as in Section 4.3. We show results in Table 5.1.4a, for term 10 (age 55 at entry), and Table 5.1.4b, for term 40 (age 25 at entry). One can observe how substantial the improvements in mortality have been, and how a rate of £111 per £1,000, which might have seemed tolerably out of the money in 1985, had become very much in the money by 2002, both because of the reduction in interest rates and because of the improvement in mortality. Thus for term 10 the at-the-money annuity rate fell from £129.9 per £1,000 in 1985 to £94.5 in 2002, but fell further to £77.9 because of the change from PMA68 to RMC92 projected mortality. For term 40 the comparable figures are £126.1, £90.7 and £74.3.

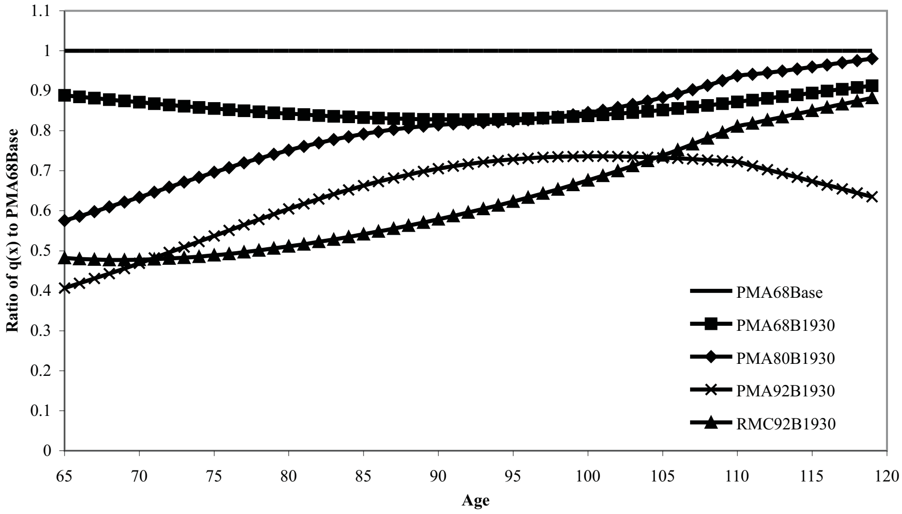

Figure 5.2. Mortality rates of tables PMA68Base, PMA68B1930, PMA80B1930, PMA92B1930 and RMC92B1930, all expressed as relative to PMA68Base; ages 65 to 119.

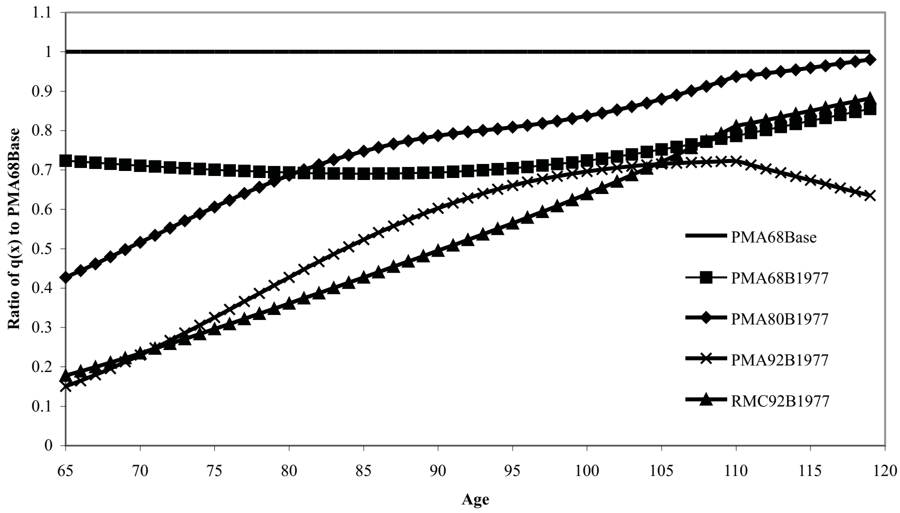

Figure 5.3. Mortality rates of tables PMA68Base, PMA8B1977, PMA80B1977, PMA92B1977 and RMC92B1977, all expressed as relative to PMA68Base; ages 65 to 119.

Table 5.1.4a. At-the-money annuity rates per £1,000, for term 10 years, i.e. aged 55 at entry, calculated using the noted mortality basis (PMA68U*yyyy*, PMA80U*yyyy*, PMA92U*yyyy* and RMC92U*yyyy*, and the specified consols yield.

Interest rates at 31 December	Policies entering, yyyy	Consols yield %	PMA68	PMA80	PMA92	RMC92
1984	1985	9.90	129.9	122.6	116.7	115.9
1985	1986	9.80	129.1	121.7	115.7	114.8
1986	1987	10.06	130.8	123.3	117.2	116.3
1987	1988	9.21	124.8	117.4	111.1	110.1
1988	1989	8.99	123.2	115.7	109.3	108.2
1989	1990	9.66	127.6	120.1	113.5	112.6
1990	1991	10.48	133.1	125.5	118.8	118.0
1991	1992	9.71	127.7	120.2	113.3	112.3
1992	1993	8.83	121.6	114.1	107.0	106.0
1993	1994	6.52	105.6	98.4	91.2	89.8
1994	1995	8.53	119.2	111.8	104.5	103.4
1995	1996	7.78	114.0	106.6	99.1	97.9
1996	1997	7.74	113.6	106.2	98.6	97.4
1997	1998	6.39	104.2	97.1	89.3	87.9
1998	1999	4.55	91.6	84.8	77.0	75.3
1999	2000	4.89	93.8	86.9	79.0	77.3
2000	2001	4.62	91.8	85.1	77.0	75.3
2001	2002	5.04	94.5	87.7	79.6	77.9

Table 5.1.4b. At-the-money annuity rates per £1,000, for term 40 years, i.e. aged 25 at entry, calculated using the noted mortality basis (PMA68U*yyyy*, PMA80U*yyyy*, PMA92U*yyyy* and RMC92U*yyyy*, and the specified consols yield.

Interest rates at 31 December	Policies entering, yyyy	Consols yield %	PMA68	PMA80	PMA92	RMC92
1984	1985	9.90	126.1	119.4	110.2	109.2
1985	1986	9.80	125.3	118.7	109.4	108.4
1986	1987	10.06	127.0	120.4	111.0	110.0
1987	1988	9.21	121.0	114.6	105.1	104.0
1988	1989	8.99	119.4	113.1	103.5	102.4
1989	1990	9.66	123.9	117.6	108.0	106.9
1990	1991	10.48	129.4	123.1	113.5	112.5
1991	1992	9.71	124.0	117.8	108.1	107.1
1992	1993	8.83	117.8	111.9	102.0	100.8
1993	1994	6.52	101.8	96.3	86.3	84.9
1994	1995	8.53	115.5	109.8	99.7	98.6
1995	1996	7.78	110.2	104.7	94.6	93.3
1996	1997	7.74	109.8	104.4	94.2	92.9
1997	1998	6.39	100.4	95.3	85.1	83.6
1998	1999	4.55	87.8	83.2	72.9	71.3
1999	2000	4.89	89.9	85.4	75.0	73.4
2000	2001	4.62	88.0	83.6	73.2	71.6
2001	2002	5.04	90.7	86.3	75.8	74.3

5.2 *Results Using the PMA80 Projected Mortality*

5.2.1 Tables 5.2f to 5.2r show the same results as Tables 4.2f to 4.2r, but using PMA80 projected mortality. For policies entering in year *zzzz* we use mortality PA80U*zzzz* and market investment conditions as at 31 December of year $zzzz - 1$. The subscript letters (f to r) allow one to match up results for corresponding years. Comparison of Tables 4.2f and 5.2f shows that the change in mortality basis caused the mean value of the GAO at term 10 to increase from 0.31 to 1.30 and for the portfolio from 1.47 to 3.50; the quantile reserves at a 99% level increased for the portfolio, from 13.28 to 21.52. The increases were proportionately greater for shorter terms than for longer terms, and also proportionately greater for lower quantiles than for higher ones. The same remarks are true for other entry years.

Table 5.2f. Present value of cost of GAO per £100 single premium: 1984 Wilkie model, initial conditions of 31 December 1989, mortality PMA80U1990.

Term	NZ%	Mean	Q_{95}	$Q_{97.5}$	Q_{99}	$Q_{99.9}$	$C_{99,1}$	$C_{99,2}$	$C_{99.9,1}$	$C_{99.9,2}$
10	22.51	1.30	8.51	12.10	16.98	24.23	2.72	3.88	3.36	5.06
20	35.90	3.01	16.43	20.84	26.30	40.89	6.83	9.50	9.15	13.40
30	41.96	4.26	20.62	26.60	32.74	50.70	10.66	14.57	14.53	20.69
40	46.49	4.99	22.49	27.81	34.70	56.27	13.22	17.69	18.86	26.27
Portfolio	82.03	3.50	13.59	17.00	21.52	32.89	7.04	9.37	9.22	12.93
Sum	46.49	3.50	17.52	22.40	28.79	43.89	8.59	11.79	11.53	16.49

Table 5.2g. Present value of cost of GAO per £100 single premium: 1984 Wilkie model, initial conditions of 31 December 1990, mortality PMA80U1991.

Term	NZ%	Mean	Q_{95}	$Q_{97.5}$	Q_{99}	$Q_{99.9}$	$C_{99,1}$	$C_{99,2}$	$C_{99.9,1}$	$C_{99.9,2}$
10	17.89	0.98	6.96	10.43	15.14	22.32	2.26	3.31	2.90	4.48
20	33.17	2.71	15.47	19.82	25.33	39.82	6.42	9.02	8.73	12.90
30	40.61	4.07	20.05	26.04	32.11	49.96	10.37	14.23	14.22	20.32
40	45.77	4.87	22.15	27.47	34.32	55.83	13.02	17.46	18.66	26.02
Portfolio	80.12	3.25	13.00	16.26	20.82	31.83	6.71	8.97	8.82	12.43
Sum	45.77	3.25	16.72	21.56	27.89	42.87	8.24	11.37	11.16	16.04

Table 5.2h. Present value of cost of GAO per £100 single premium: 1984 Wilkie model, initial conditions of 31 December 1991, mortality PMA80U1992.

Term	NZ%	Mean	Q_{95}	$Q_{97.5}$	Q_{99}	$Q_{99.9}$	$C_{99,1}$	$C_{99,2}$	$C_{99.9,1}$	$C_{99.9,2}$
10	27.85	1.73	10.03	14.05	18.91	26.51	3.28	4.56	3.96	5.79
20	38.89	3.40	17.59	22.09	27.71	42.56	7.38	10.16	9.74	14.12
30	43.64	4.53	21.36	27.42	33.64	51.83	11.07	15.06	14.98	21.25
40	47.41	5.17	22.95	28.29	35.27	57.01	13.50	18.03	19.18	26.66
Portfolio	83.92	3.82	14.44	17.91	22.50	34.28	7.49	9.90	9.74	13.58
Sum	47.49	3.82	18.53	23.49	29.99	45.32	9.06	12.36	12.04	17.12

Table 5.2i. Present value of cost of GAO per £100 single premium: 1984 Wilkie model, initial conditions of 31 December 1992, mortality PMA80U1993.

Term	NZ%	Mean	Q_{95}	$Q_{97.5}$	Q_{99}	$Q_{99.9}$	$C_{99,1}$	$C_{99,2}$	$C_{99.9,1}$	$C_{99.9,2}$
10	36.82	2.53	12.97	16.76	21.95	29.63	4.28	5.72	4.96	6.95
20	42.93	3.98	19.13	23.81	29.59	44.79	8.17	11.09	10.58	15.13
30	45.98	4.90	22.33	28.40	34.84	53.23	11.62	15.72	15.56	21.96
40	48.59	5.40	23.52	28.88	35.96	57.88	13.85	18.45	19.58	27.14
Portfolio	86.81	4.31	15.66	19.19	23.87	36.08	8.15	10.67	10.48	14.47
Sum	48.83	4.31	19.90	24.94	31.60	47.18	9.74	13.15	12.75	17.97

Table 5.2j. Present value of cost of GAO per £100 single premium: 1984 Wilkie model, initial conditions of 31 December 1993, mortality PMA80U1994.

Term	NZ%	Mean	Q_{95}	$Q_{97.5}$	Q_{99}	$Q_{99.9}$	$C_{99,1}$	$C_{99,2}$	$C_{99.9,1}$	$C_{99.9,2}$
10	81.00	9.03	24.75	28.45	33.73	42.94	11.25	13.07	12.07	14.54
20	62.31	7.04	25.61	30.55	37.02	51.70	11.93	15.33	14.25	19.20
30	54.67	6.33	25.46	31.77	38.70	57.63	13.57	17.99	17.62	24.37
40	52.38	6.08	25.07	30.64	37.73	59.78	14.83	19.58	20.57	28.29
Portfolio	95.86	6.93	20.73	24.66	29.40	42.77	11.33	14.20	13.86	18.32
Sum	79.92	6.93	24.92	30.16	37.13	44.08	12.88	16.64	15.98	21.61

Table 5.2k. Present value of cost of GAO per £100 single premium: 1984 Wilkie model, initial conditions of 31 December 1994, mortality PMA80U1995.

Term	NZ%	Mean	Q_{95}	$Q_{97.5}$	Q_{99}	$Q_{99.9}$	$C_{99,1}$	$C_{99,2}$	$C_{99.9,1}$	$C_{99.9,2}$
10	28.84	1.93	11.39	15.24	20.56	28.46	3.61	5.00	4.32	6.27
20	41.08	3.81	19.00	23.67	29.73	45.21	8.05	11.01	10.51	15.12
30	45.92	4.95	22.57	28.89	35.24	54.02	11.75	15.89	15.77	22.25
40	49.10	5.51	23.84	29.34	36.37	58.63	14.05	18.68	19.85	27.50
Portfolio	85.70	4.20	15.43	18.96	23.68	35.22	8.03	10.53	10.23	14.13
Sum	49.20	4.20	19.74	24.88	31.60	47.53	9.66	13.09	12.73	18.00

Table 5.2l. Present value of cost of GAO per £100 single premium: 1984 Wilkie model, initial conditions of 31 December 1995, mortality PMA80U1996.

Term	NZ%	Mean	Q_{95}	$Q_{97.5}$	Q_{99}	$Q_{99.9}$	$C_{99,1}$	$C_{99,2}$	$C_{99.9,1}$	$C_{99.9,2}$
10	63.67	5.87	20.16	24.11	29.43	38.06	7.99	9.72	8.75	11.10
20	54.01	5.72	23.22	28.26	34.50	49.74	10.42	13.69	12.83	17.71
30	51.40	5.83	24.59	30.88	37.63	56.32	12.96	17.30	16.95	23.60
40	51.45	5.92	24.73	30.32	37.45	59.69	14.63	19.36	20.43	28.15
Portfolio	92.58	5.82	18.79	22.66	27.39	40.34	10.05	12.81	12.50	16.81
Sum	63.67	5.82	23.37	28.62	35.59	51.73	11.61	15.36	14.61	20.22

Table 5.2m. Present value of cost of GAO per £100 single premium: 1984 Wilkie model, initial conditions of 31 December 1996, mortality PMA80U1997.

Term	NZ%	Mean	Q_{95}	$Q_{97.5}$	Q_{99}	$Q_{99.9}$	$C_{99,1}$	$C_{99,2}$	$C_{99.9,1}$	$C_{99.9,2}$
10	53.03	4.41	17.66	21.64	27.35	35.52	6.48	8.17	7.20	9.48
20	50.24	5.16	22.12	27.10	33.18	49.02	9.74	12.93	12.24	17.11
30	50.08	5.64	24.22	30.50	37.21	56.09	12.71	17.02	16.75	23.40
40	51.08	5.86	24.68	30.16	37.36	59.73	14.57	19.30	20.40	28.14
Portfolio	90.93	5.34	17.94	21.70	26.53	39.34	9.49	12.21	11.93	16.18
Sum	53.03	5.34	22.49	27.74	34.67	50.85	11.10	14.74	14.21	19.71

Table 5.2n. Present value of cost of GAO per £100 single premium: 1984 Wilkie model, initial conditions of 31 December 1997, mortality PMA80U1998.

Term	NZ%	Mean	Q_{95}	$Q_{97.5}$	Q_{99}	$Q_{99.9}$	$C_{99,1}$	$C_{99,2}$	$C_{99.9,1}$	$C_{99.9,2}$
10	79.02	8.85	24.80	28.77	34.00	43.53	11.11	12.96	11.96	14.48
20	61.40	7.03	25.83	30.87	37.50	52.47	12.00	15.45	14.35	19.39
30	54.85	6.43	25.84	32.23	39.18	58.25	13.76	18.22	17.83	24.64
40	52.80	6.21	25.39	30.95	38.09	60.54	15.01	19.80	20.86	28.66
Portfolio	95.65	6.95	20.87	24.76	29.68	42.61	11.40	14.31	13.85	18.29
Sum	79.02	6.95	25.49	30.81	37.96	54.42	12.99	16.80	16.13	21.82

Table 5.2o. Present value of cost of GAO per £100 single premium: 1984 Wilkie model, initial conditions of 31 December 1998, mortality PMA80U1999.

Term	NZ%	Mean	Q_{95}	$Q_{97.5}$	Q_{99}	$Q_{99.9}$	$C_{99,1}$	$C_{99,2}$	$C_{99.9,1}$	$C_{99.9,2}$
10	94.45	16.51	35.85	40.40	46.61	57.69	19.21	21.41	20.18	23.16
20	74.88	10.59	32.48	38.00	45.39	61.91	16.24	20.17	18.83	24.46
30	63.07	8.30	29.91	36.69	44.08	64.51	16.29	21.14	20.62	27.94
40	57.59	7.28	27.86	33.67	41.15	64.38	16.61	21.68	22.63	30.78
Portfolio	98.85	10.10	26.22	30.47	35.67	49.87	15.10	18.34	17.77	22.68
Sum	94.45	10.10	31.22	36.98	44.73	62.64	16.75	20.96	20.12	26.33

Table 5.2p. Present value of cost of GAO per £100 single premium: 1984 Wilkie model, initial conditions of 31 December 1999, mortality PMA80U2000.

Term	NZ%	Mean	Q_{95}	$Q_{97.5}$	Q_{99}	$Q_{99.9}$	$C_{99,1}$	$C_{99,2}$	$C_{99.9,1}$	$C_{99.9,2}$
10	89.82	13.99	33.11	37.88	44.24	54.99	16.70	18.92	17.65	20.61
20	70.87	9.66	31.24	36.85	43.94	61.09	15.23	19.09	17.91	23.55
30	61.57	8.02	29.46	36.29	43.60	63.85	15.96	20.78	20.25	27.53
40	57.15	7.21	27.75	33.59	41.06	64.47	16.54	21.60	22.60	30.77
Portfolio	98.25	9.31	25.07	29.32	34.54	49.03	14.23	17.44	16.96	21.87
Sum	89.82	9.31	30.21	36.01	43.70	61.60	15.92	20.10	19.29	25.47

Table 5.2q. Present value of cost of GAO per £100 single premium: 1984 Wilkie model, initial conditions of 31 December 2000, mortality PMA80U2001.

Term	NZ%	Mean	Q_{95}	$Q_{97.5}$	Q_{99}	$Q_{99.9}$	$C_{99,1}$	$C_{99,2}$	$C_{99.9,1}$	$C_{99.9,2}$
10	95.02	17.35	37.05	41.71	47.98	59.27	20.10	22.34	21.09	24.12
20	75.94	10.99	33.22	38.81	46.28	63.01	16.72	20.70	19.34	25.03
30	63.79	8.52	30.36	37.20	44.65	65.24	16.58	21.47	20.94	28.32
40	58.10	7.40	28.14	33.98	41.49	64.85	16.80	21.89	22.84	31.04
Portfolio	98.96	10.46	26.80	31.09	36.33	50.66	15.51	18.79	18.20	23.16
Sum	95.02	10.46	31.86	37.68	45.50	63.60	17.18	21.43	20.57	26.84

Table 5.2r. Present value of cost of GAO per £100 single premium: 1984 Wilkie model, initial conditions of 31 December 2001, mortality PMA80U2002.

Term	NZ%	Mean	Q_{95}	$Q_{97.5}$	Q_{99}	$Q_{99.9}$	$C_{99,1}$	$C_{99,2}$	$C_{99.9,1}$	$C_{99.9,2}$
10	90.32	14.34	33.64	38.47	44.97	55.74	17.08	19.32	18.03	21.02
20	71.36	9.84	31.60	37.28	44.38	61.70	15.45	19.35	18.16	23.84
30	62.00	8.12	29.70	36.56	43.91	64.25	16.12	20.96	20.42	27.74
40	57.38	7.28	27.91	33.76	41.26	64.75	16.64	21.72	22.72	30.92
Portfolio	98.34	9.47	25.36	29.63	34.88	49.46	14.42	17.66	17.17	22.11
Sum	90.32	9.47	30.52	36.36	44.10	62.12	16.12	20.33	19.51	25.73

5.3 *Results Using the PMA92 Projected Mortality*

5.3.1 Tables 5.3o to 5.2r show the same results as Tables 4.2o to 4.2r and Tables 5.2o to 5.2r, but using PMA92 projected mortality. For policies entering in year *zzzz* we use mortality PMA92U*zzzz* and market investment conditions as at 31 December of year *zzzz* − 1. Comparison of Tables 5.3o and 5.2o again shows that the change in mortality basis caused all the values to increase again. For this year the increases were proportionately greater for longer terms than for shorter terms, but as before were proportionately greater for lower quantiles than for higher ones.

Table 5.3o. Present value of cost of GAO per £100 single premium: 1984 Wilkie model, initial conditions of 31 December 1998, mortality PMA92U1999.

Term	NZ%	Mean	Q_{95}	$Q_{97.5}$	Q_{99}	$Q_{99.9}$	$C_{99,1}$	$C_{99,2}$	$C_{99.9,1}$	$C_{99.9,2}$
10	98.94	27.35	51.69	57.47	65.39	79.65	30.74	33.51	31.99	35.72
20	90.28	20.47	50.28	57.61	67.48	89.86	28.04	33.23	31.46	38.84
30	82.84	17.65	48.80	58.04	68.21	96.79	28.79	35.45	34.62	44.47
40	79.28	16.52	47.44	55.49	65.96	99.15	29.92	37.09	38.12	49.28
Portfolio	99.96	19.86	42.74	48.42	55.48	74.80	26.75	31.19	30.30	36.88
Sum	98.94	19.86	49.34	57.07	67.56	92.24	29.01	34.71	33.53	41.80

Table 5.3p. Present value of cost of GAO per £100 single premium: 1984 Wilkie model, initial conditions of 31 December 1999, mortality PMA92U2000.

Term	NZ%	Mean	Q_{95}	$Q_{97.5}$	Q_{99}	$Q_{99.9}$	$C_{99,1}$	$C_{99,2}$	$C_{99.9,1}$	$C_{99.9,2}$
10	97.40	24.18	48.46	54.53	62.65	76.49	27.62	30.40	28.82	32.56
20	87.75	19.11	48.82	56.29	65.77	89.05	26.63	31.78	30.19	37.62
30	81.56	17.23	48.34	57.67	67.74	96.12	28.35	35.01	34.15	43.97
40	78.91	16.44	47.40	55.51	65.98	99.50	29.87	37.05	38.15	49.35
Portfolio	99.88	18.76	41.39	47.24	54.29	73.87	25.63	30.06	29.23	35.83
Sum	97.40	18.76	58.19	55.99	66.42	74.34	27.91	33.60	32.44	40.71

Table 5.3q. Present value of cost of GAO per £100 single premium: 1984 Wilkie model, initial conditions of 31 December 2000, mortality PMA92U2001.

Term	NZ%	Mean	Q_{95}	$Q_{97.5}$	Q_{99}	$Q_{99.9}$	$C_{99,1}$	$C_{99,2}$	$C_{99.9,1}$	$C_{99.9,2}$
10	99.09	28.78	53.72	59.68	67.74	82.40	32.26	35.09	33.54	37.36
20	90.92	21.30	51.65	59.12	69.18	91.99	29.00	34.28	32.49	39.99
30	83.63	18.16	49.73	59.09	69.39	98.35	29.44	36.18	35.34	45.30
40	79.71	16.85	48.07	56.19	66.75	100.27	30.37	37.60	38.64	49.89
Portfolio	99.98	20.59	43.86	49.60	56.71	76.36	27.57	32.07	31.17	37.85
Sum	99.09	20.59	50.54	58.39	69.05	94.12	29.87	35.64	34.44	42.83

Table 5.3r. Present value of cost of GAO per £100 single premium: 1984 Wilkie model, initial conditions of 31 December 2001, mortality PMA92U2002.

Term	NZ%	Mean	Q_{95}	$Q_{97.5}$	Q_{99}	$Q_{99.9}$	$C_{99,1}$	$C_{99,2}$	$C_{99.9,1}$	$C_{99.9,2}$
10	97.61	24.95	49.61	55.79	64.15	78.13	28.45	31.29	29.67	33.46
20	88.27	19.61	49.70	57.29	66.84	90.49	27.21	32.43	30.82	38.35
30	82.03	17.56	48.98	58.38	68.56	97.20	28.79	35.50	34.63	44.54
40	79.22	16.68	47.86	56.01	66.57	100.32	30.20	37.42	38.52	49.79
Portfolio	99.89	19.20	42.11	48.00	55.13	74.96	26.15	30.62	29.78	36.45
Sum	97.61	19.20	48.96	56.84	67.39	92.39	28.44	34.18	33.01	41.36

5.4 *Results Using the RMC92 Projected Mortality*

5.4.1 Tables 5.4o to 5.4r show the same results as Tables 4.2o to 4.2r, Tables 5.2o to 5.2r, and Tables 5.3o to 5.3r, but using RMC92 projected mortality. For policies entering in year *zzzz* we use mortality RMC92U*zzzz* and market investment conditions as at 31 December of year $zzzz - 1$. Comparison of Tables 5.4o and 5.3o again shows that the change in mortality basis caused all the values to increase again, but this time by a relatively small amount. Observe, however, that for terms 30 and 40 in 1999 and later the 99.9% quantile reserve now exceeds £100 per £100 single premium, and that in 2000 and later the possible extra premium on a 99.9%, 2% basis exceeds £50.

Table 5.4o. Present value of cost of GAO per £100 single premium: 1984 Wilkie model, initial conditions of 31 December 1998, mortality RMC92U1999.

Term	NZ%	Mean	Q_{95}	$Q_{97.5}$	Q_{99}	$Q_{99.9}$	$C_{99,1}$	$C_{99,2}$	$C_{99.9,1}$	$C_{99.9,2}$
10	99.12	29.07	54.50	60.59	68.99	84.17	32.64	35.52	33.96	37.88
20	91.28	21.75	52.64	60.31	70.69	94.41	29.62	35.01	33.23	40.92
30	84.43	18.81	51.11	60.78	71.47	101.78	30.39	37.29	36.54	46.79
40	80.90	17.66	49.80	58.24	69.25	104.50	31.61	39.06	40.87	51.87
Portfolio	99.98	21.14	44.89	50.81	58.21	78.56	28.30	32.90	32.02	38.87
Sum	99.12	21.14	51.73	59.83	70.87	97.06	30.66	36.57	35.43	44.04

Table 5.4p. Present value of cost of GAO per £100 single premium: 1984 Wilkie model, initial conditions of 31 December 1999, mortality RMC92U2000.

Term	NZ%	Mean	Q_{95}	$Q_{97.5}$	Q_{99}	$Q_{99.9}$	$C_{99,1}$	$C_{99,2}$	$C_{99.9,1}$	$C_{99.9,2}$
10	97.89	25.78	51.11	57.51	66.09	80.83	29.38	32.30	30.66	34.58
20	89.07	20.34	51.14	58.96	68.92	93.58	28.16	33.51	31.91	39.65
30	83.20	18.37	50.66	60.42	71.01	101.11	29.94	36.85	36.05	46.28
40	80.56	17.58	49.78	58.27	69.29	104.89	31.56	39.03	40.29	51.96
Portfolio	99.93	19.99	43.52	49.57	56.96	77.66	27.14	31.72	30.92	37.79
Sum	97.89	19.99	50.55	58.71	69.69	95.90	29.51	35.42	34.29	42.91

Table 5.4q. Present value of cost of GAO per £100 single premium: 1984 Wilkie model, initial conditions of 31 December 2000, mortality RMC92U2001.

Term	NZ%	Mean	Q_{95}	$Q_{97.5}$	Q_{99}	$Q_{99.9}$	$C_{99,1}$	$C_{99,2}$	$C_{99.9,1}$	$C_{99.9,2}$
10	99.31	30.59	56.66	62.97	71.52	87.16	34.24	37.20	35.60	39.62
20	91.91	22.63	54.12	61.96	72.55	96.76	30.65	36.14	34.33	42.15
30	84.88	19.36	52.13	61.93	72.77	103.52	31.09	38.09	37.32	47.69
40	81.40	18.01	50.49	59.00	70.11	105.74	32.10	39.61	40.82	52.53
Portfolio	99.98	21.91	46.09	52.09	59.62	80.26	29.19	33.87	32.96	39.91
Sum	99.31	21.91	53.04	61.27	72.50	99.14	31.58	37.58	36.41	45.15

Table 5.4r. Present value of cost of GAO per £100 single premium: 1984 Wilkie model, initial conditions of 31 December 2001, mortality RMC92U2002.

Term	NZ%	Mean	Q_{95}	$Q_{97.5}$	Q_{99}	$Q_{99.9}$	$C_{99,1}$	$C_{99,2}$	$C_{99.9,1}$	$C_{99.9,2}$
10	98.11	26.61	52.35	58.87	67.72	82.62	30.27	33.25	31.57	35.55
20	89.50	20.88	52.10	60.05	70.11	95.19	28.79	34.21	32.61	40.45
30	83.67	18.74	51.36	61.21	71.92	102.33	30.42	37.39	36.58	46.90
40	80.86	17.84	50.28	58.83	69.95	105.83	31.92	39.44	40.80	52.44
Portfolio	99.94	20.47	44.28	50.41	57.91	78.87	27.70	32.35	31.53	38.48
Sum	98.11	20.47	51.39	59.66	70.77	97.31	30.09	36.06	34.92	43.62

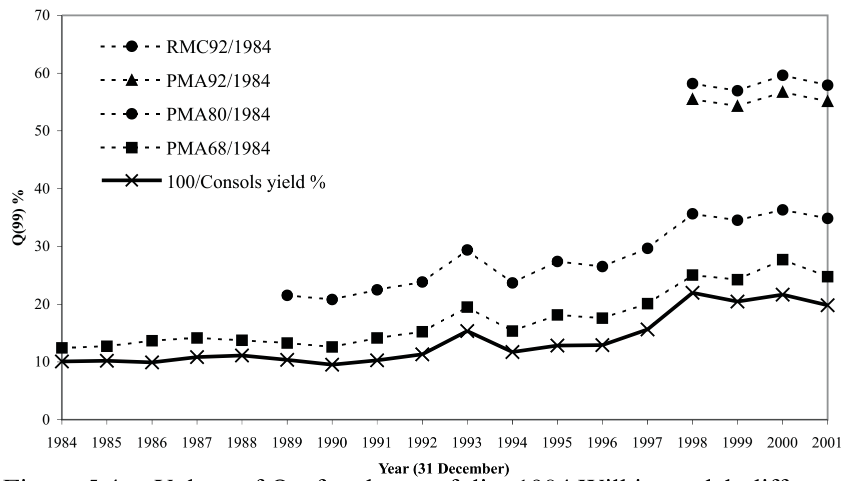

Figure 5.4. Values of Q_{99} for the portfolio, 1984 Wilkie model, different mortality tables and of 100/consols yield %.

5.5 *Comparison of Mortality Effects*

5.5.1 A comparison of the effects of different mortality rates can be seen from Figure 5.4, which shows values of Q_{99} for the Portfolio, on the 1984 Wilkie model, for the different mortality tables discussed in Sections 4 and 5, and also the values of 100/consols yield %. Observe how the values of Q_{99} rise as each new mortality table is introduced. The overall effect of mortality on Q_{99} is greater than the overall effect of the fall in interest rates.

6. The 1995 Wilkie Model

6.1 *Introduction*

6.1.1 In Wilkie (1995) the parameters of his 1984 model were updated, the structure was changed slightly, and another relevant variable, short-term interest rates, denoted B(t), was added. Changes in the model stated in Section 2.4 were no more than putting CA2 = CA3 = 0.0 and redefining CA1 as CA. The new model for B(t) can be stated as:

$$\mathrm{BE}(t) = \mathrm{BSD.BZ}(t)$$
$$\mathrm{BN}(t) = \mathrm{BA.BN}(t-1) + \mathrm{BE}(t)$$
$$\mathrm{B}(t) = \mathrm{C}(t).\exp(\mathrm{BN}(t) + \mathrm{BMU})$$

6.1.2 The new parameter values, including those for B(t), are taken from Wilkie (1995) and are: QMU = 0.047, QA = 0.58, QSD = 0.0425, YW = 1.8, YMU = 0.04, YA = 0.55, YSD = 0.155, DD = 0.13, DW = 0.58, DMU = 0.016, DY = −0.175, DB = 0.57, DSD = 0.07, CD = 0.045, CMU = 0.0305, CA = 0.9, CY = 0.34, CSD = 0.185, BA = 0.74, BMU = −0.23, BSD = 0.18.

6.1.3 Because a short-term interest rate, B(t), is modelled, it is possible to allow for the effects of a yield curve in the calculation of reserves for GAOs. How this is done is described in Appendix B. The values of B(t) and the 'consols' yield, C(t) are used, together with a parameter, β, which is explained in Appendix B. It is given the value 0.39 throughout our calculations. Note that Yang (2001) used a value of 0.5.

6.1.4 In fact the use of a yield curve, although theoretically preferable, makes rather little difference to the values of annuities at age 65 (though it would affect the values of annuities at older ages more). We have calculated the 'at-the-money' rates for an annuity at age 65, in the same way as in Sections 4.3 and 5.1, but allowing for the current value of the base rate, B(t) and a value of β of 0.39, for all terms, all mortality tables, and all December entry dates from December 1984. The differences from using the consols yield throughout may be in either direction, depending on whether short-term interest rates are higher or lower than long-term ones, but for the values calculated the differences are never more than £1.50 per £1,000 purchase price different from those shown in Tables 4.3.1 and 5.1.4a and b. Since 1990 they have never been more than £1.00 per £1,000, and the differences are less for the more recent, lighter, mortality tables. Of course, a different yield curve model might make more difference.

6.1.5 We have calculated results using the 1995 model for PMA80, PMA92 and RMC92 mortality, allowing for initial conditions as at 31 December 1994 and later years. These are discussed in Sections 6.2, 6.3 and 6.4,

6.2 *Results Using the 1995 Wilkie Model and PMA80 Mortality*

6.2.1 Tables 6.2k to 6.2r show results on the same lines as before using the 1995 investment model, PMA80 mortality, and initial conditions as at 31 December for 1994 to 2001 inclusive. Careful comparison of corresponding tables shows that the new model sometimes produces higher values, sometimes lower.

Table 6.2k. Present value of cost of GAO per £100 single premium: 1995 Wilkie model, initial conditions of 31 December 1994, mortality PMA80U1995.

Term	NZ%	Mean	Q_{95}	$Q_{97.5}$	Q_{99}	$Q_{99.9}$	$C_{99,1}$	$C_{99,2}$	$C_{99.9,1}$	$C_{99.9,2}$
10	51.69	3.84	15.73	18.87	23.31	30.60	5.59	7.04	6.24	8.21
20	57.31	5.61	21.57	25.59	30.58	43.83	9.70	12.55	11.81	16.09
30	60.75	6.79	24.69	29.87	35.63	50.86	13.27	17.23	16.55	22.43
40	62.52	7.44	25.89	30.53	36.92	54.77	15.61	20.05	20.30	27.19
Portfolio	95.77	6.06	17.71	21.03	24.97	33.58	9.78	12.21	11.42	14.90
Sum	63.17	6.06	22.29	26.71	32.40	45.07	11.27	14.56	13.73	18.50

Table 6.2l. Present value of cost of GAO per £100 single premium: 1995 Wilkie model, initial conditions of 31 December 1995, mortality PMA80U1996.

Term	NZ%	Mean	Q_{95}	$Q_{97.5}$	Q_{99}	$Q_{99.9}$	$C_{99,1}$	$C_{99,2}$	$C_{99.9,1}$	$C_{99.9,2}$
10	60.52	4.96	17.64	20.96	25.76	33.17	6.83	8.37	7.49	9.56
20	60.73	6.16	22.60	26.69	31.78	45.47	10.35	13.28	12.48	16.84
30	62.03	7.06	25.18	30.38	36.21	51.50	13.61	17.61	16.91	22.83
40	63.20	7.59	26.16	30.82	37.24	55.13	15.80	20.26	20.50	27.42
Portfolio	96.68	6.53	18.54	21.96	25.84	34.72	10.32	12.80	12.01	15.58
Sum	63.82	6.53	23.16	27.61	33.36	46.12	11.82	15.16	14.29	19.12

Table 6.2m. Present value of cost of GAO per £100 single premium: 1995 Wilkie model, initial conditions of 31 December 1996, mortality PMA80U1997.

Term	NZ%	Mean	Q_{95}	$Q_{97.5}$	Q_{99}	$Q_{99.9}$	$C_{99,1}$	$C_{99,2}$	$C_{99.9,1}$	$C_{99.9,2}$
10	62.49	5.38	18.55	21.89	26.82	34.32	7.31	8.89	7.98	10.10
20	62.04	6.48	23.29	27.42	32.54	46.12	10.74	13.72	12.90	17.33
30	62.96	7.29	25.67	30.92	36.79	52.23	13.92	17.96	17.24	23.23
40	63.73	7.44	26.49	31.17	37.63	55.62	16.01	20.52	20.74	27.71
Portfolio	97.03	6.81	19.05	22.50	26.44	35.45	10.66	13.19	12.39	16.00
Sum	64.54	6.81	23.75	28.24	34.06	46.97	12.17	15.56	14.66	19.55

Table 6.2n. Present value of cost of GAO per £100 single premium: 1995 Wilkie model, initial conditions of 31 December 1997, mortality PMA80U1998.

Term	NZ%	Mean	Q_{95}	$Q_{97.5}$	Q_{99}	$Q_{99.9}$	$C_{99,1}$	$C_{99,2}$	$C_{99.9,1}$	$C_{99.9,2}$
10	81.82	8.84	23.07	26.37	31.03	38.11	10.83	12.47	11.46	13.61
20	68.95	7.79	25.26	29.55	34.57	48.23	12.16	15.22	14.33	18.85
30	65.51	7.80	26.49	31.78	37.67	52.96	14.51	18.59	17.80	23.81
40	64.77	7.94	26.80	31.48	37.99	55.93	16.26	20.78	20.97	27.95
Portfolio	98.39	7.95	20.87	24.28	28.28	37.76	11.94	14.55	13.75	17.51
Sum	81.82	7.95	25.47	29.96	35.80	48.75	13.41	16.86	15.91	20.87

Table 6.2o. Present value of cost of GAO per £100 single premium: 1995 Wilkie model, initial conditions of 31 December 1998, mortality PMA80U1999.

Term	NZ%	Mean	Q_{95}	$Q_{97.5}$	Q_{99}	$Q_{99.9}$	$C_{99,1}$	$C_{99,2}$	$C_{99.9,1}$	$C_{99.9,2}$
10	96.73	17.63	34.63	38.56	43.51	53.20	19.95	21.86	20.81	23.40
20	82.04	11.99	32.15	36.83	42.65	57.02	16.99	20.47	19.24	24.24
30	73.26	10.06	30.60	36.31	42.52	58.66	17.33	21.75	20.78	27.21
40	69.48	9.26	29.29	34.19	41.00	59.50	18.03	22.79	22.86	30.14
Portfolio	99.79	11.67	26.48	30.15	34.56	45.11	16.15	19.08	18.15	22.34
Sum	96.73	11.67	31.36	36.25	42.58	56.63	17.66	21.45	20.33	25.72

Table 6.2p. Present value of cost of GAO per £100 single premium: 1995 Wilkie model, initial conditions of 31 December 1999, mortality PMA80U2000.

Term	NZ%	Mean	Q_{95}	$Q_{97.5}$	Q_{99}	$Q_{99.9}$	$C_{99,1}$	$C_{99,2}$	$C_{99.9,1}$	$C_{99.9,2}$
10	96.06	16.46	33.09	36.92	41.78	51.27	18.74	20.60	19.58	22.11
20	80.80	11.41	31.23	35.84	41.59	55.74	16.33	19.76	18.57	23.48
30	72.23	9.74	30.03	35.69	41.86	57.84	16.93	21.31	20.36	26.72
40	68.67	9.07	28.93	33.81	40.59	58.99	17.78	22.51	22.59	29.82
Portfolio	99.67	11.16	25.74	29.36	33.69	44.16	15.57	18.45	17.56	21.69
Sum	96.06	11.16	30.56	35.39	41.66	55.54	17.07	20.82	19.72	25.05

Table 6.2q. Present value of cost of GAO per £100 single premium: 1995 Wilkie model, initial conditions of 31 December 2000, mortality PMA80U2001.

Term	NZ%	Mean	Q_{95}	$Q_{97.5}$	Q_{99}	$Q_{99.9}$	$C_{99,1}$	$C_{99,2}$	$C_{99.9,1}$	$C_{99.9,2}$
10	96.67	17.49	34.45	38.38	43.32	53.00	19.81	21.71	20.66	23.25
20	81.85	11.92	32.06	36.74	42.56	56.92	16.92	20.39	19.18	24.17
30	73.11	10.02	30.55	36.26	42.47	58.60	17.29	21.71	20.74	27.17
40	69.29	9.24	29.26	34.16	40.97	59.47	18.00	22.77	22.84	30.11
Portfolio	99.79	11.61	26.41	30.07	34.48	45.03	16.09	19.01	18.09	22.27
Sum	96.67	11.61	31.28	36.17	42.50	56.54	17.59	21.38	20.26	25.66

Table 6.2r. Present value of cost of GAO per £100 single premium: 1995 Wilkie model, initial conditions of 31 December 2001, mortality PMA80U2002.

Term	NZ%	Mean	Q_{95}	$Q_{97.5}$	Q_{99}	$Q_{99.9}$	$C_{99,1}$	$C_{99,2}$	$C_{99.9,1}$	$C_{99.9,2}$
10	95.89	16.35	32.94	36.78	41.63	51.10	18.62	20.49	19.46	21.99
20	80.63	11.35	31.15	35.76	41.51	55.66	16.27	19.69	18.50	23.42
30	72.09	9.71	29.98	35.65	41.82	57.80	16.90	21.28	20.32	26.69
40	68.61	9.05	28.91	33.78	40.56	58.96	17.76	22.49	22.57	29.80
Portfolio	99.66	11.11	25.67	26.29	33.62	44.09	15.52	18.39	17.51	21.64
Sum	95.89	11.11	30.49	35.33	41.59	55.47	17.02	20.76	19.67	25.00

6.3 *Results Using the 1995 Wilkie Model and PMA92 Mortality*

6.3.1 Tables 6.3o to 6.2r show results, again on the same lines as before, using the 1995 investment model, PMA92 mortality, and initial conditions as at 31 December for 1998 to 2001 inclusive. Comparison of corresponding tables shows that in general the new model produces slightly lower values for these years than the 1984 model does.

Table 6.3o. Present value of cost of GAO per £100 single premium: 1995 Wilkie model, initial conditions of 31 December 1998, mortality PMA92U1999.

Term	NZ%	Mean	Q_{95}	$Q_{97.5}$	Q_{99}	$Q_{99.9}$	$C_{99,1}$	$C_{99,2}$	$C_{99.9,1}$	$C_{99.9,2}$
10	99.43	28.75	50.07	55.03	61.46	73.70	31.68	34.06	32.75	35.99
20	93.71	22.64	49.73	56.01	63.69	83.17	29.28	33.86	32.29	38.83
30	89.56	20.68	49.65	57.35	65.87	88.48	30.68	36.70	35.37	44.01
40	87.68	20.14	49.34	56.13	65.77	91.98	32.56	39.23	39.15	49.08
Portfolio	99.99	22.42	42.95	47.93	53.72	68.10	28.50	32.43	31.18	36.75
Sum	99.43	22.42	49.46	56.04	64.61	83.90	30.58	35.68	34.18	41.36

Table 6.3p. Present value of cost of GAO per £100 single premium: 1995 Wilkie model, initial conditions of 31 December 1999, mortality PMA92U2000.

Term	NZ%	Mean	Q_{95}	$Q_{97.5}$	Q_{99}	$Q_{99.9}$	$C_{99,1}$	$C_{99,2}$	$C_{99.9,1}$	$C_{99.9,2}$
10	99.31	27.45	48.37	53.23	59.51	71.53	30.32	32.66	31.37	34.55
20	93.19	21.94	48.70	54.90	62.50	81.72	28.50	33.03	31.48	37.94
30	89.16	20.28	49.02	56.68	65.14	87.58	30.21	36.19	34.87	43.45
40	87.41	19.91	48.97	55.74	65.34	91.44	32.27	38.91	38.84	48.74
Portfolio	99.99	21.81	42.13	47.12	52.86	67.05	27.84	31.75	30.49	36.01
Sum	99.31	21.81	48.58	55.10	63.58	82.68	29.91	34.96	33.47	40.59

Table 6.3q. Present value of cost of GAO per £100 single premium: 1995 Wilkie model, initial conditions of 31 December 2000, mortality PMA92U2001.

Term	NZ%	Mean	Q_{95}	$Q_{97.5}$	Q_{99}	$Q_{99.9}$	$C_{99,1}$	$C_{99,2}$	$C_{99.9,1}$	$C_{99.9,2}$
10	99.43	28.92	50.35	55.33	61.78	74.09	31.87	34.27	32.95	36.20
20	93.78	22.80	50.00	56.32	64.03	83.60	29.47	34.06	32.49	39.05
30	89.67	20.81	49.89	57.63	66.18	88.89	30.85	36.89	35.56	44.23
40	87.78	20.25	49.54	56.35	66.03	92.34	32.71	39.40	39.31	49.28
Portfolio	99.99	22.57	43.17	48.17	53.99	68.44	28.67	32.62	31.35	36.95
Sum	99.43	22.57	49.71	56.32	64.93	84.39	30.76	35.87	34.37	41.57

Table 6.3r. Present value of cost of GAO per £100 single premium: 1995 Wilkie model, initial conditions of 31 December 2001, mortality PMA92U2002.

Term	NZ%	Mean	Q_{95}	$Q_{97.5}$	Q_{99}	$Q_{99.9}$	$C_{99,1}$	$C_{99,2}$	$C_{99.9,1}$	$C_{99.9,2}$
10	99.34	27.64	48.66	53.54	59.85	71.92	30.53	32.88	31.59	34.78
20	93.24	22.10	48.98	55.20	62.84	82.16	28.69	33.23	31.68	38.17
30	89.25	20.41	49.27	56.96	65.45	87.99	30.38	36.38	35.06	43.67
40	87.48	20.01	49.17	55.96	65.59	91.79	32.42	39.08	39.00	48.93
Portfolio	99.99	21.96	42.36	47.37	53.15	67.39	28.02	31.94	30.67	36.22
Sum	99.34	21.96	48.84	55.38	63.90	83.08	30.08	35.15	33.66	40.81

6.4 *Results Using the 1995 Wilkie Model and RMC92 Mortality*

6.4.1 Tables 6.4o to 6.4r show results, again on the same lines as before, using the 1995 investment model, RMC92 mortality, and initial conditions as at 31 December for 1998 to 2001 inclusive. Comparison of corresponding tables shows that, for this mortality table too, the new model produces lower values for these years than the 1984 model does.

Table 6.4o. Present value of cost of GAO per £100 single premium: 1995 Wilkie model, initial conditions of 31 December 1998, mortality RMC92U1999.

Term	NZ%	Mean	Q_{95}	$Q_{97.5}$	Q_{99}	$Q_{99.9}$	$C_{99,1}$	$C_{99,2}$	$C_{99.9,1}$	$C_{99.9,2}$
10	99.51	30.50	52.77	57.98	64.83	77.77	33.57	36.07	34.70	38.10
20	94.49	23.97	52.03	58.61	66.67	87.28	30.87	35.62	34.05	40.85
30	90.56	21.95	51.96	60.03	68.96	92.93	32.33	38.57	37.28	46.27
40	88.85	21.43	51.77	58.85	69.07	96.81	34.37	41.30	41.29	51.64
Portfolio	99.99	23.78	45.05	50.31	56.40	71.47	30.11	34.19	32.90	38.70
Sum	99.51	23.78	51.83	58.72	67.73	88.16	32.27	37.56	36.05	43.53

Table 6.4p. Present value of cost of GAO per £100 single premium: 1995 Wilkie model, initial conditions of 31 December 1999, mortality RMC92U2000.

Term	NZ%	Mean	Q_{95}	$Q_{97.5}$	Q_{99}	$Q_{99.9}$	$C_{99,1}$	$C_{99,2}$	$C_{99.9,1}$	$C_{99.9,2}$
10	99.43	29.15	50.99	56.10	62.78	75.47	32.16	34.61	33.27	36.60
20	94.00	23.26	50.98	57.48	65.44	85.78	30.08	34.77	33.21	39.94
30	90.26	21.54	51.34	59.35	68.21	92.00	31.86	38.05	36.77	45.70
40	88.59	21.19	51.41	58.45	68.63	96.26	34.08	40.99	40.98	51.29
Portfolio	99.99	23.16	44.22	49.47	55.54	70.34	29.44	33.50	32.18	37.93
Sum	99.43	23.16	50.93	57.76	66.67	86.89	31.57	36.82	35.33	42.75

Table 6.4q. Present value of cost of GAO per £100 single premium: 1995 Wilkie model, initial conditions of 31 December 2000, mortality RMC92U2001.

Term	NZ%	Mean	Q_{95}	$Q_{97.5}$	Q_{99}	$Q_{99.9}$	$C_{99,1}$	$C_{99,2}$	$C_{99.9,1}$	$C_{99.9,2}$
10	99.51	30.70	53.09	58.32	65.21	78.22	33.79	36.30	34.93	38.34
20	94.56	24.16	52.36	58.98	67.08	87.82	31.09	35.86	34.29	41.12
30	90.59	22.10	52.62	60.36	69.33	93.42	32.54	38.80	37.51	46.53
40	88.91	21.56	52.01	59.12	69.39	97.24	34.54	41.50	41.49	51.88
Portfolio	99.99	23.95	45.31	50.61	56.71	71.89	30.30	34.41	33.11	38.94
Sum	99.51	23.95	52.13	59.07	68.12	88.65	32.47	37.79	36.28	43.79

Table 6.4r. Present value of cost of GAO per £100 single premium: 1995 Wilkie model, initial conditions of 31 December 2001, mortality RMC92U2002.

Term	NZ%	Mean	Q_{95}	$Q_{97.5}$	Q_{99}	$Q_{99.9}$	$C_{99,1}$	$C_{99,2}$	$C_{99.9,1}$	$C_{99.9,2}$
10	99.44	29.37	51.32	56.45	63.18	75.92	32.40	34.86	33.52	36.86
20	94.02	23.44	51.31	57.85	65.86	86.32	30.30	35.02	33.45	40.21
30	90.34	21.69	51.63	59.68	68.59	92.49	32.06	38.28	36.99	45.96
40	88.67	21.32	51.65	58.72	68.94	96.68	34.25	41.18	41.18	51.52
Portfolio	99.99	23.33	44.49	49.77	55.86	70.74	29.64	33.71	32.40	38.17
Sum	99.44	23.33	51.23	58.10	67.06	87.38	31.78	37.05	35.55	43.00

6.5 *Comparison of the 1984 and 1995 Models*

6.5.1 The 1984 and 1995 models do not produce very different results. In general, but not always, the means under the 1995 model are higher. This could result from a lower mean rate of inflation, QMU being 0.047 instead of 0.05, and a lower mean real consols yield, CMU being 3.05% instead of 3.5%. But the standard deviations, and hence the quantiles, are generally rather lower under the 1995 model. The values of C, the premium that the policyholder might be charged, may be either higher or lower.

6.5.2 Figure 6.1 shows values of Q_{99} for the Portfolio, for different mortality tables, for both the 1984 and 1995 models and of 100/consols yield %. It can be compared with Figure 5.4. In most cases the quantile is lower under the more recent model.

6.6 *An ARCH Model*

6.6.1 In his 1995 paper, Wilkie also described a possible ARCH (autoregressive conditional heteroskedastic) model, in which the standard

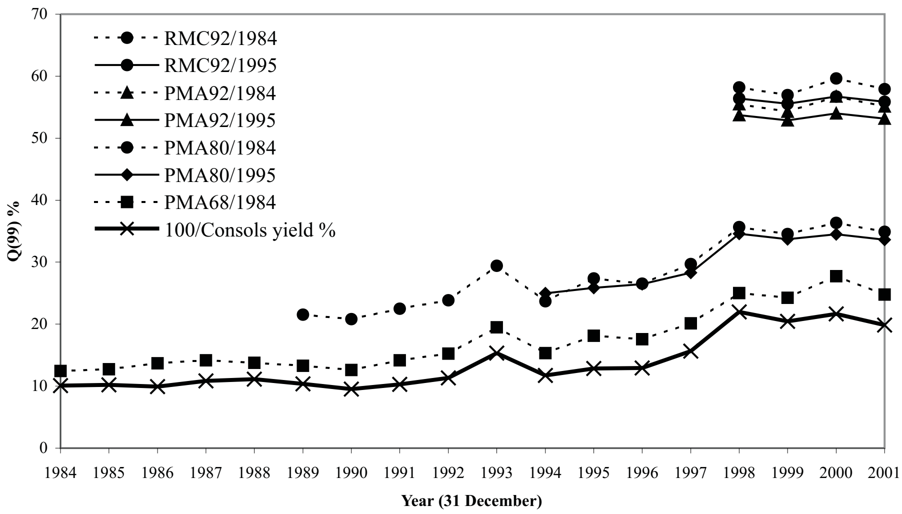

Figure 6.1. Values of Q_{99} for the portfolio, 1984 and 1995 Wilkie models, different mortality tables and of 100/consols yield %.

deviation of the rate of inflation varied with time. The formulae for inflation are now:

$$\mathrm{QSD}(t)^2 = \mathrm{QSA}^2 + \mathrm{QSB}.(\mathrm{I}(t-1) - \mathrm{QSC})^2$$
$$\mathrm{QE}(t) = \mathrm{QSD}(t).\mathrm{QZ}(t)$$
$$\mathrm{I}(t) = \mathrm{QMU} + \mathrm{QA}.(\mathrm{I}(t-1) - \mathrm{QMU}) + \mathrm{QE}(t)$$
$$\mathrm{Q}(t) = \mathrm{Q}(t-1).\exp(\mathrm{I}(t))$$

with parameters: QMU = 0.04, QA = 0.62, QSA = 0.0256, QSB = 0.55 and QSC = 0.04. Because the value of QSD(t) can become extremely large, it is convenient to limit it to 2.0 (which is a very high limit for a lognormal distribution).

6.6.2 We have used this model only with initial conditions as at 31 December 2001 and with PMA92U2002 mortality. The results are shown in Table 6.6r. The results show much higher quantiles than for the non-ARCH model, but with a tiny respite at the other extreme of the distribution: there are slightly more simulations where the guarantee does not 'bite'.

6.6.3 Because inflation is now very variable, the problem described in ¶2.4.5, with negative inflation and the value of C(t) needing to be limited to 0.05 and the problem described in Appendix B of inconsistent values for the zero-coupon discount factors (which in practice are set to zero), occur

Table 6.6r. Present value of cost of GAO per £100 single premium: 1995 Wilkie ARCH model, initial conditions of 31 December 2001, mortality PMA92U2002.

Term	NZ%	Mean	Q_{95}	$Q_{97.5}$	Q_{99}	$Q_{99.9}$	$C_{99,1}$	$C_{99,2}$	$C_{99.9,1}$	$C_{99.9,2}$
10	98.95	30.28	51.46	60.99	81.61	111.76	34.84	38.51	37.42	43.04
20	94.68	26.26	54.28	68.24	96.31	118.11	37.36	44.80	40.58	49.98
30	91.72	25.52	58.56	71.86	109.10	123.56	43.36	53.68	46.17	57.90
40	90.40	25.65	59.81	73.26	104.32	127.72	46.31	57.03	51.77	64.95
Portfolio	99.92	26.44	49.75	59.71	77.15	107.80	36.12	42.25	41.52	50.75
Sum	98.95	26.44	56.16	69.05	97.37	120.52	39.95	48.11	43.68	53.93

frequently. This does not seem to affect the results adversely. Although a model with such features is uncomfortable, the assumption in the Wilkie model (as implemented here) of normally distributed innovations is also at variance with observation, and some model with a fatter-tailed distribution for the innovations would be desirable. The example shown here gives some indication of the possible effects.

7. Reserving Since 1985

7.1 *Introduction*

7.1.1 In this Section we consider how the individual policies and the whole portfolio that we assumed to have entered in 1985 would have turned out since that date. Thus we consider the same set of policies as in Section 2, namely one each for terms of 10 to 40 years, plus the portfolio consisting of one each of these policies, but the total divided by 31. If the policies were all new in 1985, the first one would have reached age 65 in 1995, with one new policy reaching the vesting date in each year thereafter. We assume that all policies were written on 1 January 1985, and reach age 65 on 1 January in subsequent years.

7.1.2 We consider both marking-to-market each year, at a 99% level. We also consider the MGWP method, of starting with contingency reserves at a 99% level, recalculating each year, strengthening back to the 99% level if the available reserves are below the 98% level, and releasing reserves, reverting to 99%, if the available reserves exceed the 99.9% level. We first investigate what would have happened if the same mortality basis, namely PMA68U, and simulation basis, namely the 1984 Wilkie model, had been used throughout. Then we consider the effect of changes in the basis from time to time.

7.1.3 It is convenient to work entirely in term of 'units' of the invested funds, treating the 'share' price as a *numeraire*. The policyholder's funds therefore remain at 100 units throughout, and since we assume that the contingency reserves are invested in the same units, their value remains unchanged from year to year. This simplifies the calculations greatly. Thus a 10-year policy, written in 1985 for a male then aged 55, has, by its anniversary in 1986, become a 9-year policy for a male aged 56. The required contingency reserves can therefore be assessed as if it were a new 9-year policy in 1986, but we then need to adjust for the fact that some policyholders have died during the year, and their policies can no longer claim the benefit of the GAO. Over a small number of years the effect of this is not large, and, for some of the results presented, we have, for simplicity, ignored it.

7.2 *The 1985 Basis*

7.2.1 Tables 7.2.1a and 7.2.1b show the experience for a single 10-year policy, assuming PMA68B1930 mortality throughout. The left-hand section of Table 7.2.1a shows the quantile reserves, on a per policy basis. Sometimes the quantile reserve was zero; if so, we show the mean, putting the figure into italic type; sometimes the mean was less than 0.005, and it is shown as *0.00*. The next columns show selected quantile reserves reduced by survival factors, ${}_tp_{55}$. The required reserve each year shown in the column headed ${}_tp_{55} \times Q_{99}$. One can see that the initial reserve was 7.84 units per policy of 100 units invested. We do not need to consider the charge to the policyholder for the GAO here; some of the 7.84 units should initially have been provided by the policyholder, some by the shareholders; but thereafter profits and losses fall to the shareholders.

7.2.2 In the next column of Table 7.2.1a we show the change in the reserve from year to year. In many years there is a small release of reserves.

Table 7.2.1a. Experience for 10-year policy written in 1985, in units, basis: PMA68U, 1984 model, with marking to market.

Year 1 Jan	$Q_{97.5}$	Q_{98}	Q_{99}	$Q_{99.9}$	${}_tp_{55} \times Q_{98}$	${}_tp_{55} \times Q_{99}$	${}_tp_{55} \times Q_{99.9}$	Change in 99% reserve	Consols yield %
1985	3.91	4.89	7.84	13.69	4.89	7.84	13.69		9.90
1986	3.23	4.12	7.04	13.45	4.08	6.97	13.32	−0.87	9.80
1987	4.32	4.97	7.15	14.63	4.87	7.00	14.33	0.03	10.06
1988	4.08	4.87	6.82	12.90	4.71	6.59	12.47	−0.41	9.21
1989	2.13	2.92	4.85	10.27	2.79	4.63	9.80	−1.97	8.99
1990	*0.04*	*0.04*	1.29	5.76	0.04	1.21	5.41	−3.41	9.66
1991	*0.00*	*0.00*	*0.00*	1.08	0.00	0.00	1.00	−1.21	10.48
1992	*0.00*	*0.00*	*0.00*	2.27	0.00	0.00	2.06	0.00	9.71
1993	*0.00*	*0.00*	*0.00*	2.10	0.00	0.00	1.87	0.00	8.83
1994	8.36	8.52	9.02	10.15	7.43	7.87	8.86	7.87	6.52
1995	Payoff:		0.00			0.00		−7.87	8.53

Table 7.2.1b. Experience for 10-year policy written in 1985, in units, basis: PMA68U, 1984 model, MGWP method.

Year 1 Jan	${}_tp_{55} \times Q_{98}$	${}_tp_{55} \times Q_{99}$	${}_tp_{55} \times Q_{99.9}$	MGWP reserve	Change in MGWP reserve
1985	4.89	7.84	13.69	7.84	
1986	4.08	6.97	13.32	7.84	0.00
1987	4.87	7.00	14.33	7.84	0.00
1988	4.71	6.59	12.47	7.84	0.00
1989	2.79	4.63	9.80	7.84	0.00
1990	0.04	1.21	5.41	1.21	−6.63
1991	0.00	0.00	1.00	0.00	−1.21
1992	0.00	0.00	2.06	0.00	0.00
1993	0.00	0.00	1.87	0.00	0.00
1994	7.43	7.87	8.86	7.87	7.87
1995	Payoff:	0.00		0.00	−7.87

Indeed from 1991 to 1993 the 99% quantile reserves reduce to zero. However, by 1994 interest rates had fallen, and the reserves have to be increased considerably, to just over their original level. By maturity (retirement) on 1 January 1995 interest rates had risen again, so the option would have expired out of the money, and been worth actually zero, as shown in the last row. As an indication of interest rates, the value of the 'consols yield' (actually the yield on the *FT*-Actuaries irredeemables index) on 31 December of the preceding year is shown in the final column.

7.2.3 In Table 7.2.1b we continue the story, repeating the selected quantile reserves reduced by the survival factors. Then we show the reserve under the MGWP basis. This remains at the level of the previous year unless it outside the range of the 98% and 99.9% reserves, in either direction; if it is, it is altered back to the 99% reserve. Thus in 1990 there is a considerable release, followed by another small release in 1991; but in 1994 there is a big increase in the reserve, followed by the release of all of it the following year when the option expires out of the money. It can be seen that the profits or losses in total are the same under either method, but the marking-to-market method recognises them sooner than the MGWP method. On balance we believe that the marking-to-market method is more realistic, and it is more in conformity with modern accounting ideas. Neither method, however, adequately reserved for the fall in interest rates by 31 December 1993, though keeping the original contingency reserves unchanged would have done so.

7.2.4 Note that the tables show numbers of units, not pound amounts. The actual cash releases from or charges to reserves would have to be calculated by multiplying by the current unit price at each date, whatever that was.

Table 7.2.2. Value of GAO for contract maturing on 1 January of year shown; basis: PMA68U, 1984 model.

Year 1 Jan	Consols yield %	Option value %
1995	8.53	0
1996	7.78	0
1997	7.74	0
1998	6.39	5.13
1999	4.55	19.36
2000	4.89	16.66
2001	4.62	19.11
2002	5.04	15.77

7.2.5 The 10-year policy expired out of the money. Not all polices would have done so; indeed a 9-year policy would not have done so, though we do not show that. Table 7.2.2 shows the values of the option at expiry, on the PMA68 mortality basis at 1 January each year from 1995 to 2002, based on interest rates on 31 December of the preceding year, and using the 1984 Wilkie model basis, i.e. using only the long-term 'consols yield'. There is no allowance for the fact that not all policies would have survived. The option would have expired out of the money from 1995 to 1997.

7.2.7 In Tables 7.2.3a and 7.2.3b we show the experience for the 17-year policy, maturing on 1 January 2002, with a value, on the basis of PMA68U and the 1984 model, of 15.77%. The policyholder would have been aged 48 at entry, so quantiles are multiplied by ${}_tp_{48}$. With either reserving method the fall in interest rates by the end of 1993 would have required a big increase in the reserve, which could have been released the following year, and reversed the year after. The reserving in the last few years of the option's life is rather unstable. As it happens, the original 99% reserve would have met the final result quite neatly.

7.2.8 Other policies would have shown similar patterns. For those maturing on 1 January 1997 and 1998 the previous year's reserves would not have been enough, even at a 99.9% level. The successive reductions in interest rates surprised the model considerably. We discuss this further in Section 7.4. However, all other policies would have shown a release of reserves on expiry of the option, whether the option was in or out of the money.

7.2.9 In Table 7.2.4 we show the experience for the whole portfolio of 31 policies, one each for terms 10, 11, ... 40, and each for an amount of 100/31 units. For simplicity, we make no allowance for the benefits of survival. The left-hand columns show the quantile reserves for the surviving (i.e. not yet matured) policies at each date. Then the 'payoffs' column shows the cost of the option (for a policy of 100/31 units). We assume that this is charged to

Table 7.2.3a. Experience for 17-year policy written in 1985, in units, basis: PMA68U, 1984 model, with marking to market.

Year 1 Jan	$Q_{97.5}$	Q_{98}	Q_{99}	$Q_{99.9}$	${}_tp_{48} \times Q_{98}$	${}_tp_{48} \times Q_{99}$	${}_tp_{48} \times Q_{99.9}$	Change in 99% reserve	Consols yield %
1985	9.34	10.11	13.56	22.64	10.11	13.56	22.64		9.90
1986	8.91	9.81	12.90	22.98	9.77	12.85	22.90	−0.71	9.80
1987	9.61	10.81	13.98	23.45	10.73	13.87	23.27	1.02	10.06
1988	9.59	10.54	13.75	24.62	10.41	13.58	24.32	−0.29	9.21
1989	8.20	9.47	12.35	22.18	9.30	12.13	21.79	−1.45	8.99
1990	6.52	7.54	10.27	20.07	7.36	10.03	19.59	−2.11	9.66
1991	4.33	5.15	8.09	16.12	4.99	7.84	15.63	−2.18	10.48
1992	6.25	7.32	10.25	16.50	7.04	9.85	15.86	2.01	9.71
1993	7.33	8.43	11.52	18.52	8.03	10.97	17.64	1.12	8.83
1994	17.40	18.18	20.22	30.24	17.13	19.06	28.50	8.08	6.52
1995	3.76	4.57	6.86	12.80	4.26	6.39	11.92	−12.67	8.53
1996	12.12	12.91	14.98	21.22	11.86	13.76	19.50	7.38	7.78
1997	8.16	8.97	10.98	15.71	8.12	9.94	14.23	−3.82	7.74
1998	14.09	14.74	16.36	20.79	13.14	14.58	18.53	4.64	6.39
1999	25.08	25.48	26.87	30.90	22.33	23.55	27.08	8.96	4.55
2000	20.35	20.73	21.75	25.22	17.83	18.71	21.69	−4.84	4.89
2001	25.08	25.74	25.93	27.20	21.69	21.85	22.92	3.15	4.62
2002	Payoff:		15.77			13.00		−8.85	5.04

Table 7.2.3b. Experience for 17-year policy written in 1985, in units, basis: PMA68U, 1984 model, MGWP method.

Year 1 Jan	${}_tp_{48} \times Q_{98}$	${}_tp_{48} \times Q_{99}$	${}_tp_{48} \times Q_{99.9}$	MGWP reserve	Change in MGWP reserve
1985	10.11	13.56	22.64	13.56	
1986	9.77	12.85	22.90	13.56	0.00
1987	10.73	13.87	23.27	13.56	0.00
1988	10.41	13.58	24.32	13.56	0.00
1989	9.30	12.13	21.79	13.56	0.00
1990	7.36	10.03	19.59	13.56	0.00
1991	4.99	7.84	15.63	13.56	0.00
1992	7.04	9.85	15.86	13.56	0.00
1993	8.03	10.97	17.64	13.56	0.00
1994	17.13	19.06	28.50	19.06	5.50
1995	4.26	6.39	11.92	6.39	−12.67
1996	11.86	13.76	19.50	13.76	7.38
1997	8.12	9.94	14.23	13.76	0.00
1998	13.14	14.58	18.53	13.76	0.00
1999	22.33	23.55	27.08	23.55	9.78
2000	17.83	18.71	21.69	18.71	−4.84
2001	21.69	21.85	22.92	21.85	3.15
2002	Payoff:	13.00		13.00	−8.85

Table 7.2.4. Experience for portfolio written in 1985, in units, basis: PMA68U, 1984 model.

Year 1 Jan	$Q_{97.5}$	Q_{98}	Q_{99}	$Q_{99.9}$	Payoffs	Change in 99% reserve, with payoffs	MGWP reserve	Change in MGWP reserve with payoffs	Consols yield %
1985	9.27	10.00	12.43	20.95			12.43		9.90
1986	9.09	9.72	12.30	20.16		−0.13	12.43	0.00	9.80
1987	9.36	10.12	12.77	20.90		0.47	12.43	0.00	10.06
1988	9.37	10.08	12.63	20.64		−0.14	12.43	0.00	9.21
1989	8.64	9.36	11.94	19.22		−0.69	12.43	0.00	8.99
1990	7.83	8.48	10.81	17.96		−1.13	12.43	0.00	9.66
1991	7.02	7.71	9.83	16.55		−0.98	12.43	0.00	10.48
1992	7.86	8.37	10.80	18.32		0.97	12.43	0.00	9.71
1993	8.47	9.10	11.22	18.32		0.42	12.43	0.00	8.83
1994	14.04	14.83	17.33	23.70		6.11	17.33	4.90	6.52
1995	7.49	8.15	9.94	16.77	0.00	−7.39	9.94	−7.39	8.53
1996	10.86	11.54	13.53	19.93	0.00	3.59	13.53	3.59	7.78
1997	9.19	9.86	11.61	18.13	0.00	−1.92	13.53	0.00	7.74
1998	12.78	13.47	15.22	21.48	5.13	3.78	15.22	1.86	6.39
1999	18.98	19.58	21.69	28.78	19.36	7.09	21.69	7.09	4.55
2000	16.81	17.41	19.49	25.85	16.66	−1.66	21.15	0.00	4.89
2001	20.16	20.75	22.89	29.36	19.11	4.02	22.89	2.36	4.62
2002	15.97	16.55	18.56	23.93	15.77	−3.83	22.38	0.00	5.04

the contingency reserve. The corresponding release of reserves on expiry is allowed for by the absence of that policy in the quantile reserves. Then follow columns showing the change in the 99% quantile reserve, and the experience according to the MGWP method, in both cases after taking the required payoffs into account. It can be seen that the experience of the portfolio is much more stable than that of the individual policies, and although the contingency reserve has crept upwards as interest rates have fallen, it has not behaved unreasonably, except that perhaps the model was 'fooled' by the increase in interest rates by the end of 1994.

7.3 *Changes of Basis*

7.3.1 We now consider the effect of possible changes in the basis for calculating the reserves since 1985. We have already discussed four mortality bases: PMA68 from 1985, PMA80 starting in 1989/90 (i.e. 31 December 1989, 1 January 1990), and PMA92 and RMC92 starting in 1998/99. We have discussed two simulation bases: the 1984 basis from 1985 and the 1995 basis starting in 1994/95. If we assume that changes in the reserving basis were introduced at the earliest date we get the set of bases in Table 7.3.1, the last two being alternatives.

Table 7.3.1. Calculation bases, and effective dates.

Date from: 31Dec/1 Jan	Date to: 31Dec/1 Jan	Mortality	Simulation
1984/85	1989/90	PMA68	1984 model
1989/90	1994/95	PMA80	1984 model
1994/95	1998/99	PMA80	1995 model
1998/99	2001/02	PMA92	1995 model
1998/99	2001/02	RMC92	1995 model

7.3.2 The different bases give different annuity values at retirement, and thus different payoffs for the option, which are shown in Table 7.3.2. The difference between the 1995 and 1984 simulation models is in whether the yield curve is or is not allowed for, as this affects the annuity values for current annuities. Values are shown even for years prior to the introduction of the basis. It is clear, as we already knew, that the later mortality bases are 'stronger' than the earlier ones, and also that the effect of the simulation model on the payoff is small, and may be in either direction.

Table 7.3.2. Values on different bases of GAO for contracts maturing on 1 January of year shown.

Year 1 Jan	Consols yield %	PMA68 1984 model	PMA80 1995 model	PMA80 1995 model	PMA92 1995 model	RMC92 1995 model
1995	8.53	0	0	0	3.75	4.76
1996	7.78	0	2.68	2.95	9.10	10.36
1997	7.74	0	3.11	3.48	9.83	11.12
1998	6.39	5.13	12.79	12.57	20.49	22.37
1999	4.55	19.36	28.97	28.43	39.15	42.17
2000	4.89	16.66	25.93	25.74	36.16	38.95
2001	4.62	19.11	28.73	28.30	39.45	42.46
2002	5.04	15.77	24.93	25.25	35.92	38.62

7.3.3 In Tables 7.3.3a and 7.3.3b we show results for the 17-year policy, allowing for changes in basis. In any year when there is a change of basis two lines are shown with 'a' and 'b' after the year showing the values before and after the change, and hence the effect of the change. Mortality during the deferred period is allowed for, but for simplicity on the PMA68 basis throughout. As compared with the fixed basis in Table 7.2.1a the payoff (allowing for survival) was 13.00 and is now 29.61 on PMA92 (31.84 on RMC92), an increase of 16.61 (18.84). The two changes in the mortality basis increase the 99% reserve by 19.31 (23.33). The change in the simulation model adds 4.03. One cannot wholly separate the various effects of: changes

Table 7.3.3a. Experience for 17-year policy written in 1985, in units, bases changing as shown.

Year 1 Jan	Basis	$Q_{97.5}$	Q_{98}	Q_{99}	$Q_{99.9}$	${}_tp_{48} \times Q_{98}$	${}_tp_{48} \times Q_{99}$	${}_tp_{48} \times Q_{99.9}$	Change in 99% reserve
1985	PMA68/84	9.34	10.11	13.56	22.64	10.11	13.56	22.64	
1986	PMA68/84	8.91	9.81	12.90	22.98	9.77	12.85	22.90	−0.71
1987	PMA68/84	9.61	10.81	13.98	23.45	10.73	13.87	23.27	1.02
1988	PMA68/84	9.59	10.54	13.75	24.62	10.41	13.58	24.32	−0.29
1989	PMA68/84	8.20	9.47	12.35	22.18	9.30	12.13	21.79	−1.45
1990a	PMA68/84	6.52	7.54	10.27	20.07	7.36	10.03	19.59	−2.11
1990b	PMA80/84	14.36	15.62	18.95	30.95	15.25	18.50	30.22	8.47
1991	PMA80/84	11.72	12.71	16.29	26.08	12.32	15.79	25.28	−2.71
1992	PMA80/84	14.05	15.35	18.91	26.51	14.76	18.18	25.49	2.39
1993	PMA80/84	15.37	16.69	20.44	28.93	15.90	19.47	27.56	1.29
1994	PMA80/84	27.50	28.43	30.91	43.07	26.79	29.13	40.59	9.66
1995a	PMA80/84	11.09	12.06	14.79	21.90	11.23	13.77	20.39	−15.36
1995b	PMA80/95	16.08	16.58	19.12	21.22	15.44	17.80	19.76	4.03
1996	PMA80/95	17.40	18.07	20.55	26.86	16.60	18.88	24.68	1.08
1997	PMA80/95	17.17	17.93	20.13	25.56	16.24	18.23	23.15	−0.65
1998	PMA80/95	21.86	22.43	23.95	28.26	20.00	21.35	25.19	3.12
1999a	PMA80/95	35.82	36.19	37.72	42.11	31.71	33.05	36.90	11.70
1999b	PMA92/95	48.93	49.41	51.23	56.62	43.30	44.89	49.62	11.84
2000	PMA92/95	43.88	44.31	45.66	48.90	38.11	39.27	42.06	−5.62
2001	PMA92/95	43.84	44.06	44.87	46.84	37.13	37.82	39.48	−1.46
2002	PMA92/95	Payoff:		35.92			29.61		−8.20
or									
1999a	PMA80/95	35.82	36.19	37.72	42.11	31.71	33.05	36.90	
1999b	PMA92/95	52.22	52.75	54.64	60.47	46.22	47.88	52.99	14.83
2000	PMA92/95	46.95	47.42	48.78	52.28	40.79	41.96	44.97	−5.93
2001	PMA92/95	47.02	47.25	48.11	50.21	39.82	40.55	42.32	−1.41
2002	PMA92/95	Payoff:		38.62			31.84		−8.71

in interest rates, shortening of term, change in mortality basis, and change in simulation model, because they interact. A change in interest rates may have differing effects on different mortality bases, but it is clear that the improvement in mortality has had a very large effect on the results.

7.3.4 In Table 7.3.4 we show results for the portfolio, also allowing for changes in basis. The 99% contingency reserve was initially 12.43. Had the basis remained unchanged, the reserve would have been 18.56. Instead it has increased to 38.22 on PMA92 (40.30 on RMC92). The two changes in mortality basis have added explicitly $21.62 (= 7.55 + 14.07)$ on PMA92 $(23.96 = 7.55 + 16.41$ on RMC92). The change in simulation model added 2.05. Further, payoffs for expiring options have taken out 5.47 (5.85) from the reserve. Again, it is clear that the effect of improving mortality assumptions has probably been greater than the effect of interest rate changes over the period.

Table 7.3.3b. Experience for 17-year policy written in 1985, in units, bases changing as shown.

Year 1 Jan	Basis	${}_tp_{48} \times Q_{98}$	${}_tp_{48} \times Q_{99}$	${}_tp_{48} \times Q_{99.9}$	MGWP reserve	Change in MGWP reserve
1985	PMA68/84	10.11	13.56	22.64	13.56	
1986	PMA68/84	9.77	12.85	22.90	13.56	0.00
1987	PMA68/84	10.73	13.87	23.27	13.56	0.00
1988	PMA68/84	10.41	13.58	24.32	13.56	0.00
1989	PMA68/84	9.30	12.13	21.79	13.56	0.00
1990a	PMA68/84	7.36	10.03	19.59	13.56	0.00
1990b	PMA80/84	15.25	18.50	30.22	18.50	4.94
1991	PMA80/84	12.32	15.79	25.28	18.50	0.00
1992	PMA80/84	14.76	18.18	25.49	18.50	0.00
1993	PMA80/84	15.90	19.47	27.56	18.50	0.00
1994	PMA80/84	26.79	29.13	40.59	29.13	10.63
1995a	PMA80/84	11.23	13.77	20.39	13.77	−15.36
1995b	PMA80/95	15.44	17.80	19.76	17.80	4.03
1996	PMA80/95	16.60	18.88	24.68	17.80	0.00
1997	PMA80/95	16.24	18.23	23.15	17.80	0.00
1998	PMA80/95	20.00	21.35	25.19	21.35	3.55
1999a	PMA80/95	31.71	33.05	36.90	33.05	11.70
1999b	PMA92/95	43.30	44.89	49.62	44.89	11.84
2000	PMA92/95	38.11	39.27	42.06	39.27	−5.62
2001	PMA92/95	37.13	37.82	39.48	39.27	0.00
2002	PMA92/95	Payoff:	29.61		29.61	−9.66
or						
1999a	PMA80/95	31.71	33.05	36.90	33.05	
1999b	PMA92/95	46.22	47.88	52.99	47.88	14.83
2000	PMA92/95	40.79	41.96	44.97	41.96	−5.93
2001	PMA92/95	39.82	40.55	42.32	41.96	0.00
2002	PMA92/95	Payoff:	31.84		31.84	−10.12

7.4 *Further Observations*

7.4.1 It appears that the Wilkie model, especially in the 1984 version, leads to instability of reserves as the terms of policies shorten, and has tended to underestimate the reductions in interest rates that have actually taken place. There may be three reasons for this, which we now discuss.

7.4.2 First, the 1984 model includes a third order autoregressive model for the real consols yield, i.e. net of the effect of inflation. This means that changes in interest rates have an effect on the forecasts in the three subsequent years. This was intended to improve the forecasts, and indeed did so in the period over which the model was fitted, but it now appears to be not working so well. It is possibly an example of 'over-parameterisation', and the 1995 model reduced the consols yield model to a first order one, which seems to be more stable.

Table 7.3.4. Experience for portfolio written in 1985, in units, bases changing as shown.

Year 1 Jan	Basis	$Q_{97.5}$	Q_{98}	Q_{99}	$Q_{99.9}$	Payoffs	Change in 99% reserve, with payoffs	MGWP reserve	Change in MGWP reserve with payoffs
1985	PMA68/84	9.27	10.00	12.43	20.95			12.43	
1986	PMA68/84	9.09	9.72	12.30	20.16		−0.13	12.43	0.00
1987	PMA68/84	9.36	10.12	12.77	20.90		0.47	12.43	0.00
1988	PMA68/84	9.37	10.08	12.63	20.64		−0.14	12.43	0.00
1989	PMA68/84	8.64	9.36	11.94	19.22		−0.69	12.43	0.00
1990a	PMA68/84	7.83	8.48	10.81	17.96		−1.13	12.43	0.00
1990b	PMA80/84	14.24	15.15	18.36	27.17		7.55	18.36	5.93
1991	PMA80/84	12.89	13.73	16.78	25.24		−1.58	18.36	0.00
1992	PMA80/84	14.32	15.09	18.06	26.92		1.28	18.36	0.00
1993	PMA80/84	15.26	16.21	19.01	27.99		0.95	18.36	0.00
1994	PMA80/84	23.19	24.22	27.28	35.06		8.27	27.28	8.92
1995a	PMA80/84	13.34	14.37	16.72	25.74		−10.56	16.72	−10.56
1995b	PMA80/95	15.88	16.72	18.77	24.94	0.00	2.05	16.72	0.00
1996	PMA80/95	16.82	17.61	19.52	25.63	0.10	0.75	19.52	2.90
1997	PMA80/95	16.91	17.48	19.48	25.41	0.11	−0.04	19.41	0.00
1998	PMA80/95	19.62	20.19	22.04	27.44	0.41	2.56	22.04	3.04
1999a	PMA80/95	27.29	27.96	29.94	36.36		7.90	29.94	7.90
1999b	PMA92/95	40.53	41.44	44.02	52.46	1.26	14.07	44.02	15.34
2000	PMA92/95	37.78	38.70	41.12	48.61	1.17	−2.89	42.85	0.00
2001	PMA92/95	37.79	38.73	41.05	48.39	1.27	−0.07	41.58	0.00
2002	PMA92/95	35.17	35.90	38.22	45.41	1.16	−2.83	40.42	0.00
or									
1999a	PMA80/95	27.29	27.96	29.94	36.36			29.94	
1999b	PMA92/95	42.71	43.65	46.35	55.26	1.36	16.41	46.35	17.77
2000	PMA92/95	39.80	40.79	43.33	51.20	1.26	−3.02	45.09	0.00
2001	PMA92/95	39.87	40.85	43.29	51.05	1.37	−0.05	43.72	0.00
2002	PMA92/95	37.10	37.87	40.30	47.90	1.25	−2.98	42.48	0.00

7.4.3 Secondly, the real data has 'fatter-tailed' changes than would occur with a normal distribution, so larger changes occur more frequently than expected. This suggests that an allowance for fatter-tailed innovations should be included in the model, as we discuss further in Section 11.2.

7.4.4 Thirdly, the models pull the expected consols yield towards a mean value, which is equal to the mean rate of inflation plus the mean consols yield. The mean 'forces' of inflation (means of the logarithms of the changes) are 0.05 and 0.47 in the two models. The mean consols yields are 0.035 and 0.0305. Thus the central consols yield (not strictly the mean) is 8.5% in the 1984 model, and 7.75% in the 1995 model. As interest rates get well below these levels, the model expects them to move up a little, and underestimates the probabilities of further falls. We have used the parameters from Wilkie's

1986 and 1995 papers unchanged, since that involved no introduction of subjective judgement on our part. But in practice we would nowadays be much more likely to use a model with a mean rate of inflation of 2.5% or 3%. This of course would increase the required contingency reserves and the initial costs of GAOs even more. But it would perhaps be more realistic.

7.4.5 It is not obvious whether marking to market, the MGWP 'bands' method, or some other reserving method is best. On balance, we favour marking to market, as we have explained in ¶7.2.3. This is question that deserves further discussion.

8. Theoretical Developments Since 1985

8.1 *Introduction*

8.1.1 So far the theoretical model that we have used has been based on the Report of the Maturity Guarantees Working Party (Ford *et al.*, 1980) and other papers of that date, although we have updated the basis of calculation to take into account changes since then both in mortality rates and in the investment model that we have been using. We now consider some theoretical developments that have taken place since 1985.

8.1.2 In Section 8.2 we consider an alternative way of assessing policyholder premiums, instead of the method described in Section 2.3; this alternative has, we feel, certain theoretical advantages, though the numerical results are similar. In Section 9 we discuss the use of 'conditional tail expectations' instead of quantile reserves; the former have many theoretical and practical advantages over the latter. In Section 10 we introduce the very important topic of option pricing methodology. In Section 11 we discuss making allowance for the uncertainty of mortality projections.

8.2 *An Alternative Way of Assessing Premiums*

8.2.1 In Section 2.3 we described how policyholders might be charged for a guaranteed annuity option (or indeed any other benefit that would require significant contingency reserves). The method used the expected value of the benefit, A, and the desired quantile reserve, Q_α. The parameters to be chosen were α, the security level of the quantile reserve, and h, the excess return required by the shareholders on their investment. The resulting charge, B, was added to A to give a total charge of $C_{\alpha,h}$.

8.2.2 This method takes account of only two statistics of the total distribution of the cost of the options, A and Q_α. It ignores the rest of the distribution, including the shape of the distribution both below Q_α and above it. As discussed in ¶2.3.10, it is not, we feel, satisfactory to assume that limited liability will cut in as soon as the costs of guarantees exceed Q_α. The costs might fall on other reserves of the insurer, perhaps the 'free' assets or the estate, or on the bonus rate on all policies, or on assets of the parent company of the insurer, if there is one. It is therefore appropriate to consider the total distribution.

8.2.3 We look at this from the point of view of an investment by the 'shareholders'. We assume that a contingency reserve of Q_α has to be set up, and we assume, as before, that it is invested in the same funds as the policyholder's investment. At maturity, the invested amount should be sufficient (if all our assumptions are correct) to pay for the guarantees in a large fraction, α, of the outcomes. In some cases, perhaps many cases, the cost of the guarantee will prove to be nil, and the full value of the reserve will fall to the shareholders. In other cases the guarantee will cost less than the proceeds of Q_α and rather less will fall to the shareholders. In a small fraction, $1 - \alpha$, of cases the accumulated contingency reserves will prove insufficient and the guarantees will have to be financed by the shareholders from other funds. The contingency reserve can be seen as a risky investment of the shareholders, which will often have a positive return, but might have a negative one. But how much might shareholders be willing to pay for such an investment?

8.2.4 As we have discussed in Section 3.2, some might look at the '*beta*' of such an investment in relation to some market index; but we do not believe that investors will readily do this. Instead we believe that the methodology we now describe may be more acceptable. We assume that the shareholders are happy to receive whatever returns are available on the investment of the contingency reserve, Q_α, but that they would like some extra return on any positive amounts after the guarantee claims have been paid for, and would value negative returns adversely.

8.2.5 We use the notation of Section 2.3. If Q_α is invested till time T it will have accumulated to $Q_\alpha.P(T)/P(0)$. The claim amount is $\mathrm{Max}(0,S(T).(g.a(T)-1))$, which we denote as $G.P(T)/P(0)$. The shareholders receive $(Q_\alpha - G).P(T)/P(0)$ if this is positive, and pay $(G - Q_\alpha).P(T)/P(0)$, if the former expression is negative. We suggest that they discount any positive amounts at a rate per annum of j more than the normal return, and discount any negative amounts at a rate per annum of k less than the normal return. Thus they value their investment at:

$$(Q_\alpha - G).P(T)/P(0).P(0)/P(T).1/(1+j)^T = (Q_\alpha - G)(1+j)^T \quad \text{if } Q_\alpha > G$$
$$(Q_\alpha - G).P(T)/P(0).P(0)/P(T).1/(1-k)^T = (Q_\alpha - G)/(1-k)^T \quad \text{if } Q_\alpha < G$$

which can be expressed as:

$$\int_0^{Q_\alpha} (Q_\alpha - G)/(1+j)^T.p(G).dG + \int_{Q_\alpha}^{\infty} (Q_\alpha - G)/(1-k)^T.p(G).d(G,)$$

where $p(G)$ is the density function of G. This can be denoted as $V(Q_\alpha)$, and is a function, *inter alia*, of α, j and k. In practice this amount can be calculated during the simulations by discounting relevant positive amounts at rate j and relevant negative amounts at rate $-k$.

8.2.6 Having calculated the value to the shareholders of their investment, we now assume that they are willing to put an amount $V(Q_\alpha)$, leaving the policyholder to provide $Q_\alpha - V(Q_\alpha)$, which we shall denote $D_{\alpha,j,k}$. It can be compared with $C_{\alpha,h}$, noting that α has the same function, and that h and j have a similar function; k can also be considered to have a similar function to h, but in reverse, penalising deficits.

8.2.7 Just as we do not know what values of α and h investors would like, we do not know what values of α, j and k they might use. We quote figures in Tables 8.2a and 8.2b for $\alpha = 99\%$ and 99.9%, $j = 1\%$ and 2% and $k = 1\%$ and 2%. We also show the values of $C_{\alpha,h}$ previously calculated, for

Table 8.2a. Policyholder's premiums for GAO per £100 single premium: 1984 Wilkie model, initial conditions of 31 December 1984, mortality PMA68U1985.

Term	Q_{99}	$C_{99,1}$	$C_{99,2}$	$D_{99,1,1}$	$D_{99,1,2}$	$D_{99,2,1}$	$D_{99,2,2}$
10	7.84	0.94	1.51	0.98	0.98	1.62	1.63
20	15.36	3.36	5.03	3.60	3.61	5.71	5.73
30	21.08	6.12	8.82	6.76	6.79	10.45	10.48
40	23.69	8.28	11.55	9.38	9.44	14.08	14.13
	$Q_{99.9}$	$C_{99.9,1}$	$C_{99.9,2}$	$D_{99.9,1,1}$	$D_{99.9,1,2}$	$D_{99.9,2,1}$	$D_{99.9,2,2}$
10	13.69	1.47	2.47	1.53	1.53	2.67	2.67
20	26.87	5.22	8.18	5.66	5.66	9.45	9.45
30	35.39	9.27	13.84	10.42	10.43	16.81	16.82
40	41.29	12.99	18.76	15.11	15.11	23.64	23.64

Table 8.2b. Policyholder's premiums for GAO per £100 single premium: 1995 Wilkie model, initial conditions of 31 December 2001, mortality RMC92U2002.

Term	Q_{99}	$C_{99,1}$	$C_{99,2}$	$D_{99,1,1}$	$D_{99,1,2}$	$D_{99,2,1}$	$D_{99,2,2}$
10	63.18	32.40	34.86	32.59	32.59	35.46	35.47
20	65.86	30.30	35.02	31.13	31.15	37.36	37.38
30	68.79	32.06	38.28	33.86	33.91	42.78	42.83
40	68.94	34.25	41.18	37.04	37.13	47.49	47.57
	$Q_{99.9}$	$C_{99.9,1}$	$C_{99.9,2}$	$D_{99.9,1,1}$	$D_{99.9,1,2}$	$D_{99.9,2,1}$	$D_{99.9,2,2}$
10	75.92	33.52	36.86	33.78	33.78	37.74	37.74
20	86.32	33.45	40.21	34.78	33.79	44.01	44.01
30	92.49	36.99	45.96	39.97	39.97	53.41	53.41
40	96.68	41.18	51.52	46.07	46.07	62.55	62.56

$\alpha = 99\%$ and 99.9% and $h = 1\%$ and 2%. We show these for 1985 entrants, using initial conditions as at 31 December 1984, on the 1984 Wilkie model, with mortality PMA68U1985 (see Table 4.2a) and for 2002 entrants, using initial conditions as at 31 December 2001, on the 1995 Wilkie model, with mortality RMC92U2002 (see Table 6.4r). We omit the portfolio and the 'Sum'.

8.2.8 One can see from these tables that the values of D vary very little with the 'penalty' rate k, especially when $\alpha = 99.9\%$. The 'tail' of the distribution beyond the quantile level is necessarily small (1% or 0.1% of the total), and the tail is not very long, because the value of a life annuity is limited, with non-negative interest rates, to the expectation of life. In practice we have limited the long-term interest rates to a minimum of 0.5%, which constrains the possible tail even further.

8.2.9 One can also see that the values of D for particular values of α and j are not very different from the values of C for the same value of α and $h = j$. The principles involved in the calculations are comparable, even though the methodology is different. Thus the 'D' method might be just as satisfactory as the 'C' method (or even more so) for calculating the premiums that should be charged to policyholders for any risky contract that requires substantial contingency reserves.

9. Conditional Tail Expectations

9.1 *Introduction*

9.1.1 We observed in ¶3.2.1 that a quantile reserve was an alternative name for the concept also described as 'Value at Risk'. Value at Risk has, however, been criticised, for example by Artzner (1998), and by Wirch & Hardy (1999) for being 'incoherent'. We can, for example, observe from Table 2.5.1 that Q_{90} for term 10 is zero. This can occur for any risk that is sufficiently out of the money or is so unlikely that the less extreme quantiles are zero. Consider, as another example, a one-year term assurance on a younger life, for whom the probability of death within one year is small, possibly less than 0.001. The quantile reserve, even on a 99.9% basis, would be zero. Thus it is possible for a quantile reserve to be smaller than the mean value of the claim. This is unsatisfactory.

9.1.2 Another problem is that, when risks are combined into a portfolio, it is possible for the quantile for the portfolio to be greater than the sum of the corresponding quantiles for the individual risks. Consider a large portfolio of one-year term assurances: the quantile reserve on say a 99% basis would allow for a small number of deaths occurring. But the sum of the zero quantiles would still be zero. This inconsistency is less troublesome when the individual quantiles are greater than their respective means, and the results are positively, but less than perfectly, correlated (or are independent).

We have not in fact had any result where the quantile for the portfolio is less than the sum of the individual quantiles, but the possibility of inconsistency remains.

9.1.3 A way in which to produce a 'coherent' contingency reserve is to use 'tail VAR' or 'conditional tail expectation' (CTE). Just as the quantile at level α is defined as the value of any X (the risk) such that $P(X < Q_\alpha) = \alpha$, so the CTE, which we denote as T_α, is defined as: $E[X \mid X \geq Q_\alpha]$, that is, the expected value of all those claims greater than or equal to the corresponding quantile. It is easily calculated during our simulations. For 10,000 simulations we defined for example the 99% quantile as the value of $V0_{9901}$. We can calculate the CTE by taking the average of the 100 largest values of V0, from $V0_{9901}$ to $V0_{10000}$ inclusive. (This gave us an incentive to define e.g. Q_{99} as $V0_{9901}$.)

9.1.4 The value of the CTE can never be less than the mean (because Q_0 is itself equal to the mean, and $Q_\alpha \geq Q_\beta$ if $\alpha > \beta$), and it can be shown (Artzner, 1998) that, when risks are combined into a portfolio, the portfolio CTE cannot be greater than the sum of the individual CTEs. Further, it gives an easy way to partition the portfolio CTE amongst the individual contributors to the risk (see Panjer & Jing, 2001). If the claim for individual i is denoted X_i and the total claim is denoted $X = \Sigma X_i$ with a CTE at level α of T_α and corresponding quantile of Q_α, then the individual contribution to T_α can be taken as $T_{i,\alpha} = E[X_i \mid X \geq Q_\alpha]$. It follows that the individual contributions sum to the total, $T_\alpha = \Sigma T_{i,\alpha}$, and the whole system is coherent.

9.1.5 The CTE can be considered also as equal to the quantile, plus the expected value of the excess over the quantile, or $T_\alpha = Q_\alpha + E[X - Q_\alpha \mid X \geq Q_\alpha]$. The excess can be thought of as the pure 'stop loss' or 'excess of loss' premium. However, it would not normally be possible to obtain reassurance cover at such a premium, because the reassurer would need to set up contingency reserves to cover his risk in the same way as the primary insurer. In practice, therefore, the CTE just gives a level of security higher than the quantile reserve does, for the same value of α. The value of T_α for any chosen α is equal to Q_β for some $\beta > \alpha$. Therefore to hold a CTE reserve of T_α, and to take no further action, is equivalent to holding a quantile reserve at level β. This can be seen, for example, from the detailed figures for a 10 year term policy in 1984 using PMA68 mortality (as in Table 2.5.1). The quantile reserve on a 99% basis, Q_{99}, estimated by the value of $V0_{9901}$ is 7.84. The CTE, T_{99}, estimated as the average of the 100 largest values of V0, is 10.64. But we have $V0_{9963} = 10.63$ and $V0_{9964} = 10.76$, so a 99% CTE reserve of 10.64 is almost equivalent to a 99.63% quantile reserve.

9.2 *Results on a CTE Basis*

9.2.1 Table 2.5.1 showed certain quantile measures for policies entering in 1985, using the initial conditions of 31 December 1984 and PMA68U1985

Table 9.1a. Quantile reserves and CTEs for GAO per £100 single premium: 1984 Wilkie model, initial conditions of 31 December 1984, mortality PMA68U1985.

Term	NZ%	Mean	Q_{95}	$Q_{97.5}$	Q_{99}	$Q_{99.9}$	T_{95}	$T_{97.5}$	T_{99}	$T_{99.9}$
10	6.57	0.26	1.00	3.91	7.84	13.69	4.93	7.55	10.64	15.93
20	16.51	0.99	7.52	11.03	15.36	26.87	12.49	15.91	20.35	28.79
30	24.05	1.74	11.37	16.17	21.08	35.39	17.59	21.78	27.08	38.64
40	29.74	2.30	13.70	18.05	23.69	41.29	20.12	24.67	31.05	44.76
Portfolio	59.65	1.33	6.72	9.27	12.43	20.95	10.29	12.78	16.10	22.72
Sum	29.74	1.33	8.74	12.66	17.78	29.84	14.26	18.06	22.96	33.03

Table 9.1b. Quantile reserves and CTEs for GAO per £100 single premium: 1995 Wilkie model, initial conditions of 31 December 2001, mortality RMC92U2002.

Term	NZ%	Mean	Q_{95}	$Q_{97.5}$	Q_{99}	$Q_{99.9}$	T_{95}	$T_{97.5}$	T_{99}	$T_{99.9}$
10	99.44	29.37	51.32	56.45	63.18	75.92	58.44	63.28	69.14	79.29
20	94.02	23.44	51.31	57.85	65.86	86.32	60.25	66.42	74.08	89.05
30	90.34	21.69	51.63	59.68	68.59	92.49	62.36	69.51	78.72	98.94
40	88.67	21.32	51.65	58.72	68.94	96.68	62.22	69.65	79.94	104.22
Portfolio	99.99	23.33	44.49	49.77	55.86	70.74	51.56	56.23	62.21	73.28
Sum	99.44	23.33	51.23	58.10	67.06	87.38	60.85	67.41	75.82	93.34

mortality. Table 9.1a shows the same quantile reserves (though only for the terms shown in Table 4.2a), and the corresponding CTE reserves, for $\alpha = 95\%$, 97.5%, 99% and 99.1%. Table 9.2b shows the same results using the 1995 Wilkie model, initial conditions of 31 December 2001 and RMC92U mortality (see Table 6.4r)

9.2.2 It can be seen from these two examples that the CTE is always greater than the quantile reserve, that the increase is less, both absolutely and proportionately, for higher values of α (so that, e.g. $T_{99.9} - Q_{99.9} < T_{99} - Q_{99}$), and that the increase is absolutely greater but proportionately less in Table 9.2b than in Table 9.2a, i.e. when the values are generally higher.

9.2.3 To show how to calculate the individual contributions to the portfolio CTE we add a little notation. Put $V0_j$ as the cost of the guarantee for the portfolio for simulation j, after sorting, so that $V0_j > V0_{j-1}$. $V0_j$ has been calculated as the sum of the individual costs for terms 10 to 40 inclusive, which we denote as $V0_{j,i}$, and then divided by 31 (because there are 31 terms). We have already defined e.g. T_{99} for the portfolio as $T_{99} = \Sigma_{j=9901,10000} V0_j/100$. We now go through (sorted) simulations $j = 9{,}901$ to 10,000 and calculate, for each term, i, $U_{99,i} = \Sigma_{j=9901,10000} V0_{j,i}/100$. Note that these simulations are those with the 100 largest values of the portfolio cost, which are not necessarily the same as the 100 simulations with the largest values of $V0_{j,i}$, though it is likely that there is some overlap. Then $U_{99} = \Sigma_i U_{99,i}/31$.

9.2.4 We show the results of these calculations in Tables 9.2a and 9.2b, using the same extreme sets of data as in Tables 9.1a and 9.1b. We describe the values $U_{\alpha,i}$ as 'partial CTEs'. We can see that, in every case, $U_\alpha < T_\alpha$, as we would expect. But the reduction is not uniform, either absolutely or proportionately for all terms. Indeed the sequence of values becomes different. In both cases the values of T increase with term (though this is not true for all sets of data), while in both cases the value of U is largest (among the terms shown) for term 30. The average of the values of U for all 31 terms equals the portfolio value of T.

Table 9.2a. Partial CTEs for GAO per £100 single premium: 1984 Wilkie model, initial conditions of 31 December 1984, mortality PMA68U1985.

Term	T_{99}	U_{99}	$T_{99.9}$	$U_{99.9}$
10	10.64	2.13	15.93	4.24
20	20.35	12.26	28.79	19.19
30	27.08	23.73	38.64	29.09
40	31.05	20.87	44.76	27.81
Portfolio	16.10	16.10	22.72	22.72

Table 9.2b. Partial CTEs for GAO per £100 single premium: 1995 Wilkie model, initial conditions of 31 December 2001, mortality RMC92U2002.

Term	T_{99}	U_{99}	$T_{99.9}$	$U_{99.9}$
10	69.14	48.18	79.29	51.69
20	74.08	61.12	89.06	69.92
30	78.72	70.48	98.94	81.97
40	79.94	58.43	104.22	64.19
Portfolio	62.21	62.21	73.28	73.28

10. Option Pricing and Hedging

10.1 *Introduction*

10.1.1 So far we have described a system of reserving for extreme events by calculating and setting up static reserves, which are maintained, perhaps with adjustment, for the duration of the contract. However, another way of reserving for financial options, such as GAOs are in respect of the interest rate risk, is by modern option pricing and dynamic hedging. We now consider this possibility.

10.1.2 The analogy between GAOs and other types of financial option has been brought out by other authors, including Bolton *et al.* (1997), Van Bezooyen, Exley & Mehta (1998), Pelsser (2002) and Balotta & Haberman (2002). We discuss these in Section 10.5.

10.1.3 Yang (2001) also uses option pricing methodology. We follow her approach, which is to use, *inter alia*, the Black (1976) model, originally devised for options on commodity prices. The essence of the Black approach is that the price of the commodity is assumed to follow geometric Brownian motion, so that its value at any future date is lognormally distributed. However, GAOs are a type of 'quanto' option, so are more complicated. Further, we allow for the stochastic nature of interest rates before retirement (which Yang did not). A full description of our assumptions and methodology, showing how the formula for the option price and for the hedging proportions have been derived, is contained in Appendix C, which the interested reader may wish to read in full now. For those who just wish to know the results, we now give an outline of our assumptions, repeating to some extent the material in Appendix C.

10.2 *The Option Pricing Model*

10.2.1 We start by defining a new type of option, a *Maxi* option, which provides, at expiry at T, the greater in value of two assets, i.e. Max(A(T), B(T)). This in fact is what many insurance companies provide. We then show that a GAO, of the type we are considering, is a type of 'quanto' option, where the payoff at time T depends on two factors, the value of the 'units', S(T), which defines the quantity, and the relationship of the guaranteed rate to the market annuity value, so that the payoff is £S(T) $\times$ Max(1, $g \times a$(T)). Note that the Maxi option includes the basic value of the units, so the GAO now consists of the whole contract, not just the extra over and above the unit value.

10.2.2 We follow closely the methods of Baxter & Rennie (1996). We assume that there are three tradeable assets. The first is the unit fund, whose market price per unit at time t is S(t). Next we assume a zero-coupon bond, a '*zcb*', which pays 1 at time T, and prior to that has value B(t,T) or just B(t). We then assume that we can invest in a deferred life annuity, a '*dla*', whose value at time t is D(t,T) or D(t), and which provides a 'life annuity' at time T, of annual rate 1, of the required type. This is not an annuity on an individual life, but instead is an annuity certain, with the payments reducing proportionately to ${}_tp_{65}$ on some prescribed life table. We relate the price of the deferred annuity to the price of a forward life annuity, an '*fla*', whose price at time t, F(t,T) or F(t), is related by: D(t) = F(t) $\times$ B(t).

10.2.3 We assume that the prices of the tradeables are driven by three separate Brownian Motions, W_1, W_2 and W_3. W_i and W_j have instantaneous correlation ρ_{ij}. The W_is are related to three *independent* Brownian motions, Z_1, Z_2 and Z_3 through a matrix, C: dW = C.dZ. C can conveniently be made a lower triangular matrix.

10.2.4 The share price, $S(t)$, is driven by the stochastic differential equation:

$$dS(t) = \mu_S().S(t).dt + \sigma_S.S(t).dW_3,$$

where σ_S is a constant, and $\mu_S() = \mu_S$, so that the logarithm of the share price performs a random walk with constant drift.

10.2.5 The zcb price, $B(t)$, is driven by the zcb interest rate, $R(t)$, which has the stochastic differential equation:

$$dR(t) = \mu_R().dt + \sigma_R.dW_2,$$

where σ_R is a constant, and $\mu_R() = \alpha_R(\theta_R - R(t))$, so that $R(t)$ follows an Ornstein-Uhlenbeck process. The zcb price, $B(t)$ is related to the zcb interest rate, $R(t)$, by:

$$B(t) = \exp(-(T - t).R(t)).$$

10.2.6 The dla price, $D(t) = F(t) \times B(t)$, is driven by the zcb price, $B(t)$, and the fla price, $F(t)$. We assume that $F(t)$ has stochastic differential equation:

$$dF(t) = \mu_F().F(t).dt + \sigma_F.F(t).dW_1,$$

where σ_F is a constant, and $\mu_F() = \alpha_F(\theta_F - \log F(t)) + \frac{1}{2}\sigma_F^2$, implying an Ornstein-Uhlenbeck process for $\log F(t)$.

10.2.7 We then show that the price of the option at time t is:

$$V(t) = S(t).[G.N(d_1) + N(d_2)]$$

where

$$\begin{aligned}
d_1 &= \log(G)/\Sigma + \tfrac{1}{2}\Sigma \\
d_2 &= -\log(G)/\Sigma + \tfrac{1}{2}\Sigma \\
G &= g.F(t).\exp(\text{Cov}) \\
\text{Cov} &= (T - t)^2.\rho_{12}\sigma_R.\sigma_F/2 + (T - t).\rho_{13}.\sigma_S.\sigma_F
\end{aligned}$$

and

$$\Sigma = \sigma_F\sqrt{(T - t)}$$

We observe that $\mu_S()$, $\mu_R()$ and $\mu_F()$ do not enter this formula. This is similar to what is found in the Black-Scholes result for an ordinary option on a share. We note also that ρ_{23} does not come in either; any correlation between the zcb and the share does not affect the value of the option. However, ρ_{12} and ρ_{13} are very relevant.

10.2.8 The hedging proportions, the amounts to be invested at all times in the three tradeables, are:

— invested in the share: $\varphi_S(t) = V(t)$
— invested in the dla: $\varphi_D(t) = S(t).G.N(d_1)$
— invested in the zcb: $\varphi_B(t) = -\varphi_D(t)$.

The amount invested in units is the full value of the Maxi option. The amounts invested in the dla and the zcb are equal but of opposite signs, the former positive (implying a 'long' position) and the latter negative (or 'short', equivalent to borrowing). The more the option is in the money the larger are these offsetting amounts, approaching in the limit the full value of the option. If the option is very far out of the money, these amounts are both small.

10.3 *Discrete Hedging*

10.3.1 In Appendix C we show that, if the 'real world' model is the same as the model used for option pricing, with the same parameters in so far as these are relevant, and if hedging is simulated as taking place, free of transaction costs, at frequent enough intervals, then the result of the investment process closely matches the required payoff of the option in each simulation. This most important point is essential to the option pricing methodology. In order for the theoretical option price, calculated in accordance with the theoretical model, to be taken as the true or 'fair' value of the option, it is necessary that it is possible to carry out hedging in accordance with the required proportions, sufficiently frequently, and sufficiently cheaply, so that the results of investment according to the hedging strategy can be shown to replicate the desired payoff. Unless hedging can actually be carried out in this way, then theoretical option prices are, we suggest, not 'fair values', nor indeed of much use for any purpose. We revert to this in Section 12.

10.3.2 Note that for the GAO the real world model and the option pricing model need to correspond only in relation to the standard deviations and two out of the three correlation coefficients, and of course need to use the same values for the guaranteed rate, g, and the initial value of the fla, F(0).

10.3.3 The theoretical 'real world' model used for validation in Appendix C is an artificially simplified one. We now investigate what the consequences would be of a more realistic real world model being used for the simulation, with, nevertheless, the option prices and hedging proportions being calculated according to the model set out in Appendix C. As in the previous Sections we assume the Wilkie model for the real world. However, this model is defined for simulation only at annual steps, so we need to be able to simulate it at more frequent intervals. We do this by means of interpolation with 'stochastic bridges', which we describe more fully in Appendix D.

10.3.4 For these experiments we use, in the first place, a hybrid Wilkie model, with the parameters of the 1984 model (as given in ¶2.4.3), but with the addition of the model for the base rate, B(t), as in the 1995 version of the model, but with parameters: BMU $= -0.185$, BA $= 0.75$, BSD $= 0.175$. This allows us to construct a yield curve, as described in Appendix B, with a value for β of 0.39. Later we use the full 1995 version.

10.3.5 We first simulate annual values of the model, and then use a Brownian bridge to interpolate between successive annual values of the logarithm of the share Total Return Index, S(t). We use an OU bridge for the logarithm of the consols yield, C(t), and another for the 'log spread' = log(B(t)/C(t). The annual parameters we use in the first place are:

for the share total return: $\sigma_y = 0.2$;
for the log consols model: $\mu_y = -2.48$, $\alpha_y = 0.96$ and $\sigma_y = 0.08$;
for the log spread model: $\mu_y = -0.185$, $\alpha_y = 0.75$ and $\sigma_y = 0.175$.

In the 1995 Wilkie model, the log spread is simulated using an AR(1) model, and the parameters above are those for the annual hybrid model. The other parameters are based on simulations using the 1984 parameters of the model.

10.3.6 When we use the 1995 model for the annual simulations, with the parameters as given in ¶6.1.2, we use bridging parameters:

for the share total return: $\sigma_y = 0.2$;
for the log consols model: $\mu_y = -2.56$, $\alpha_y = 0.94$ and $\sigma_y = 0.095$;
for the log spread model: $\mu_y = -0.23$, $\alpha_y = 0.74$ and $\sigma_y = 0.18$.

10.3.7 We then choose parameters for the option pricing model that are based on simulations of the hybrid model, with the zero coupon rate derived from the yield curve and the value of the fla derived from a specific mortality table and the appropriate part of the yield curve. In practice we find that the particular mortality table used affects the mean value of the fla (which is not needed for the option pricing formula), but hardly affects the standard deviation (which is what we need). The zcb rate depends on the term, and on exactly how its parameters are estimated. In the option model, we assume that the zcb rate for a specific maturity date has constant parameters. But this is the track of a zcb rate that is constantly shortening its term. In practice the standard deviation of zcb rates for shorter terms is larger than that of rates for longer terms, so the assumption of constancy is not strictly valid. But the variation is not great. For simplicity, we use the same option pricing parameters for all terms and all mortality bases. It would be more precise to use different ones.

10.3.8 In our first trials we discovered that our chosen parameters gave surpluses at maturity when the hedging strategy was followed. We therefore adjusted the parameters a little, so that the investment proceeds broadly matched the required payoffs from the option. The parameters appropriate for the 1984 hybrid model were:

for the share: 0.2;
for the zcb: 0.006;
for the fla: 0.03;
correlation coefficient between share and fla: 0.3;
correlation coefficient between zcb and fla: −0.9.

10.3.9 When we used the 1995 parameters of the model, these were altered to:

for the share: 0.2;
for the zcb: 0.01;
for the fla: 0.04;
correlation coefficient between share and fla: 0.2;
correlation coefficient between zcb and fla: −0.9.

10.3.10 In carrying out our experiments it appeared that what was all-important to the success of the hedging strategy was how closely the option pricing model matched the model used for stochastic bridging. The annual model used did not seem to matter. If this is generally true, it is possibly an important result, and it deserves more investigation than is necessary for our immediate purpose.

10.3.11 We restrict ourselves to simulation with hedging being carried out twice per month. We have discovered in another context that, if reasonable transaction costs are allowed for, this frequency gives the best balance between the cost of hedging and the accuracy. In a fuller investigation we would allow for transaction costs in this case too, and find what frequency of hedging was best. Another strategy, described by Boyle and Hardy (1997), is to investigate frequently, but to alter the portfolio only when it is sufficiently far away from the desired proportions.

10.3.12 As described in Appendix C, if one starts with the correct amounts invested in shares, dla and zcb, in accordance with the correct hedging proportions at time 0, by the end of the next time step the value of investments will, in general, not have changed exactly in line with the option value. In C.11.6 we described the different investment strategies that might then be followed. Sometimes one of these strategies seems to be the best, sometimes another. We find that procedure (iv) is always amongst the best, so we have used it throughout these experiments. This involves investing the net proceeds at each step, whatever they are, in accordance with the correct hedging proportions at that point.

10.3.13 When we reach the end of the deferred period (the retirement age) for each contract, we then see whether the investment proceeds are sufficient to pay for the required payoff of the maxi option at that time. Often they are too great; often they are too little. We calculate the deficit, and we then calculate the present value of that deficit. We could discount in any of four different ways, assuming that the contingency reserve (which we shall wish to set up) is invested in: the share portfolio, the zcb, the dla, or in accordance with the same proportions as the option hedging portfolio. Sometimes one of these is better, sometimes another. Investing in accordance with the hedge portfolio seems to be, on balance, the best, and we have used this in these experiments. However, whether this is the best strategy in any other case would require investigation; in investigations of other types of option we have found that it is far from being the best.

10.3.14 We now, for any particular term, and starting conditions, have the initial option price, the same for all simulations, and the discounted present values of the deficit for each simulation. We should then allow for the fact that not all policyholders will survive to the end of the term, and so be able to take advantage of the GAO. This affects both the option premium and the value of the deficit. We therefore multiply both the excess of the option value over £100 and the value of the deficit by ${}_{T}p_{x}$, to give figures that correspond with V0 as defined in ¶2.3.6. We denote the reduced option price (in excess of £100) as ROP, and the reduced and discounted value of the deficit by DPV, as before.

10.3.15 We can calculate the mean value of the DPV, its standard deviation (and since the deficit is reasonably symmetrical this is meaningful, though the DPVs are also quite skew and very fat-tailed), and we can rank them in sequence, just as we did for the corresponding amounts, V0 in Section 2.3. We can then calculate quantile reserves and CTEs as before. The insurer needs to set up contingency reserves, in addition to the option price, at some desired level of security, based either on a quantile, or, preferably, on a CTE. The policyholder's premium (in excess of £100) should consist of the ROP, the mean value of the DPV, the two together making up the amount denoted A, together with a further amount B, calculated in the same way as before, making up a total amount C. Alternatively the office may calculate premiums in accordance with the procedure we describe in Section 8.2. The relevant question is then: does the hedging procedure allow lower contingency reserves to be set up, and does it allow lower premiums for the policyholder, both with the same level of security, and the same parameters?

10.4 *Results of hedging*

10.4.1 We start as if at the beginning of 1985, with market conditions for the investment model as at 31 December 1984, and using mortality PMA68U1985. Table 10.4.1 shows, for terms 10, 20, 30 and 40, the value of the fla, F(0) at commencement, the value of the maxi option per £100, the value of the reduced option premium, and the amounts that should be

Table 10.4.1. Values for the option per £100 single premium: initial conditions of 31 December 1984, mortality PMA68U1985.

Term	F(0)	Option price	Reduced OP	Share	Zero-coupon bond	Deferred annuity
10	7.69	100.22	0.19	100.22	−5.56	5.56
20	7.74	100.91	0.74	100.91	−13.65	13.65
30	7.85	101.39	1.13	101.39	−16.47	16.47
40	7.93	101.50	1.24	101.50	−15.72	15.72

invested in the share, the zero-coupon bond and the deferred life annuity initially (assuming the full, not the reduced, option price). The options are well out of the money, the values of the option are quite small, and the amounts invested in the zcb and dla are not enormous, though they are much larger than the value of the option might suggest.

10.4.2 Table 10.4.2 shows the same as Table 2.5.1 (for terms 10, 20, 30, 40, the total portfolio, and the 'Sum') with the mean cost and selected quantiles, all including the ROP. Careful comparison with Table 2.5.1 shows that the means for terms 10 and 20 are larger with the hedging strategy, but for terms 30 and 40 are smaller. The extreme quantiles are very much smaller now, but Q_{90} for a 10-year term, previously zero, is now larger. Previously the values all increased considerably with term. Now the values for term 40 are lower than for term 30, and the values for term 30 are not much greater than those for term 20. Of course with different parameters different results might well be obtained.

Table 10.4.2. Present value of cost of GAO per £100 single premium, with hedging: 1984 Wilkie model, initial conditions of 31 December 1984, mortality PMA68U1985.

Term	Mean	Q_{90}	Q_{95}	$Q_{97.5}$	Q_{99}	$Q_{99.5}$	$Q_{99.9}$
10	0.42	0.73	0.91	1.09	1.30	1.42	1.78
20	1.07	1.60	1.80	1.98	2.18	2.35	2.60
30	1.10	1.88	2.11	2.32	2.56	2.71	3.02
40	0.53	1.64	1.93	2.15	2.43	2.62	3.00
Portfolio	0.91	1.33	1.43	1.51	1.60	1.67	1.80
Sum	0.91	1.59	1.81	2.00	2.24	2.41	2.76

10.4.3 Table 10.4.3 shows the same as Table 2.5.3 (omitting the 'Sum'), with values of B and C for $\alpha = 99\%$ and 99.9%, $h = 1\%$ and 2%. Remember that C is the proposed premium that the policyholder should pay, in excess of the basic premium of £100. Comparison with Table 2.5.3 shows that these

Table 10.4.3. Charge to policyholders per £100 single premium, with hedging: different combinations of α and h; 1984 Wilkie model, initial conditions of 31 December 1984, mortality PMA68U1985.

Term		99%, 1%		99%, 2%		99.9%, 1%		99.9%, 2%	
	A=Mean	B	C	B	C	B	C	B	C
10	0.42	0.08	0.50	0.15	0.57	0.12	0.54	0.23	0.65
20	1.07	0.18	1.26	0.32	1.39	0.25	1.33	0.43	1.51
30	1.10	0.34	1.43	0.55	1.65	0.44	1.54	0.72	1.82
40	0.53	0.54	1.07	0.84	1.37	0.70	1.24	1.09	1.63
Portfolio	0.91	0.14	1.05	0.23	1.14	0.18	1.09	0.29	1.21

Table 10.4.4. Values for the option per £100 single premium: initial conditions of 31 December 1995, mortality PMA80U1996.

Term	F(0)	Option price	Reduced OP	Share	Zero-coupon bond	Deferred annuity
10	9.35	107.13	6.57	107.13	−65.69	65.69
20	9.46	107.62	6.90	107.62	−56.07	56.07
30	9.52	105.91	5.35	105.91	−41.13	41.13
40	9.55	103.58	3.24	103.58	−25.32	25.32

values are now much lower. It would therefore seem from these figures that the hedging strategy, although the mean cost may sometimes be greater than without it, would be beneficial to both shareholder and policyholder, in that lower contingency reserves would be required, and a lower premium is appropriate. But we have tried only one experiment.

10.4.4 We now move forward to the initial conditions of December 1995, and use PMA80U mortality and the 1995 Wilkie model. Table 10.4.4 shows details of the option pricing. The options are now a little into the money. The option prices are not much above 100, but the amounts to be invested in the zcb and the dla are quite a lot larger than before.

10.4.5 Table 10.4.5 shows the same information as Table 10.4.2 and it should be compared with Table 6.2l. The means are higher for short terms, lower for longer terms, and all the quantiles are substantially lower. There is not a great variation in the quantiles by term, whereas in Table 6.2l they rise quite a lot as the term increases.

10.4.6 Table 10.4.6 shows the same information as Table 10.4.3 and should also be compared with Table 6.2l. The values of C are almost everywhere lower than they would be without hedging. The only exception is $C_{99,1}$ and $C_{99.9,1}$ for term 10.

Table 10.4.5. Present value of cost of GAO per £100 single premium, with hedging: 1995 Wilkie model, initial conditions of 31 December 1995, mortality PMA80U1996

Term	Mean	Q_{90}	Q_{95}	$Q_{97.5}$	Q_{99}	$Q_{99.5}$	$Q_{99.9}$
10	7.28	8.42	8.74	9.04	9.35	9.65	10.28
20	7.79	9.17	9.53	9.83	10.23	10.52	11.09
30	6.56	8.03	8.42	8.81	9.35	9.68	10.46
40	4.84	6.63	7.35	8.00	8.93	9.68	11.46
Portfolio	6.88	7.77	8.00	8.19	8.41	8.61	8.96
Sum	6.88	8.31	8.75	9.13	9.61	9.95	10.72

Table 10.4.6. Charge to policyholders per £100 single premium, with hedging: different combinations of α and h; 1995 Wilkie model, initial conditions of 31 December 1995, mortality PMA80U1996.

Term		99%, 1%		99%, 2%		99.9%, 1%		99.9%, 2%	
	A=Mean	B	C	B	C	B	C	B	C
10	7.28	0.19	7.47	0.35	7.62	0.27	7.55	0.50	7.78
20	7.79	0.41	8.20	0.70	8.49	0.55	8.34	0.94	8.73
30	6.56	0.64	7.20	1.04	7.60	0.90	7.46	1.46	8.02
40	4.84	1.16	6.01	1.81	6.65	1.88	6.72	2.92	7.76
Portfolio	6.88	0.31	7.18	0.51	7.39	0.42	7.29	0.69	7.57

Table 10.4.7. Values for the option per £100 single premium: initial conditions of 31 December 2001, mortality RMC92U2002.

Term	F(0)	Option price	Reduced OP	Share	Zero-coupon bond	Deferred annuity
10	12.77	141.46	39.29	141.46	−141.09	141.09
20	13.08	139.75	37.54	139.75	−135.93	135.93
30	13.29	132.93	31.32	132.93	−120.34	120.34
40	13.45	123.30	22.33	123.30	−94.98	94.98

10.4.7 We now go to the other end of the scale to use the 1995 Wilkie model, the initial conditions of 31 December 2001, and RMC92U mortality. Table 10.4.7 shows details of the option pricing. The options are now well into the money, and correspondingly expensive. The amounts to be invested in the zcb and the dla are now almost as large as the amount to be invested in shares, especially for shorter terms.

Table 10.4.8. Present value of cost of GAO per £100 single premium, with hedging: 1995 Wilkie model, initial conditions of 31 December 2001, mortality RMC92U2002

Term	Mean	Q_{90}	Q_{95}	$Q_{97.5}$	Q_{99}	$Q_{99.5}$	$Q_{99.9}$
10	38.35	39.38	39.71	40.11	40.65	41.25	42.67
20	37.92	40.43	41.63	42.66	43.72	44.47	45.97
30	33.94	37.64	38.84	39.81	40.92	41.75	43.21
40	26.86	31.71	33.35	34.86	36.40	37.66	40.40
Portfolio	34.99	37.43	38.14	38.75	39.41	40.01	40.66
Sum	34.99	38.01	39.10	40.10	41.29	42.12	43.84

10.4.8 Table 10.4.8 shows the same information as Table 10.4.2 and it should be compared with Table 6.4r. The means are higher, but the quantiles are substantially lower. It is interesting that the mean and the quantiles do not vary very much by term, just as can be seen in Table 6.4r.

10.4.9 Table 10.4.9 shows the same information as Table 10.4.3 and should also be compared with Table 6.4r. The values of C are now generally larger than what they would be without hedging. Thus for term 10 they are now distinctly higher, for term 20 they are mostly higher, for term 40 they are lower, and for term 30 they are mixed. While the hedging would allow lower contingency reserves, it does not seem to be conspicuously cheaper for the policyholder. But the differences between the two methods are quite sensitive to the parameters and assumptions used. Further, we have not allowed for the transaction costs of hedging, which would make all the hedging methods considerably dearer.

10.4.10 In our calculations so far, both with and without using the hedging strategy, we have assumed that the real 'real world' does in fact behave according to our assumptions, our assumptions about the models, the

Table 10.4.9. Charge to policyholders per £100 single premium, with hedging: different combinations of α and h; 1995 Wilkie model, initial conditions of 31 December 2001, mortality RMC92U2002.

Term		99%, 1%		99%, 2%		99.9%, 1%		99.9%, 2%	
	A=Mean	B	C	B	C	B	C	B	C
10	38.35	0.21	38.56	0385	38.73	0.39	38.74	0.72	39.07
20	37.92	0.96	38.89	1.65	39.57	1.33	39.26	2.28	40.20
30	33.94	1.60	35.54	2.60	36.54	2.12	36.06	3.44	37.38
40	26.86	2.70	29.56	4.19	31.05	3.82	30.67	5.92	32.77
Portfolio	34.99	0.88	35.87	1.47	36.46	1.13	36.12	1.87	36.87

values of the parameters, and the distribution of innovations, both for the annual (macro) model and for the bridging (micro) process; further, we have assumed at each stage that the latest mortality forecasts would in fact be realised, whereas the record of the forecasts made by the C.M.I. from time to time has not been as accurate as one would have liked. In Section 11 we discuss a number of further enhancements to the methods we have discussed so far, and consider their possible effects.

10.5 *Other Papers on Options and GAOs*

10.5.1 Boyle (1978) seems to have been the first to suggest that option pricing methodology could be applied to the valuation of guaranteed annuity options, but this idea does not seem to have been taken up during the next 20 years. Bolton *et al.* (1997) draw an analogy between a GAO and a 'swaption', which is a financial option giving the holder the right to effect a 'swap' at some future date on pre-arranged terms. A 'swap' is, in its economic effect, the same as the forward purchase of a coupon bond at a specific time, for a specific term and coupon rate. Beyond quoting some specimen costs for such swaptions, Bolton *et al.* carry the idea no further, and assume that a life office would purchase a swaption from some counterparty.

10.5.2 Van Bezooyen, Exley and Mehta (1998) (VBEM) discuss the swaption idea much more fully, in conjunction with the quanto concept. They develop two formulae for the option price, both similar to ours, but omitting the important element of the (negative) correlation between the zcb and the dla. Their first formula uses the Black model, assuming that the prices of shares and of the fla are lognormally distributed; but they seem to omit variation of the interest rate during the deferred period, or at least its correlation with the fla price.

10.5.3 The second formula of VBEM allows for the purchase, not of an annuity, but of a further zcb at retirement, on the grounds that an annuity is simply the weighted sum of a number of such zcbs. However, an option on an annuity is not the same as the sum of separate options on such zcbs, because each separate option could be either exercised or not, whereas the GAO is either exercised all together or not at all. They model interest rates with a one-factor Hull & White (1990) model, in which (we suppose, but full details are not given) the immediate spot rate follows a Vasicek model (like our zcb rate).

10.5.4 VBEM suggest hedging by, in effect, entering into forward contracts for a portfolio of zcbs that match the required annuity; in our view this is like reinsurance and simply passes the real hedging problem on to someone else, as we explain in Section 10.7. They also demonstrate the results of discrete hedging at monthly intervals over a past period, and show how a more complicated hedging strategy than we have suggested could match up changes in the partial derivatives of the option value as well as in the option value itself; this is well worth exploring further. However, they do not indicate a way of calculating contingency reserves to cover hedging error and the corresponding transaction costs.

10.5.5 Pelsser (2002) develops the swaption idea further, showing that a life annuity (as we have defined it, i.e. an annuity certain with reducing payments) can be replicated by a series of coupon bonds equally as well as by a series of zero-coupon bonds. An option on a life annuity can be replicated by a series of swaptions on coupon bonds, which are readily available in the market (though whether for the terms and quantities required is not clear). A single option on a life annuity has a lower value (or rather, no higher a value) than the options on the replicating bonds, because one can 'cherry-pick' which of the latter to exercise whereas the GAO must be exercised in totality. But a result of Jamshidian (1989) indicates that, if rates of interest are modelled by a one-factor model, the two portfolios can have equal values at some adjusted exercise rate.

10.5.6 Pelsser also develops a theoretical model for GAOs on similar lines to what we have done, but with an interesting addition. Using the fact that, if interest rates are never negative, the annuity rate has a lower limit, he models the forward life annuity through the rate (we model the value), as if it were a shifted lognormal, i.e. the excess of the rate over the minimum value is assumed to be lognormal. If one converts the rate to a value by taking the reciprocal, one then has a distribution with an upper limit, which cannot be lognormal, and this would be intractable. By choosing the deferred annuity as the numeraire, i.e. dividing by $D(t)$ in our model, he keeps the lognormal part in the numerator of his expressions, and derives a formula, similar to ours, but with the offset recognised. This model is worthy of further consideration. However, Pellser does not then seem to use an explicitly separate model to drive the zcb part before retirement, as we do, and he gives no numerical results in this part of his paper.

10.5.7 Pellser ignores the connection between share price changes and interest rates because he is working in the framework of traditional with profits policies, with increases in the reversionary bonus from time to time. When the bonus is increased, extra swaptions are purchased, but the swaption portfolio remains static otherwise. He does not discuss how the swaptions are paid for, but obviously the cost could be treated as part of the cost of bonus.

10.5.8 The paper by Belotta & Haberman (2002) is possibly still subject to revision, because we have seen several versions, and the latest version has numerical results that do not seem wholly plausible. They are ambitious and complicated, using a single factor interest rate model, with the fullest generalisation of the Heath-Jarrow-Morton (1992) model, in which the standard deviations of forward rates are functions of time, and including the correlation between shares and interest rates. They also use Jamshidian's (1989) result. They then simplify to two specific cases, one with fixed standard deviations, the other with an exponential decay in the standard deviation. Both methods appear to miss out the correlation observed in our model between short and long term rates, though with a single factor this should be automatic. The do not derive the hedging proportions. However, their work also deserves further study.

10.5.9 None of the authors mentioned above discuss how the options should be priced to the policyholder, nor what contingency reserves might need to be kept in addition to the option value, for example for mortality uncertainty, as we discuss in Section 11.3, nor the errors and costs of empirical hedging at discrete intervals (though VBEM do give one historical example), nor the practicability of hedging, which we now discuss.

10.6 *Practical Considerations*

10.6.1 We have so far made our calculations as if it were in fact possible to carry out the hedging strategy in the way we have described. It is of course possible to invest in the 'share' portfolio; life offices do this anyway. But the zcb and the dla are a different matter. The former has to be held 'short' the latter 'long'. Zero-coupon U.K. government bonds, in the form of 'Strips', have existed since early in 1998. We do not know how easy it is to hold such a bond short for a long period, and to adjust the amount up and down as required, but so far as we know, bonds can be held short in the 'repo' market only for very short periods, such as a few days.

10.6.2 Deferred life annuities, in the form we have described, i.e. as fixed interest contract with no payment until some vesting date, and payments thereafter in proportion to the values of ${}_tp_{65}$ according to a specified life table, are not, and so far as we know never have been, issued. It would be possible to replicate such a bond with an appropriate schedule of zcbs, or similarly with deferred coupon bonds, if these existed.

10.6.3 Even though Strips have existed since 1998, their duration is not long enough to match all the payments even of an immediate annuity, and certainly not to cover a deferred annuity with a longer term. The longest U.K. Government stock in issue in 1985 with a final redemption date was 12% Exchequer 2013-17, which might be repaid as early as 12 December 2013, some 28 years ahead from 1985, or might be redeemed as late as 12 December 2017, 32 years ahead. It remained the longest dated stock until 1996, by which time its maturity dates were 17 and 21 years away. Longer stocks issued since then have been:

— on 29 February 1996:	8% Treasury 2021,	repayable 7 June 2021;
— on 29 January 1998:	6% Treasury 2028,	repayable 7 December 2028;
— on 25 May 2000:	$4\frac{1}{4}$% Treasury 2032,	repayable 7 June 2032;
— on 27 September 2001:	5% Treasury 2025,	repayable 7 March 2025.

All four of these stocks are now in strippable form, and a series of Strips exists, payable on 7 March, 7 June, 7 September and 7 December, each quarter until 7 June 2032. So-called 'irredeemable' stocks also exist, the largest of which is $3\frac{1}{2}\%$ War Stock (often called 'War Loan') but all of these are now repayable when the Government chooses, generally with just three months notice. Such an early redemption would presumably happen if interest rates fell significantly below the coupon rates on the stocks, which range from $2\frac{1}{2}\%$ to 4%. None of the irredeemables is therefore satisfactory to match with certainty a very long liability, particularly a GAO, which is of greater value when interest rates fall.

10.6.4 Some U.K. corporate bonds with longer maturities do exist, but their security is obviously less than that of Government bonds, and during the 1980s and much of the 1990s the quantities issued were not large. So far as we are aware, they are only in the form of coupon bonds. It may sometimes be possible to match a schedule of deferred annuity payments with long and short holdings of coupon bonds, but this still requires holding such bonds short for extended periods, and also of adjusting the amounts frequently to match the hedging requirement.

10.6.5 What may be required is to match a portfolio of GAO policies of different terms. The overall maturity schedule would need to be taken into account, but for any portfolio of policies the negative holdings are necessarily shorter in date overall than the positive holdings, which are longer in date. A life office could perhaps include a portfolio of immediate or vested annuities in the calculations, and by so doing offset some of the early negative holdings. But there would still be a lot of frequent adjustment, and stocks of a long enough date do not seem to be available.

10.6.6 A zcb and a dla combined form a portfolio which is exactly equivalent to a forward contract to purchase an fla. A life office might be able to arrange such a forward contract with another institution, such as an investment bank. However, that other institution would need to match its liability by holding a zcb short and a dla long, so the problem is simply passed on rather than solved.

10.6.7 A further problem about any dynamic hedging is that the very fact of hedging may affect the prices of the underlying securities in the market. This is feature of all option pricing models, all of which (so far as we are aware) assume that the prices of the underlying securities follow some stochastic process which is not affected by the action of hedging by those who have written options. But it is reasonable to assume that the attempted purchase or sale of any security, beyond some trivial quantity, acts to increase or decrease the market price of the security to some extent. Since this volume effect is not normally taken into account, theoretical option prices are therefore only marginal prices, i.e. the prices of a negligible quantity of options.

10.6.8 This aspect of option pricing was recognised by the Maturity Guarantees Working Party (Ford *et al.*, 1980), who observed (¶6.6), in the context of maturity guarantees, that if all life offices that had written policies with maturity guarantees tried to sell shares when the price of shares fell, as the hedging strategy required, this would simply push the price of shares down further, which in turn would require more selling of shares. This type of 'portfolio insurance' effect is said to have exacerbated the share price crash of October 1987.

10.6.9 In the case of GAOs the effect would be felt differently. If share prices rose, so that the amounts that should be invested in zcbs and dlas increased, this would mean more selling of (shorter) zcbs and buying of (longer) dlas, so altering the 'slope' of the yield curve. If interest rates fell, so that the prices of both zcbs and dlas rose, the slope of the yield curve would also be altered; but the effect of the change would be a differential effect on the (shorter) zcbs and the (longer) dlas, which might reinforce the effect of the change in slope of the yield curve, possibly in an unstable way.

10.6.10 The quantity of U.K. life business with GAOs attached was estimated by Bolton *et al.* (1997), to have been about £35 billion, as at the end of 1996. The value has probably increased since then because of the rise in share markets, but the quantity has probably reduced because few new policies have been written, and some have passed retirement and others have been bought out. In comparison with the size of the U.K. gilts market (about £300 billion in 2002) the amount is not overwhelming, but it would be large enough to affect prices if long enough bonds existed to make the hedging possible at all.

10.6.11 We conclude that, however, attractive the hedging strategy is conceptually and mathematically, in the case of real GAOs its practicability is severely limited. If the term to vesting were very short, say less than five years, it might be possible to hedge a large part of the dla, say the payments from age 65 to age 90, and then use quantile or CTE reserving explicitly for the balance beyond age 90 (with quantile or CTE reserving still being needed for the potential hedging error, as we have described). But as a general solution, we do not consider that hedging of actual GAOs is practicable.

10.7 *Reinsurance*

10.7.1 It may be suggested that a life office could purchase a suitable option contract from an investment bank. This we consider as equivalent to reinsurance. There are possible advantages of such reinsurance, as we discuss below, but the counterparty would require to reserve for its liability in the same way as the original insurer, and would therefore, other things being equal, need to charge the same to the original insurer as the original insurer would have to charge to the policyholder. If the counterparty charges a premium as if hedging were possible, but is not in fact able to hedge, or chooses not to, then its security is no better than that of an insurer who fails to charge sufficiently, or fails to hedge.

10.7.2 We note and comment on some advantages of reinsurance of financial risk by an insurer with a counterparty, who might well be an investment bank.

10.7.3 The counterparty may have expertise and systems that enable it to carry out hedging operations more easily and cheaply than the original insurer. On the other hand the insurer may be able to hire or develop such expertise.

10.7.4 The counterparty may be able to offset risks of options against other opposing risks, and so be able to hedge its net position more easily. The insurer may not have access to such offsetting risks. On the other hand, a GAO is what we have described as a Maxi option. The offsetting risk is a Mini option, under which the policy proceeds are converted to an annuity at the *poorer* of the market rate and the guaranteed rate. We do not know how many policyholders would like to purchase such a contract, even if there were a discount on the premium (i.e. a negative extra premium) equivalent to the positive extra premium paid by those policyholders who wished to purchase a GAO.

10.7.5 The counterparty may have access to more capital then the insurer, or may have a larger diversity of risks than the insurer, in either case enabling it to accept a lower rate of excess return on the contingency reserves that need to be set up, i.e. use lower rates of h, j or k in the methods described in Sections 2.3 and 8.2.

10.7.6 The usual advantages of life or general reinsurance, that the reinsurer has a larger pool of lives or risks insured, and so the individual risks can be spread more widely, do not apply in the case of financial options. All contracts with the same terms face the same outcome. Diversification of risks may occur to some extent if contracts expire at different dates, or have offsetting positions, or are in different markets. We have shown how a portfolio of GAOs may have smaller risks, and so require smaller contingency reserves than the sum of the individual policies, but they may still require quite large reserves, and the results from year to year may be strongly correlated, since interest rates do not bounce around enormously, but instead show long periods with rates at similar levels.

10.7.7 Whilst reinsurance has some obvious advantages, particularly for smaller life offices, we do not consider that it is, overall, a satisfactory solution to reserving for GAOs. The buck stops somewhere, and the risk from GAOs (unlike the risk of individual insurances which can be treated as almost independent) is not significantly altered by passing it on.

11. Further Enhancements

11.1 *Introduction*

11.1.1 Both in the static reserving described in Section 2 and in the dynamic hedging (with contingency reserves) described in Section 10, we have made many assumptions: first, that the models we are using are correct; then that we know the parameters of those models accurately; in particular we assume that the random innovations, both of the annual Wilkie model, and of the bridging model, are normally distributed; finally that the mortality forecasts that we have used turn out to be correct. It is possible to relax each of these assumptions in one way or another. We describe below how this could be done, without showing results.

11.2 *Fat-tailed Distributions for the Innovations*

11.2.1 The Wilkie model as we have used it assumes normally distributed random innovations, and we have used normally distributed innovations for the bridging models in Section 10.3. This is consistent with the assumptions made in Appendix C that the 'drivers' of the various investment processes are Brownian motions. However, there is no difficulty in computer simulation of any model to use one or more out of a great many different distributions for innovations. It is well known that changes of variables in the real world are leptokurtic, or 'fat-tailed', with a kurtosis well in excess of the value 3 that a normal distribution would have. If the real world is fatter-tailed than we have assumed, it is likely that the spread of results would be wider than we have assumed, and hence that the contingency reserves would be greater than we have calculated. In addition, if innovations are fat-tailed, the hedging error is likely to be much larger than we have calculated.

11.2.2 Using monthly values from December 1950 to August 2001 of the relevant variables we can calculate statistics of (a) the differences in the logarithm of a share total return index, (b) the residuals of the logarithm of the consols yield after fitting an AR(1) model with $\alpha = 0.9912$, and (c) the residuals of the 'log spread' after fitting an AR(1) model with $\alpha = 0.9534$. We obtain:

(a) skewness: 0.01 kurtosis: 11.6;
(b) skewness: -0.23 kurtosis: 4.89;
(c) skewness: 1.53 kurtosis: 10.43.

These are very far from normality.

11.2.3 It is not difficult to simulate a fatter-tailed distribution. There are many ways to do this. One that we favour is to treat each innovation, X, as the difference between two independent non-negative random variables, Y_1 and Y_2, so that $X = Y_1 - Y_2$. Y_1 and Y_2 could both be lognormally distributed. If they are lognormally distributed with $\log Y_i \sim N(\mu_i, \sigma_i^2)$, we have four parameters to play with. It is convenient to arrange that X is standardised, so that it has zero mean and unit variance. This puts two constraints on the values of the parameters. One can then choose the other two to match, within limits, any desired skewness and kurtosis. If $\mu_1 = \mu_2$ and $\sigma_1 = \sigma_2$ then the distribution is symmetrical and automatically has zero mean and zero skewness.

11.2.4 Lognormal variables are easy to simulate. So also are variables from many of the other non-negative distributions used in general insurance work, such as the Pareto, Burr and Weibull, described in Hogg & Klugman (1984), all of which have distribution functions that can be easily inverted. Others, such as the gamma, are harder to simulate, but methods are known. One could choose Y_1 and Y_2 from different distributions, if one wished to represent the shapes of the positive and negative tails in different ways. The options are greater than our knowledge of the tails of the real-world distributions might justify.

11.2.5 Another possibility is to use an α-stable (or stable Paretian) distribution (see Finkelstein, 1997). Methods for simulating random variables from these distributions are also known (see Chambers, Mallow & Stuck, 1976). Apart from the normal distribution, which is a special case, α-stable distributions have infinite variance. If α-stable distributions are used for option pricing, the values of most options become infinite. However, one can still carry out simulations, using the usual hedging strategy based on lognormal distributions for the option values, and with the real world simulated with α-stable innovations.

11.2.6 Among the distributions mentioned in ¶11.2.4 the lognormal, gamma and Weibull always have finite variance. The Pareto and Burr may have infinite variance for certain values of the parameters. If a distribution with finite variance is used for simulation, then the effect over multiple time periods still tends toward normality; this is a consequence of the central limit theorem. If a distribution with infinite variance is used, then the effect tends towards an α-stable distribution. The short-term and long term effects may need to be distinguished.

11.2.7 If a fat-tailed distribution with finite variance is used within the Wilkie model (or any similar model) for annual simulation, the effect over a longer term may be diminished because of the tendency towards normality. Over a small number of years, however, the effect may be significant. One may therefore prefer to test out the effect of an infinite variance distribution when considering static reserving, as in Section 2.

11.2.8 On the other hand, if dynamic hedging is used, then the bridging process is important (indeed it may be all-important, and it is possible that using fat-tailed innovations within the annual model would have almost no effect on the hedging method). A fat-tailed distribution for the bridging innovations will, normally, put up the hedging error considerably; the central limit theorem has no opportunity to have an effect.

11.2.9 We have not tested out reserving for GAOs using fat-tailed distributions in either context, but separate investigations in another context show that for an at-the-money option the 99% quantile may be multiplied several-fold.

11.3 *Stochastic Mortality Rates*

11.3.1 We have assumed that the forecast mortality rates, as published by the C.M.I. Bureau from time to time, turn out to be the correct rates. But the very fact that the basis of projection has changed so much between the projections based on 1967-70 mortality and those based on 1991-94 mortality shows that even the best constructed forecasts of mortality rates may prove to be wrong. One way to deal with this is to allow the future mortality rtes to be random. Lee (2000) described a possible stochastic mortality model, and Yang (2001) modified his method and used it in her thesis.

11.3.2 The method starts, like the C.M.I. Bureau's forecasts, from a base mortality table applicable to some year, say $t = 0$, with annual mortality rates $q(x,0)$. It then uses the C.M.I. projection factors, $\mathrm{RF}(x,t)$, so that the projected mortality rate at age x in year t is $q(x,t) = \mathrm{RF}(x,t) \times q(x,0)$. So far all is deterministic. It then assumes two random variables applicable to year t, $\mathrm{X}(t)$ and $\mathrm{Y}(t)$. $\mathrm{X}(t)$ performs a random walk, with in principle a zero mean but with a bias to counteract the variance, so that $\mathrm{X}(t) = \mathrm{X}(t-1) - \frac{1}{2}\sigma_{\mathrm{X}}^2 + \sigma_{\mathrm{X}}.z_{\mathrm{X}}(t)$, where the $z_{\mathrm{X}}()$s are independent, with zero mean and unit variance; they might well be distributed normally, but in view of our comments in Section 11.2, they need not be. $\mathrm{Y}(t)$ is dependent on $\mathrm{X}(t)$, $\mathrm{Y}(t) = \mathrm{X}(t) - \frac{1}{2}\sigma_{\mathrm{Y}}^2 + \sigma_{\mathrm{Y}}.z_{\mathrm{Y}}(t)$, where the $z_{\mathrm{Y}}()$s are independent of each other and of the $z_{\mathrm{X}}()$s, and are unit distributed. We start with $\mathrm{X}(0) = 0$. Then the experienced mortality rates in year t are assumed to be $q(x,t) \times \exp(\mathrm{Y}(t))$.

11.3.3 The rationale for these two variables is that $\mathrm{X}(t)$ represents the overall drift of mortality rates, that continues from year to year, while $\mathrm{Y}(t)$ includes both $\mathrm{X}(t)$ and an annual factor that is peculiar to that year, representing perhaps the effects of an epidemic, a hard winter, or some such feature. In practice Yang found, on the basis of the limited evidence available from *C.M.I. Reports*, that σ_{Y} could be taken as zero, so $\mathrm{Y}(t) = \mathrm{X}(t)$. The expected values of $\mathrm{X}(t)$ and $\mathrm{Y}(t)$ are both unity; the $-\frac{1}{2}\sigma^2$ terms ensure that this is so.

11.3.4 A limitation of this model is that, in any one year, it applies the same multiplicative factor at each age x. It might be preferable to have a model in which the adjustment factors varied smoothly with x. In recent years we have seen mortality at ages over 60 improving sharply in the U.K., whereas the mortality of males in their 30s has slightly worsened. The same effect does not apply at all ages. A more elaborate model could be constructed, where random factors, say $\mathrm{Y}_1(t)$ and $\mathrm{Y}_2(t)$ were used, similar to $\mathrm{Y}(t)$, with experience mortality equalling $q(x,t) \times \exp(\mathrm{Y}_1(t) + \mathrm{Y}_2(t).(x - x_0))$ so that the adjustment factor was linear. However, to parameterise this might, with the limited data available, be difficult.

11.3.5 While the mortality experienced in each of the years during the deferred period would affect the number of survivors to retirement age, the more important effect would be on the mortality that would be assumed at retirement for the ensuing annuity. We need to model the response of actuaries and life offices to recent mortality experience in setting annuity rates. Yang (2001) assumed that actuaries calculated the average adjustment factor for the previous four years to set mortality rates for annuities. Other assumptions are possible.

11.3.6 Yang (2001) found, not surprisingly, that using stochastic mortality increased the required quantile reserves significantly. However, the effect varied very much with the conditions, and it would be misleading to quote any particular numbers. But for realistic reserving, it would seem desirable to take stochastic mortality into account.

11.3.7 Note that this stochastic mortality method allows for uncertainty about the values of q_x at each age in each future year, but that, given those values, it is assumed that the number of deaths at each age is exactly as expected.

11.4 *An Investment Hypermodel*

11.4.1 Just as one can allow for the uncertainty of mortality projections by the stochastic mortality method described in Section 11.3, so one can allow for the uncertainty of the parameters of the Wilkie (or any other) model in the simulations. The method was originally described by Wilkie (1986b) and again by Lee & Wilkie (2000). Instead of keeping the parameters the same for each simulation, one chooses values of all the parameters at the start of each simulation from some multivariate distribution of the parameters.

11.4.2 Thus, instead of simulating, for example, the inflation rate in the Wilkie model as:

$$\mathrm{I}(t) = \mathrm{QMU} + \mathrm{QA}.(\mathrm{I}(t-1) - \mathrm{QMU}) + \mathrm{QSD}.\mathrm{QZ}(t),$$

with constant values of QMU, QA and QSD, one uses, for simulation s:

$$\mathrm{I}(t) = \mathrm{QMU}(s) + \mathrm{QA}(s).(\mathrm{I}(t-1) - \mathrm{QMU}(s)) + \mathrm{QSD}(s).\mathrm{QZ}(t).$$

where QMU(s), QA(s) and QSD(s) are picked at random at the start of simulation s, and remain constant throughout that simulation.

11.4.3 In order to generate values of QMU(s), QA(s) and QSD(s) we need 'hyperparameters', denoted (rather clumsily perhaps) as QMUMU, QMUSD, QAMU, QASD, QMUQACC, etc. We then assume that QMU(s) is normally distributed, with mean QMUMU and standard deviation QMUSD. That QA(s) is also normally distributed with mean QAMU, standard deviation QASD and with correlation coefficient between QA(s) and QMU(s) of QMUQACC, and so on.

11.4.4 The use of normal distributions can be justified, if the parameters have been estimated by maximum likelihood, because then, at least asymptotically, the parameter estimates are distributed around the maximum likelihood estimators with a multivariate normal distribution, and the variance-covariance matrix of those estimates is derived in the fitting process.

The standard deviations could be therefore taken as the standard errors of the estimates of the parameters, based on the data the parameters have been estimated from, with correlation coefficients likewise. However, the mean values of the parameters could be taken as the same as the constant values of the parameters used in the chosen version of the model, which may be whatever is now considered the appropriate values for future simulation,

11.4.5 It is desirable that $QA(s)$ should lie between 0 and 1, so it may be necessary to limit its range after picking it. Likewise $QSD(s)$ should be non-negative; it could either be restricted, or it may be thought better to simulate either $QSD(s)$ or the variance, $QV(s) = QSD(s)^2$, lognormally, since a lognormal distribution is very close to a normal if the standard deviation of the latter is small relative to the mean. Similar modifications and limitations might be needed throughout.

11.4.6 The results in Wilkie (1986b) show that the standard deviations and hence the extreme quantiles of all the variables are increased by this method, though by how much would depend on the variable and on the parameters of the hypermodel. The question deserves fuller investigation.

11.5 *Model Uncertainty*

11.5.1 Not only do we not know the true values of the parameters of any model, we do not even know whether any model is itself the correct one. We are not in a world of Newtonian mechanics, where the true model (at least in the sublunary world) has been well known for some centuries. Instead there are competing models: there are those based on the concepts of efficient markets, such as the random walk model and that of Smith (1996); there are those based on econometric ideas, such as the Wilkie model and others of the same style, like Thomson (1996), Yakoubov, Teeger & Duval (1999) and Whitton & Thomas (2000); there are those that try to bridge the gap between these two camps, such as Cairns (1999). Each has its adherents, and none is authoritative.

11.5.2 One way of reflecting the uncertainty about the true model is to use each of a number of alternative models as the real world model, and compare the results. The prudent actuary might then take the highest contingency reserves; the most optimistic might take the lowest. An alternative would be to carry out each of the simulations using a model chosen at random for that simulation. The question would then be how to weight the models and give them probabilities in the first place. We have no solution to this, and can only suggest that each actuary uses their subjective prejudices, but then explains what they have done.

11.6 *Conclusion*

11.6.1 In conclusion, we consider that each of these extensions to the basic model is both practicable and worth doing. On the other hand, if caution is put into every aspect of the basis, and if marking to market each year is practised, then a very high initial probability of solvency, the α of Section 2.3, may not be required. It is better to assess the position on as realistic a basis as is possible (and practicable), rather than just to increase some of the standard deviations in the basis arbitrarily, or to use a weak basis and contingency reserves set up on a high probability level.

12. Summary and Conclusions

12.1 In this paper we have presented, first, a system of calculating contingency reserves for guaranteed annuity options based on quantile or conditional tail expectation reserves, and, secondly, a system based similarly on quantile or CTE reserves after taking into account the possibility of dynamic hedging according to option pricing theory. We then indicated a variety of extensions that would make either of these methods more realistic.

12.2 We also showed how the required reserves would have been quite small in 1985, but with 'marking-to-market' would have increased during the rest of the 1980s and the 1990s, so that offices that had written such GAOs would have built up funds which were sufficient to pay the required benefits and they would not have been caught unexpectedly. GAOs would still have been costly, but the costs would have been recognised more in advance. In addition, the fact that larger reserves were needed than many offices seem to have held would have alerted them to the dangers of writing policies with GAOS, and they might have stopped offering such benefits sooner than they did, and might have made more realistic charges to policyholders who wished such an option, rather than giving away a valuable benefit apparently free.

12.3 Several lessons can be learned from this paper. First, the methodology that we have used in Sections 2 to 7 of the paper was all known and publicly available by 1985. Further, Guidance Note 8, first issued in October 1983, contained a relevant paragraph; we quote the version from the Faculty of Actuaries Year Book 1984-85; essentially the same paragraph still exists:

> 4.2.2: The prudent assumptions on which the reserve under Part VI must be calculated will naturally allow for stochastic variations as well as other contingencies. In determining the extent to which the actuary would consider it prudent to make provision for the more extreme stochastic variations in valuing particular categories of contact (for example in relation to mortality and morbidity fluctuations, and variations in benefits resulting from the inclusion in a unit-linked contract of a maturity guarantee) he may reasonably take into account the basis of the solvency margin that the company is required to hold on account of the liabilities under those contracts (net of the permitted deduction for reinsurance cessions). ... '.

12.4 It is reasonable then for those outside the group of Appointed Actuaries to ask why few or no offices had either charged or reserved for these options (see the survey reported by Bolton *et al.*, 1997), and why the then supervisors did not ask offices whether they had set up the required reserves. Of course, the supervisors may not have known that such benefits were being offered, but one wonders whether they have any obligation to be aware of what is on the market, and some Appointed Actuaries may not have been aware that the policies of the office they were advising contained such benefits.

12.5 Several other conclusions can be drawn from our investigations, which we list below

12.6 CTEs are preferable to quantile reserves, as we explain in Section 9.

12.7 Marking-to-market seems preferable to the MGWP's system of adjusting reserves only when they go outside specific bands, but this requires further discussion.

12.8 Calculating a theoretical option price is not enough; one must also test out empirically how well the hedging strategy might perform, particularly taking into account transaction costs and the possibility that either the parameters or the nature of the real world model are not the same as those used for calculating option prices and hedging proportions. One needs then to set up contingency reserves to allow for all the possible hedging errors. Further, the option price is only a guide to the 'fair value' if the required hedging strategy can practicably be followed and then is actually followed.

12.9 Some further comments on fair value may be helpful. There are three values that have entered our calculations:

— the mean value of the benefit, A in Section 2.3, and including any option premium (and also including the basic unit liability);
— the charge to the policyholder, C of Section 2.5 or D of Section 8.2;
— the quantile or CTE reserve, Q or T.

In our view these three concepts are relevant to all insurance contracts.

12.10 The mean value is in some sense the 'best estimate' of the liability, but it is not enough. As we understand it, the 'fair value' is meant to be the value at which a liability could be transferred between willing parties. In that case the charge to the policyholder (adjusted for expenses as appropriate) is the right value. If the liability were to be transferred between life offices, a willing acquirer of the liability would need to set up contingency reserves just as the transferring office has done, and would require an appropriate return on those reserves. The transferring office no longer needs the contingency reserves, so should not expect to receive any further reward because they are no longer at risk.

12.11 The amount of the contingency reserves minus the expected value ('Q-A' or 'T-A') 'belongs' to the shareholders, and not to the policyholders, but it is needed to support the risks that they have underwritten, and cannot be released to them until those risks have expired; at that time the amounts due to the individual policies, which may be greater or less than the expected amount, are paid (or transferred to the offices' annuity account) and the balance can be released. The contingency reserves are, in our view, as in GN8, quoted in ¶12.3, a substitute for, not an addition to, the traditional statutory solvency margin. If these concepts are understood it may help any accounting principles that are introduced.

12.12 The work on GAOs does not finish here. Further investigations and discussion could usefully be undertaken, on the following lines:

transaction costs could be included both for the basic policy and for hedging;

— fat-tailed innovations could be used for the annual model (Section 11.2);
— fat-tailed innovations could be used for the bridges (Section 11.2);
— stochastic mortality could be introduced (Section 11.3);
— a hypermodel to allow for parameter uncertainty could be used (see Section 11.4);
— alternative stochastic models could be used (Section 11.5);
— whether our suggestion that the success of hedging depends only on the bridging model, and not at all on the annual model should be investigated (Section 10.3);
— realistic portfolios could be investigated; however, it might be desirable to try to model these with a number of representative policies ('model points') rather than with the full portfolio, which might take an inordinate amount of computer time, even nowadays;
— consideration of the desirable values to use for α and h (Section 3.2);
— consideration of an appropriate method for charging for individual policies as part of a portfolio (Sections 3.3 and 9);
— consideration of how frequently marking to market should take place, and what a suitable value of α is with different frequencies.

We hope that this paper stimulates further discussion and research on these lines.

12.13 Just as our paper was being competed we received a draft of a paper by Boyle and Hardy (2002), which covers the same subject as this paper, in a rather similar way. We look forward to exchanging ideas further with them.

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APPENDIX A

INVESTMENT DATA

Table A.1. Market indices at 31 December of year shown; B(*t*) is required only for the 1995 Wilkie model; the other items are required for both the 1984 and the 1995 models

31 December	Q(*t*)	I(*t*)	Y(*t*) %	D(*t*)	C(*t*) %	B(*t*) %
1982	82.51	0.0527	5.26	20.10	10.25	10.00
1983	86.89	0.0518	4.62	21.74	9.71	9.00
1984	90.87	0.0448	4.42	26.21	9.90	9.50
1985	96.05	0.0553	4.33	29.57	9.80	11.50
1986	99.62	0.0365	4.04	33.75	10.06	11.00
1987	103.30	0.0363	4.32	37.59	9.21	8.50
1988	110.30	0.0656	4.71	43.64	8.99	13.00
1989	118.80	0.0742	4.24	51.08	9.66	15.00
1990	129.90	0.0893	5.47	56.46	10.48	14.00
1991	135.70	0.0437	5.02	59.62	9.71	10.50
1992	139.20	0.0255	4.35	59.32	8.83	7.00
1993	141.90	0.0192	3.37	56.69	6.52	5.50
1994	146.00	0.0285	4.02	61.16	8.53	6.25
1995	150.70	0.0317	3.80	68.52	7.78	6.50
1996	154.40	0.0243	3.74	75.31	7.74	6.00
1997	160.00	0.0356	3.23	77.88	6.39	7.25
1998	164.40	0.0271	2.92	78.08	4.55	6.25
1999	167.30	0.0175	2.36*	76.37	4.89	5.50
2000	172.20	0.0289	2.48*	73.93	4.62	6.00
2001	173.40	0.0069	2.92*	73.75	5.04	4.00

* gross yield calculated by grossing up 'actual yield' by 1/0.9.

Table A.2. Derived initial conditions for 1984 and 1995 Wilkie models at 31 December of year shown

	1984 model				1995 model			
	YE(t)	DM(t)	DE(t)	CM(t)	YE(t)	DM(t)	DE(t)	CM(t)
1982	0.0627	0.1087	0.0335	0.0694	0.0788	0.1033	0.0283	0.0694
1983	−0.0474	0.0973	−0.0102	0.0686	−0.0476	0.0966	−0.0180	0.0686
1984	−0.0051	0.0868	0.1030	0.0675	−0.0087	0.0899	0.1021	0.0675
1985	−0.0191	0.0805	0.0056	0.0670	−0.0309	0.0854	−0.0277	0.0670
1986	−0.0421	0.0717	0.0617	0.0656	−0.0446	0.0790	0.0655	0.0656
1987	0.0516	0.0646	0.0172	0.0643	0.0424	0.0735	−0.0112	0.0643
1988	0.0581	0.0648	0.0881	0.0644	0.0390	0.0724	0.0775	0.0644
1989	−0.0869	0.0667	0.0677	0.0648	−0.1003	0.0727	0.0307	0.0648
1990	0.2176	0.0712	−0.0174	0.0659	0.1936	0.0748	−0.0317	0.0659
1991	0.0527	0.0657	0.0432	0.0649	0.0648	0.0708	0.0310	0.0649
1992	−0.0514	0.0577	−0.0619	0.0631	−0.0436	0.0649	−0.0757	0.0631
1993	−0.2270	0.0500	−0.0763	0.0612	−0.2269	0.0590	−0.0682	0.0612
1994	0.0849	0.0457	0.0169	0.0597	0.0670	0.0550	0.0153	0.0597
1995	−0.0740	0.0429	0.0836	0.0584	−0.0829	0.0520	0.0571	0.0584
1996	−0.0435	0.0392	0.0122	0.0569	−0.0513	0.0484	−0.0068	0.0569
1997	−0.2019	0.0384	−0.0177	0.0559	−0.2170	0.0467	−0.0297	0.0559
1998	−0.1942	0.0362	−0.0655	0.0546	−0.2107	0.0442	−0.0715	0.0546
1999	−0.3423	0.0324	−0.0659	0.0530	−0.3610	0.0407	−0.0652	0.0530
2000	−0.1860	0.0317	−0.1073	0.0519	−0.2223	0.0392	−0.1093	0.0519
2001	−0.0126	0.0268	−0.0222	0.0499	−0.0345	0.0350	−0.0183	0.0499

APPENDIX B

A YIELD CURVE FOR THE WILKIE MODEL

B.1 The Wilkie model as published in 1984 had only a yield for 'consols' (by which term is meant a true perpetuity), denoted $C(t)$, so nothing other then a level yield curve could be derived. In 1995 a model for 'base rate' (by which term is meant either an immediate short-term interest rate, or a one-year rate, as required by circumstances), denoted $B(t)$, was added. This could give two ends of a yield curve. However, no model for the intermediate values was derived. Many other interest models have been published, but few are 'integrated', in the sense of including consistent long-term models for share prices and inflation as well as interest rates, such as we need in this investigation, though the models of Smith (1996) and Cairns (2000) could be used for this purpose. For the calculation of annuity values it is convenient to have a full yield curve available, and in this Appendix we describe how we construct a full yield curve model based on the $C(t)$ and $B(t)$ of the Wilkie model.

B.2 The yield curve used for the FTSE Actuaries British Government Securities (BGS) Indices (see Dobbie & Wilkie, 1978) described redemption yields on issued coupon stocks, and was modelled by the formula:

$$Y(t,n) = A(t) + B(t).\exp(-C(t).n) + D(t).\exp(-F(t).n),$$

where $Y(t,n)$ is the redemption yield at time (day) t on a stock of term n and $A(t)$, $B(t)$, $C(t)$, $D(t)$ and $F(t)$ are constants which vary from day to day. Such a curve starts at $n = 0$ with value $A(t) + B(t) + D(t)$, and as $n \to \infty$ it tends to a value of $A(t)$. In between there can be one minimum or maximum value. The yield curve for Debenture and Unsecured Loan Stocks, published from 31 December 1977 to 30 December 1994, was simpler (see Wilkie, 1985), with only one exponential term:

$$Y(t,n) = A(t) + B(t).\exp(-C(t).n).$$

In the absence of data on which to build a more complex model, we have chosen to use the simpler formula.

B.3 The yield curve described above is fitted to the redemption yields of redeemable (and 'irredeemable') coupon stocks. To calculate annuity values we require the rates on zero-coupon stocks. We start by assuming that the redemption yield curve is in fact one for stocks standing at 'par', with a price of 100, so that the redemption yield equals the coupon. We also assume coupons payable annually in arrears (although interest on real BGS are almost all paid half-yearly). We use Wilkie's $B(t)$ and $C(t)$ as the end points, and put:

$$Y(t, n) = C(t) + (B(t) - C(t)).\exp(-\beta.n),$$

where $Y(t, n)$ is the par yield at time t for term n, and β is a constant, not changing from year to year. We then derive the zero-coupon rates, at annual intervals, recursively, as follows:

Let $v(t, n)$ be the value at time (year) t of a zero-coupon stock of term n.

Then the value of a coupon stock of term n, standing at a par of unity, with coupon $Y(t, n)$, and redeemable at par, can be derived as:

$$1 = Y(t, n).\Sigma_{m=1,n} v(t, m) + 1.v(t, n).$$

B.4 We start at term $n = 1$, where we have:

$$1 = Y(t, 1).\Sigma_{m=1,1} v(t, m) + 1.v(t, 1)$$

whence $v(t, 1) = 1/\{Y(t, 1) + 1\}$.
We continue, year by year:

$$1 = Y(t, n).\Sigma_{m=1,n-1} v(t, m) + \{1 + Y(t, n)\}.v(t, n),$$

whence $v(t, n) = \{1 - Y(t, n).\Sigma_{m=1,n-1} v(t, m)\}/\{1 + Y(t, n)\}$.

B.5 From the values of $v(t, n)$ we can derive, if we wish, a zero-coupon yield curve:

$$Z(t, n) = 1/v(t, n)^{1/n} - 1,$$

and the forward rate discount factors and yields:

$$f(t, n) = v(t, n + 1)/v(t, n)$$

and

$$F(t, n) = 1/f(t, n) - 1.$$

B.6 A problem about this approach, which would apply whatever form is used for a par yield curve, is that, when calculating the zero-coupon discount factor $v(t, n)$, the sum of the coupons from years 1 to $n - 1$, $Y(t, n).\Sigma_{m=1,n-1} v(t, m)$, may exceed unity, so that the calculated value of the zero-coupon discount factor, $v(t, n)$, is negative. This is unsatisfactory. It happens when, for longer terms, the par yield curve is still rising noticeably, and this happens when, with our formula, the value of β is too low, for the particular values of $B(t)$ and $C(t)$. We have therefore had to choose a value of

β that is large enough for this anomaly to not happen, at least within the first 55 years (allowing for an annuity for a male aged 65 being calculated with a terminal age of 120), for the eight sets of initial conditions we have used (from December 1994 to December 2001) and the 10,000 simulations we have used. We find that a value of $\beta = 0.39$ is large enough, but that a value of 0.38 is not large enough. We have therefore used the former value, $\beta = 0.39$, throughout. It is possible, of course, that this value might not be large enough for a different set of initial conditions, or for further simulations. Indeed Yang (2001) used a value of β of 0.5. However, because a high value of β produces a very flat yield curve, rather little different from using a constant interest rate of C(t), as we have done for the 1984 Wilkie model, we have used the lowest value that did not give us inconsistencies.

B.7 A further inconsistency that could occur, but does not seem to happen except in conjunction with the first inconsistency in our experiments, is that the value of the forward discount factor, $f(t, n)$, could be greater than unity, so that the forward interest rate, $F(t, n)$, is negative. This need not upset the calculation of an annuity value in the form $\Sigma_{m=1,\omega} \cdot {}_mp_{65} . v(t, m)$, although it is still theoretically undesirable.

APPENDIX C

GAOS AND QUANTO OPTIONS

C.1 *Introduction*

C.1.1 A Guaranteed Annuity Option in the form that we have described is analogous to a 'quanto option', though it is not the same as the 'typical' quanto option. Quanto options seem to get rather slight reference in the main textbooks, though what is there is helpful. Thus quantos are not referred to at all in the Second Edition of Hull (1993), but are discussed briefly in the Third Edition (1997) and Fourth Edition (2000). There is no reference in Rebonato (1998). Baxter & Rennie (1996) discuss the typical quanto clearly, but not comprehensively.

C.2 *Maxis and Minis*

C.2.1 It is first useful to define some new terms. The normal or 'vanilla' option is a *call* option, 'on A with B', allowing the holder to obtain A at time T, paying for it with B. Thus its value, allowing the values of both A and B to vary with time, is Max(0, A(T) − B(T)). The corresponding *put* option, 'of A on B', has value Max(0, B(T) − A(T)). It is clear that a put option of A on B is the same as a call option on B with A, and in the foreign currency options market there are only call options, not puts.

C.2.2 What insurance companies often provide is a 'guarantee', which we describe as a '*maxi*' option. It pays Max(A(T), B(T)) at time T. We consider only European options, with fixed T, but the notation would work for American options too. The complement of a maxi option is a '*mini*' option, that pays Min(A(T), B(T)). A maxi plus a mini are the same as A plus B. Since the holder of the maxi gets an advantage, and the holder of a mini a disadvantage, the maxi should cost more than Max(A(0), B(0)) and the mini should cost less than Min(A(0), B(0)). They may be designed so that A(0) = B(0), but they need not be. A GAO is a maxi option, as is a 'maturity guarantee' on a unit-linked policy.

C.2.3 A maxi is also the same as B plus a call on A with B, which in turn is the same as A plus a put of A on B. This is just the 'put-call parity' formula, often expressed as:

$$cash + call = share + put$$

but we can also add: $= maxi$.

C.2.4 Likewise $cash + share = maxi + mini$, so we get:

$$mini = share - call = cash - put.$$

These formulae allow us to get the value of any such option from the value of any other.

C.3 *Quanto Options*

C.3.1 Now quantos. The 'typical' quanto option is expressed in terms of a share price and two currencies. Consider the Nikkei index, with value $S(t)$, and an exchange rate: $\$1 = ¥F(t)$. The Nikkei index is denominated in yen (¥), but the option pretends that it is denominated in dollars. Thus a quanto call option on the Nikkei pays $\$\text{Max}(0, S(T) - K)$. This is equivalent to $¥\text{Max}(0, S(T) - K) \times F(T)$. But this is not the only sort of quanto option.

C.3.2 Baxter & Rennie (1996) define a quanto forward as paying $\$S(T)$ at time T, equivalent to $¥S(T) \times F(T)$, and show how this can be hedged and hence valued. But they show that it is only $S(T) \times F(T)$ at a fixed T that can be hedged exactly, not $S(t) \times F(t)$ for general t. They also give the value of a quanto option that pays $\$\text{Max}(0, S(T) - K)$ as above.

C.3.3 The corresponding maxi quanto option would pay $\$\text{Max}(S(T), K)$, or more generally $\$\text{Max}(S(T), K(T))$, or $¥\text{Max}(S(T), K(T)) \times F(T)$.

C.3.4 There is, however, another type of quanto option, which is what one might like for hedging a foreign currency investment. This is a maxi option on the currency for a quantum determined by the value of shares, which would pay: $¥S(T) \times \text{Max}(K, F(T))$, or $\$S(T) \times \text{Max}(K/F(T), 1)$. Thus if the yen has risen against the dollar (F(T) has fallen), one gets the advantage of $K/F(T)$, whereas if the yen has fallen (more yen to the dollar, so F(T) is bigger), one converts at 'par'. The whole contract would need to take into account S(0), F(0) and the amount of dollars or yen originally invested.

C.3.5 We call the original quanto option a Type A quanto option, and the one just defined as a type B quanto option. They are symmetrical, in that a type A maxi can be expressed as paying $¥F(T) \times \text{Max}(S(T), K_A(T))$ and a type B as paying $¥S(T) \times \text{Max}(F(T), K_B(T))$. The 'K(T)s' are probably different so we have denoted them as K_A and K_B, but they play the same role, and might well be constants, not functions of T.

C.3.6 Baxter & Rennie (1996) give formulae for the Type A option. It is plausible that the Type B option can be valued with the same formula, *mutatis mutandis*, but this would need proof. In fact our GAO model is a little different from this 'vanilla' quanto.

C.4 *Guaranteed Annuity Options*

C.4.1 Guaranteed annuity options are similar to type B quanto options, in that the proceeds of some investment at some retirement date, say S(T), can be taken either as 'cash' (strictly cash which must be applied to purchase a life annuity at market rates), or for the purchase of an annuity at time T. This value depends on the guaranteed annuity rate, the age of the assured, the conditions of payment of the annuity, the mortality table assumed at time T and, most importantly, the market interest rates at time T. It can be expressed as $V(T) = g \times a(T)$ where g is the guaranteed rate (e.g. £111 per year per £1,000 cash purchase price, so $g = 0.111$) and $a(T)$ is the value of a suitable annuity (of unit amount) at time T. The cash amount would correspondingly buy an annuity at a market rate of $1/a(T)$. Thus at time T the GAO pays £$S(T) \times \mathrm{Max}(1, V(T))$ = £$S(T) \times \mathrm{Max}(1, g \times a(T))$.

C.4.2 Prior to time T, at say time t, the annuity could be purchased, either as a 'deferred life annuity', being paid for immediately at time t, or as a 'forward life annuity', at a price which is determined at time t, but with payment at time T. 'Cash' at time T can likewise be purchased in advance, at a price $B(t, T)$, because it is just a zero-coupon bond (zcb) maturing at T. The price of the deferred annuity, $D(t, T)$ equals the price of the forward life annuity times the zcb price, $D(t, T) = B(t, T) \times F(t, T)$.

C.5 *Fundamentals*

C.5.1 We now derive a formula for the GAO. We follow closely the methods of Baxter & Rennie (1996), using the principles of the Black (1976) model for the option.

C.5.2 We start by considering what the tradeable assets are. We have assumed a unit fund, which we refer to as a 'share' without specifying how it is invested. The market price per unit at time t is $S(t)$, and at maturity (the retirement date), T, it has a value $S(T)$. Next we assume a zero-coupon bond, a '*zcb*', which pays 1 at time T, and prior to that has value $B(t, T)$ or just $B(t)$.

C.5.3 We also assume that we can invest in a deferred life annuity, a '*dla*', whose value at time t is $D(t, T)$ or $D(t)$, and which provides a 'life annuity' at time T, of annual rate 1, of the required type. We can relate the price of the deferred annuity to the price of a forward life annuity, an '*fla*', (which is a derivative contract, not directly investible), whose price at time t, $F(t, T)$ or $F(t)$, is related by: $D(t) = F(t) \times B(t)$. Note that the dla and fla are not annuities on individual lives, but instead are annuities certain, with the payments reducing proportionately to ${}_tp_{65}$ on some prescribed life table. Such an annuity should match the expected payments on a portfolio of annuities. However they will not necessarily meet the actual payments on such a portfolio, because of (a) the random individual lifetimes of the group of insured and (b) the overall mortality rates of the class of insured, which may be different from what was expected originally. We do not consider these problems in this Appendix.

C.5.4 We now consider the Brownian motions 'driving' the prices. We assume three separate Brownian Motions, W_1, W_2 and W_3. W_i and W_j have instantaneous correlation ρ_{ij}. The W_is are related to three *independent* Brownian motions, Z_1, Z_2 and Z_3 through a matrix, C:

$$dW = C.dZ$$

where we shall choose the values for the elements of C appropriately in due course.

C.5.5 We assume that the share price, S(t), is driven by the stochastic differential equation:

$$dS(t) = \mu_S().S(t).dt + \sigma_S.S(t).dW_3,$$

where σ_S is a constant, and $\mu_S()$ is some function of t and S(t), to be defined later. We could have $\mu_S() = 0$ (a random walk with no drift), or $\mu_S() = \mu_S$ (a random walk with constant drift), or $\mu_S() = \alpha_S(\theta_S - S(t))$ (an Ornstein-Uhlenbeck process, analogous to a first order discrete autoregressive, or AR(1), process), or some combination of these. For shares in practice we shall choose $\mu_S() = \mu_S$.

C.5.6 We let the zcb price, B(t), be driven by the zcb interest rate, R(t), which has the stochastic differential equation:

$$dR(t) = \mu_R().dt + \sigma_R.dW_2,$$

where σ_R is a constant, and $\mu_R()$ is some function of t and R(t), similar to $\mu_S()$. In practice we shall choose $\mu_R() = \alpha_R(\theta_R - R(t))$, an Ornstein-Uhlenbeck process. This is similar to the Vasicek (1977) model, but this is usually applied to the spot rate, rather than to a zcb rate for constant maturity date. The zcb price, B(t), is related to the zcb interest rate, R(t), by:

$$B(t) = \exp(-(T - t).R(t)).$$

The differential equation for B(t) will be derived below from that for R(t) through Ito's formula.

C.5.7 The dla price, D(t) = F(t) × B(t), is driven by the zcb price, B(t), and the fla price, F(t). We assume that F(t) has stochastic differential equation:

$$dF(t) = \mu_F().F(t).dt + \sigma_F.F(t).dW_1,$$

where σ_F is a constant, and $\mu_F()$ is some function of t and F(t), similar to $\mu_S()$. F(t) follows a form of logarithmic Brownian motion. In practice we shall choose an Ornstein-Uhlenbeck process for log F(t), the details of which are given in Section C.11.

C.6 *Ito's Formula*

C.6.1 We now introduce Ito's formula (Baxter & Rennie, 1996, page 59, but extended) in order to get formulae for the differential equations of 1/B(t) and F(t). If Y(t) = f(X(t)), and $dX(t) = \mu_X().dt + \sigma_X.dW$, then:

$$dY(t) = f'().dX + \tfrac{1}{2}\sigma_X^2.f''().dt + \partial Y/\partial t.dt$$

where $f'() = \partial Y/\partial X$ and $f''() = \partial^2 Y/\partial X^2$.

C.6.2 We first put $C(t) = 1/B(t)$, and express it as a function of $R(t)$:

$$C(t) = 1/B(t) = \exp((T - t).R(t)).$$

so:

$$f'() = \partial C/\partial R = (T - t).\exp((T - t).R(t)) = (T - t).C(t)$$

and:

$$f''() = \partial^2 C/\partial R^2 = (T - t)^2.\exp((T - t).R(t)) = (T - t)^2.C(t)$$

and also:

$$\partial C/\partial t = -R(t).\exp((T - t).R(t)) = -R(t).C(t).$$

We also have:

$$dR(t) = \mu_R().dt + \sigma_R.dW_2.$$

Hence:

$$\begin{aligned} dC(t) &= (T - t).C(t).dR + \tfrac{1}{2}\sigma_R^2.(T - t)^2.C(t).dt - R(t).C(t).dt \\ &= (T - t).C(t).(\mu_R().dt + \sigma_R.dW_2) + \tfrac{1}{2}\sigma_R^2.(T - t)^2.C(t).dt - R(t).C(t).dt \\ &= C(t).\{(T - t).\mu_R() + \tfrac{1}{2}\sigma_R^2.(T - t)^2 - R(t)\}.dt + (T - t).C(t).\sigma_R.dW_2 \end{aligned}$$

C.7 *Change of Numeraire*

C.7.1 It is convenient to choose a *numeraire* other than pounds, and for this we choose the zcb. The value of the dla relative to the zcb is $D(t)/B(t) = F(t)$. The value of the share relative to the zcb is $S(t)/B(t) = S(t).C(t)$, which we shall denote as $H(t)$. The value of the zcb relative to itself is, of course, unity.

C.7.2 We now need the stochastic differential equation for $H(t)$. We get this from the product rule (Baxter & Rennie, 1996, page 62):

$$d(XY) = X.dY + Y.dX + \rho_{XY}.\sigma_X.\sigma_Y.dt$$

where ρ_{XY} is the instantaneous correlation coefficient between X and Y, which we have to derive from the component parts of the stochastic differential equations. We have:

$$\begin{aligned} &H(t) = S(t).C(t) \\ &\quad dS(t) = \mu_S().S(t).dt + \sigma_S.S(t).dW_3 \\ &\quad dC(t) = C(t).\{(T - t).\mu_R() + \tfrac{1}{2}\sigma_R^2.(T - t)^2 - R(t)\}.dt + (T - t).C(t).\sigma_R.dW_2 \end{aligned}$$

C.7.3 It is now convenient to define the independent Brownian motions, Z_i. We start by putting:

$$dW_1 = dZ_1$$

so that $c_{11} = 1, c_{12} = 0, c_{13} = 0$, and then:

$$dW_2 = c_{21}dZ_1 + c_{22}.dZ_2$$

where $c_{21} = \rho_{12}$ and $c_{22}^2 = (1 - c_{21}^2)$, so that $E[dW_2] = 1$ and $\mathrm{Corr}(dW_1, dW_2) = \rho_{12}$, and so that $c_{23} = 0$, and finally:

$$dW_3 = c_{31}.dZ_1 + c_{32}.dZ_2 + c_{33}.dZ_3$$

where $c_{31} = \rho_{13}, c_{32} = (\rho_{23} - c_{21}.c_{31})/c_{22}$ and $c_{33}^2 = (1 - c_{31}^2 - c_{32}^2)$, so that $E[dW_3] = 1$, $\mathrm{Corr}(dW_1, dW_3) = \rho_{13}$ and $\mathrm{Corr}(dW_2, dW_3) = \rho_{23}$. C is the Choleski decomposition of the correlation matrix.

C.7.4 We now re-express $dS(t)$ and $dC(t)$ in terms of the Zs:

$$\begin{aligned}
dS(t) &= \mu_S().S(t).dt + \sigma_S.S(t).(c_{31}.dZ_1 + c_{32}.dZ_2 + c_{33}.dZ_3) \\
dC(t) &= C(t).\{(T-t).\mu_R() + \tfrac{1}{2}\sigma_R^2.(T-t)^2 - R(t)\}.dt \\
&\quad + (T-t).C(t).\sigma_R(c_{21}.dZ_1 + c_{22}.dZ_2)
\end{aligned}$$

C.7.5 We now have:

$$\begin{aligned}
dH(t) &= d(S(t).C(t)) \\
&= S(t).dC(t) + C(t).dS(t) \\
&\quad + \{\sigma_S.S(t).c_{31}.(T-t).C(t).\sigma_R.c_{21} + \sigma_S.S(t).c_{32}.(T-t).C(t).\sigma_R.c_{22}\}.dt \\
&= S(t).\{C(t)\{(T-t)\mu_R() + \tfrac{1}{2}\sigma_R^2(T-t)^2 - R(t)\}dt \\
&\quad + (T-t).C(t).\sigma_R(c_{21}.dZ_1 + c_{22}.dZ_2)\} \\
&\quad + C(t).\{\mu_S().S(t).dt + \sigma_S.S(t).(c_{31}.dZ_1 + c_{32}.dZ_2 + c_{33}.dZ_3)\} \\
&\quad + \{\sigma_S.S(t).c_{31}.(T-t).C(t).\sigma_R.c_{21} + \sigma_S.S(t).c_{32}.(T-t).C(t).\sigma_R.c_{22}\}.dt \\
&= S(t).C(t)[(T-t).\mu_R() + \mu_S() + \tfrac{1}{2}\sigma_R^2(T-t)^2 - R(t) \\
&\quad + (T-t).\sigma_S.\sigma_R.(c_{31}.c_{21} + c_{32}.c_{22})].dt \\
&\quad + S(t).C(t).[\{(T-t).\sigma_R.c_{21} + \sigma_S.c_{31}\}.dZ_1 \\
&\quad + \{(T-t).\sigma_R.c_{22} + \sigma_S.c_{32}\}.dZ_2 + \sigma_S.c_{33}.dZ_3] \\
&= H(t)[(T-t).\mu_R() + \mu_S() + \tfrac{1}{2}\sigma_R^2(T-t)^2 - R(t) + (T-t).\sigma_S.\sigma_R.\rho_{23}].dt \\
&\quad + H(t).[\{(T-t)\sigma_R c_{21} + \sigma_S c_{31}\}dZ_1 + \{(T-t)\sigma_R c_{22} + \sigma_S c_{32}\}.dZ_2 \\
&\quad + \sigma_S c_{33}.dZ_3]
\end{aligned}$$

C.8 *Equivalent Martingales*

C.8.1 To calculate the value of the option we need to change the stochastic differential equations so that the process for each of the tradeables (relative to the new numeraire) is a martingale. We do this by adjusting the Zs to new values $Z_i^*(t) = Z_i(t) + \int \gamma_i(t).dt$ or

$$dZ_i^* = dZ_i + \gamma_i(t)$$

in such a way that the tradeables are martingales. We denote adjusted values by an asterisk.

C.8.2 The zcb is easy, since its value relative to the new numeraire is always 1.

C.8.3 The value of the dla, relative to the new numeraire, is $F(t)$ with stochastic differential equation:

$$\begin{aligned} dF(t) &= \mu_F(\,).F(t).dt + \sigma_F.F(t).dW_1 \\ &= \mu_F(\,).F(t).dt + \sigma_F.F(t).dZ_1 \end{aligned}$$

If we put

$$\begin{aligned} dZ_1^* &= dZ_1 + \gamma_1(t)dt \quad \text{with:} \\ \gamma_1(t) &= \{\mu_F(\,).F(t)\}/\{\sigma_F.F(t)\} \\ &= \mu_F()/\sigma_F \end{aligned}$$

we get:

$$dF^*(t) = \sigma_F.F^*(t).dZ_1^*$$

and $F^*(t)$ is a martingale, since it has no drift (dt) term.

C.8.4 The value of the share, relative to the new numeraire, is $H(t)$ with stochastic differential equation:

$$\begin{aligned} dH(t) = {} & H(t).[(T-t).\mu_R(\,) + \mu_S(\,) + \tfrac{1}{2}\sigma_R^2(T-t)^2 - R(t) + (T-t).\sigma_S.\sigma_R.\rho_{23}].dt \\ & + H(t).[\{(T-t).\sigma_R.c_{21} + \sigma_S.c_{31}\}.dZ_1 + \{(T-t).\sigma_R.c_{22} + \sigma_S.c_{32}\}.dZ_2 \\ & + \sigma_S.c_{33}.dZ_3] \end{aligned}$$

We first replace dZ_1 by $dZ_1^* - \gamma_1(t)$ to give dH(t) (partially adjusted)

$$
\begin{aligned}
&= \mathrm{H}(t).[(\mathrm{T}-t).\mu_{\mathrm{R}}() + \mu_{\mathrm{S}}() + \tfrac{1}{2}\sigma_{\mathrm{R}}^2(\mathrm{T}-t)^2 - \mathrm{R}(t) + (\mathrm{T}-t).\sigma_{\mathrm{S}}.\sigma_{\mathrm{R}}.\rho_{23}].\mathrm{d}t \\
&\quad + \mathrm{H}(t).[\{(\mathrm{T}-t).\sigma_{\mathrm{R}}.c_{21} + \sigma_{\mathrm{S}}.c_{31}\}\{\mathrm{dZ}_1^* - \gamma_1(t)\} \\
&\quad + \{(\mathrm{T}-t).\sigma_{\mathrm{R}}.c_{22} + \sigma_{\mathrm{S}}.c_{32}\}.\mathrm{dZ}_2 + \sigma_{\mathrm{S}}.c_{33}.\mathrm{dZ}_3] \\
&= \mathrm{H}(t).[(\mathrm{T}-t).\mu_{\mathrm{R}}() + \mu_{\mathrm{S}}() + \tfrac{1}{2}\sigma_{\mathrm{R}}^2(\mathrm{T}-t)^2 - \mathrm{R}(t) + (\mathrm{T}-t).\sigma_{\mathrm{S}}.\sigma_{\mathrm{R}}.\rho_{23}].\mathrm{d}t \\
&\quad + \mathrm{H}(t).[\{(\mathrm{T}-t).\sigma_{\mathrm{R}}.c_{21} + \sigma_{\mathrm{S}}.c_{31}\}.\{\mathrm{dZ}_1^* - \mu_{\mathrm{F}}()/\sigma_{\mathrm{F}}.\mathrm{d}t\} \\
&\quad + \{(\mathrm{T}-t).\sigma_{\mathrm{R}}.c_{22} + \sigma_{\mathrm{S}}.c_{32}\}.\mathrm{dZ}_2 + \sigma_{\mathrm{S}}.c_{33}.\mathrm{dZ}_3] \\
&= \mathrm{H}(t).[(\mathrm{T}-t).\mu_{\mathrm{R}}() + \mu_{\mathrm{S}}() + \tfrac{1}{2}\sigma_{\mathrm{R}}^2(\mathrm{T}-t)^2 - \mathrm{R}(t) + (\mathrm{T}-t).\sigma_{\mathrm{S}}.\sigma_{\mathrm{R}}.\rho_{23} \\
&\quad - \{(\mathrm{T}-t).\sigma_{\mathrm{R}}.c_{21} + \sigma_{\mathrm{S}}.c_{31}\}.\mu_{\mathrm{F}}()/\sigma_{\mathrm{F}}].\mathrm{d}t \\
&\quad + \mathrm{H}(t).[\{(\mathrm{T}-t).\sigma_{\mathrm{R}}.c_{21} + \sigma_{\mathrm{S}}.c_{31}\}.\mathrm{dZ}_1^* \\
&\quad + \{(\mathrm{T}-t).\sigma_{\mathrm{R}}.c_{22} + \sigma_{\mathrm{S}}.c_{32}\}.\mathrm{dZ}_2 + \sigma_{\mathrm{S}}.c_{33}.\mathrm{dZ}_3]
\end{aligned}
$$

C.8.5 We then define a new Brownian motion, Z_4, with:

$$\sigma_4(t).\mathrm{dZ}_4 = \{(\mathrm{T}-t).\sigma_{\mathrm{R}}.c_{22} + \sigma_{\mathrm{S}}.c_{32}\}.\mathrm{dZ}_2 + \sigma_{\mathrm{S}}.c_{33}.\mathrm{dZ}_3$$

and:

$$
\begin{aligned}
\sigma_4(t)^2 &= \{(\mathrm{T}-t).\sigma_{\mathrm{R}}.c_{22} + \sigma_{\mathrm{S}}.c_{32}\}^2 + \{\sigma_{\mathrm{S}}.c_{33}\}^2 \\
&= (\mathrm{T}-t)^2.c_{22}^2.\sigma_{\mathrm{R}}^2 + 2(\mathrm{T}-t).c_{22}.c_{32}.\sigma_{\mathrm{R}}.\sigma_{\mathrm{S}} + (c_{32}^2 + c_{33}^2).\sigma_{\mathrm{S}}^2
\end{aligned}
$$

so that:

dH(t) (partially adjusted)

$$
\begin{aligned}
&= \mathrm{H}(t).[(\mathrm{T}-t).\mu_{\mathrm{R}}() + \mu_{\mathrm{S}}() + \tfrac{1}{2}\sigma_{\mathrm{R}}^2(\mathrm{T}-t)^2 - \mathrm{R}(t) + (\mathrm{T}-t).\sigma_{\mathrm{S}}.\sigma_{\mathrm{R}}.\rho_{23} \\
&\quad - \{(\mathrm{T}-t).\sigma_{\mathrm{R}}.c_{21} + \sigma_{\mathrm{S}}.c_{31}\}.\mu_{\mathrm{F}}()/\sigma_{\mathrm{F}}].\mathrm{d}t \\
&\quad + \mathrm{H}(t).[\{(\mathrm{T}-t).\sigma_{\mathrm{R}}.c_{21} + \sigma_{\mathrm{S}}.c_{31}\}.\mathrm{dZ}_1^* + \sigma_4(t).\mathrm{dZ}_4]
\end{aligned}
$$

and we then replace dZ_4 by $\mathrm{dZ}_4^* - \gamma_4(t)\mathrm{d}t$ with

$$
\begin{aligned}
\gamma_4(t) = [&(\mathrm{T}-t).\mu_{\mathrm{R}}() + \mu_{\mathrm{S}}() + \tfrac{1}{2}\sigma_{\mathrm{R}}^2(\mathrm{T}-t)^2 - \mathrm{R}(t) + (\mathrm{T}-t).\sigma_{\mathrm{S}}.\sigma_{\mathrm{R}}.\rho_{23} \\
&- \{(\mathrm{T}-t).\sigma_{\mathrm{R}}.c_{21} + \sigma_{\mathrm{S}}.c_{31}\}.\mu_{\mathrm{F}}()/\sigma_{\mathrm{F}}]/\sigma_4(t)
\end{aligned}
$$

to give:

$$\mathrm{dH}^*(t) = \mathrm{H}^*(t).[\{(\mathrm{T}-t).\sigma_{\mathrm{R}}.c_{21} + \sigma_{\mathrm{S}}.c_{31}\}.\mathrm{dZ}_1^* + \sigma_4(t).\mathrm{dZ}_4^*],$$

so that $\mathrm{H}^*(t)$ is a martingale

C.8.6 There are two ways of thinking about the new starred variables we have defined. Any starred variable, say $X^*(t)$, related to the original variable, $X(t)$, starts with the same value as $X(t)$, so $X^*(0) = X(0)$, but then, in any particular realisation, follows a track closely associated with that of $X(t)$, in that the random part is the same, but it has a different, zero, drift. A starred variable can be thought of either as a variable subject to a different probability measure, often called the 'Q measure', or else as a variable in the same probability space as $X(t)$, the 'real world probability space', that 'shadows' $X(t)$, using only the stochastic parts of the differential equation defining $X(t)$, i.e. using the dZ parts, and ignoring the dt parts. If $X(t)$ is a random walk with upwards drift, $X^*(t)$ will be centred around the value of $X(0)$, with the same 'scatter' as $X(t)$; however, if $X(t)$ is autoregressive, so that it stays close to some mean value, $X^*(t)$ will scatter like a random walk.

C.9 *The Guaranteed Annuity Option*

C.9.1 The value of any payoff, X, at time T, which is a function of $F(t)$ and $H(t)$, can now be calculated as the expected value of X, expressed as a function of $F^*(t)$ and $H^*(t)$, i.e. under the equivalent martingale measure. The combination of the basic policy and the GAO is a maxi option, and we can express the payoff (see ¶2.3.5) as:

$$X(T) = S(T).\max(g.F(T), 1).$$

However, we wish to express this in terms of the numeraire, so that the payoff is:

$$\begin{aligned} X(T)/B(T) &= S(T)/B(T).\max(g.F(T), 1) \\ &= H(T).\max(g.F(T), 1) \end{aligned}$$

Thus the value of the maxi option in terms of the numeraire is:

$$E^*[X(T)/B(T)] = E[H^*(T).\max(g.F^*(T), 1)].$$

Re-expressed in pounds at time 0, it is:

$$V(0) = B(0).E^*[X(T)/B(T)] = B(0).E[H^*(T).\max(g.F^*(T), 1)].$$

Note that we treat time 0 as the starting point of the option, with it being exercised T time units later.

C.9.2 We now need to consider the joint distribution of $H^*(T)$ and $F^*(T)$. We have:

$$\mathrm{dF}^*(t) = \sigma_\mathrm{F}.\mathrm{F}^*(t).\mathrm{dZ}_1^*$$
$$\mathrm{dH}^*(t) = \mathrm{H}^*(t).[\{(\mathrm{T}-t).\sigma_\mathrm{R}.c_{21} + \sigma_\mathrm{S}.c_{31}\}.\mathrm{dZ}_1^* + \sigma_4(t).\mathrm{dZ}_4^*]$$

with:

$$\sigma_4(t)^2 = (\mathrm{T}-t)^2.c_{22}^2.\sigma_\mathrm{R}^2 + 2(\mathrm{T}-t).c_{22}.c_{32}.\sigma_\mathrm{R}.\sigma_\mathrm{S} + (c_{32}^2 + c_{33}^2).\sigma_\mathrm{S}^2.$$

C.9.3 To go further we use Ito's formula again. Let $\mathrm{K}(t) = \log(\mathrm{F}^*(t))$. $\mathrm{f}'() = 1/\mathrm{F}^*(t)$, $\mathrm{f}''() = -1/\mathrm{F}^*(t)^2$, and $\partial\mathrm{K}/\partial t = 0$. Hence:

$$\begin{aligned}\mathrm{dK}(t) &= 1/\mathrm{F}^*(t).(\sigma_\mathrm{F}.\mathrm{F}^*(t).\mathrm{dZ}_1^*) + \tfrac{1}{2}(\sigma_\mathrm{F}.\mathrm{F}^*(t))^2.(-1/\mathrm{F}^*(t)^2).\mathrm{d}t + 0.\mathrm{d}t\\ &= -\tfrac{1}{2}\sigma_\mathrm{F}^2.\mathrm{d}t + \sigma_\mathrm{F}.\mathrm{dZ}_1^*\end{aligned}$$

Similarly, putting $\mathrm{L} = \log(\mathrm{H}^*(t))$, we get:

$$\begin{aligned}\mathrm{dL}(t) &= \{1/\mathrm{H}^*(t)\}.\mathrm{H}^*(t).[\{(\mathrm{T}-t).\sigma_\mathrm{R}.c_{21} + \sigma_\mathrm{S}.c_{31}\}.\mathrm{dZ}_1^* + \sigma_4(t).\mathrm{dZ}_4^*]\\ &\quad + \tfrac{1}{2}\mathrm{H}^*(t)^2.[\{(\mathrm{T}-t).\sigma_\mathrm{R}.c_{21} + \sigma_\mathrm{S}.c_{31}\}^2 + \sigma_4(t)^2].(-1/\mathrm{H}^*(t)^2).\mathrm{d}t\\ &= -\tfrac{1}{2}[\{(\mathrm{T}-t).\sigma_\mathrm{R}.c_{21} + \sigma_\mathrm{S}.c_{31}\}^2 + \sigma_4(t)^2].\mathrm{d}t\\ &\quad + \{(\mathrm{T}-t).\sigma_\mathrm{R}.c_{21} + \sigma_\mathrm{S}.c_{31}\}.\mathrm{dZ}_1^* + \sigma_4(t).\mathrm{dZ}_4^*\end{aligned}$$

C.9.4 Now $\mathrm{K}(\mathrm{T}) = \mathrm{K}(0) + \int_0^\mathrm{T}[-\tfrac{1}{2}\sigma_\mathrm{F}^2.\mathrm{d}t + \sigma_\mathrm{F}.\mathrm{dZ}_1^*]$
The integral consists of two parts, one deterministic, $\int_0^\mathrm{T} -\tfrac{1}{2}\sigma_\mathrm{F}^2.\mathrm{d}t$, which can be integrated in the usual way, to give $\int_0^\mathrm{T} -\tfrac{1}{2}\sigma_\mathrm{F}^2.\mathrm{d}t = -\tfrac{1}{2}\sigma_\mathrm{F}^2.\mathrm{T}$ and the other stochastic, $\int_0^\mathrm{T} \sigma_\mathrm{F}.\mathrm{dZ}_1^*$, which is normally distributed with mean zero, and variance $\int_0^\mathrm{T} \sigma_\mathrm{F}^2.\mathrm{d}t = \sigma_\mathrm{F}^2.\mathrm{T}$. Therefore K(T) is normally distributed with mean:

$$\mathrm{E}[\mathrm{K}(\mathrm{T})] = \mathrm{K}(0) + \int_0^\mathrm{T} -\tfrac{1}{2}\sigma_\mathrm{F}^2.\mathrm{d}t = \mathrm{K}(0) - \tfrac{1}{2}\sigma_\mathrm{F}^2.\mathrm{T}$$

and variance:

$$\mathrm{Var}[\mathrm{K}(\mathrm{T})] = \sigma_\mathrm{F}^2.\mathrm{T}$$

C.9.5 Also

$$\begin{aligned}\mathrm{L}(\mathrm{T}) = \mathrm{L}(0) - \int_0^\mathrm{T} &\{\tfrac{1}{2}[\{(\mathrm{T}-t).\sigma_\mathrm{R}.c_{21} + \sigma_\mathrm{S}.c_{31}\}^2 + \sigma_4(t)^2].\mathrm{d}t\\ &+ \{(\mathrm{T}-t).\sigma_\mathrm{R}.c_{21} + \sigma_\mathrm{S}.c_{31}\}.\mathrm{dZ}_1^* + \sigma_4(t).\mathrm{dZ}_4^*\}\end{aligned}$$

so L(T) is normally distributed with mean:

$$\begin{aligned}
\mathrm{E}[\mathrm{L}(\mathrm{T})] &= \mathrm{L}(0) - \int_0^{\mathrm{T}} \tfrac{1}{2}[\{(\mathrm{T}-t).\sigma_{\mathrm{R}}.c_{21} + \sigma_{\mathrm{S}}.c_{31}\}^2 + \sigma_4(t)^2].\mathrm{d}t \\
&= \mathrm{L}(0) - \int_0^{\mathrm{T}} \tfrac{1}{2}[(\mathrm{T}-t)^2.\sigma_{\mathrm{R}}^2.c_{21}^2 + 2(\mathrm{T}-t).\sigma_{\mathrm{R}}.c_{21}.\sigma_{\mathrm{S}}.c_{31} + \sigma_{\mathrm{S}}^2.c_{31}^2 \\
&\quad + (\mathrm{T}-t)^2.c_{22}^2.\sigma_{\mathrm{R}}^2 + 2(\mathrm{T}-t).c_{22}.c_{32}.\sigma_{\mathrm{R}}.\sigma_{\mathrm{S}} + (c_{32}^2 + .c_{33}^2).\sigma_{\mathrm{S}}^2].\mathrm{d}t \\
&= \mathrm{L}(0) - \int_0^{\mathrm{T}} \tfrac{1}{2}[(\mathrm{T}-t)^2.(c_{21}^2 + c_{22}^2).\sigma_{\mathrm{R}}^2 + 2(\mathrm{T}-t).(c_{21}.c_{31} + c_{22}.c_{32}).\sigma_{\mathrm{R}}.\sigma_{\mathrm{S}} \\
&\quad + (c_{31}^2 + c_{32}^2 + c_{33}^2).\sigma_{\mathrm{S}}^2].\mathrm{d}t \\
&= \mathrm{L}(0) - \tfrac{1}{2}[\mathrm{T}^3/3.(c_{21}^2 + c_{22}^2).\sigma_{\mathrm{R}}^2 + 2\mathrm{T}^2/2.(c_{21}.c_{31} + c_{22}.c_{32}).\sigma_{\mathrm{R}}.\sigma_{\mathrm{S}} \\
&\quad + \mathrm{T}.(c_{31}^2 + c_{32}^2 + c_{33}^2).\sigma_{\mathrm{S}}^2] \\
&= \mathrm{L}(0) - \tfrac{1}{2}[\mathrm{T}^3.\sigma_{\mathrm{R}}^2/3 + \mathrm{T}^2.\rho_{23}.\sigma_{\mathrm{R}}.\sigma_{\mathrm{S}} + \mathrm{T}.\sigma_{\mathrm{S}}^2]
\end{aligned}$$

and variance:

$$\begin{aligned}
\mathrm{Var}[\mathrm{L}(\mathrm{T})] &= \int_0^{\mathrm{T}} \{(\mathrm{T}-t).\sigma_{\mathrm{R}}.c_{12} + \sigma_{\mathrm{S}}.c_{13}\}^2 + \sigma_4(t)^2.\mathrm{d}t \\
&= [\mathrm{T}^3.\sigma_{\mathrm{R}}^2/3 + \mathrm{T}^2.\rho_{23}.\sigma_{\mathrm{R}}.\sigma_{\mathrm{S}} + \mathrm{T}.\sigma_{\mathrm{S}}^2]
\end{aligned}$$

C.9.6 Further, K(T) and L(T) have covariance (which comes only through Z_1^*):

$$\begin{aligned}
\mathrm{Covar}[\mathrm{K}(\mathrm{T}), \mathrm{L}(\mathrm{T})] &= \int_0^{\mathrm{T}} \sigma_{\mathrm{F}}.[\{(\mathrm{T}-t).\sigma_{\mathrm{R}}.c_{12} + \sigma_{\mathrm{S}}.c_{31}\}].\mathrm{d}t \\
&= \mathrm{T}^2.c_{21}.\sigma_{\mathrm{R}}.\sigma_{\mathrm{F}}/2 + \mathrm{T}.c_{31}.\sigma_{\mathrm{S}}.\sigma_{\mathrm{F}} \\
&= \mathrm{T}^2.\rho_{12}.\sigma_{\mathrm{R}}.\sigma_{\mathrm{F}}/2 + \mathrm{T}.\rho_{13}.\sigma_{\mathrm{S}}.\sigma_{\mathrm{F}}
\end{aligned}$$

Since $\mathrm{K}(\mathrm{T}) = \log \mathrm{F}^*(\mathrm{T})$ and $\mathrm{L}(\mathrm{T}) = \log \mathrm{H}^*(\mathrm{T})$ are jointly normally distributed, $\mathrm{F}^*(\mathrm{T})$ and $\mathrm{H}^*(\mathrm{T})$ are jointly lognormally distributed.

C.9.7. To calculate the value of the maxi option we need another result, relating to bivariate lognormal distributions. If X_1 and X_2 are jointly normally distributed, with means μ_1 and μ_2, variances σ_1^2 and σ_2^2 and correlation coefficient ρ, their density function can be written:

$$f_X(x_1, x_2) = \frac{1}{2\pi\sigma_1\sigma_2(1-\rho^2)^{1/2}}$$
$$\exp\left\{-\frac{1}{2(1-\rho^2)}\left(\frac{(x_1-\mu_1)^2}{\sigma_1^2} - \frac{2\rho(x_1-\mu_1)(x_2-\mu_2)}{\sigma_1\sigma_2} + \frac{(x_2-\mu_2)^2}{\sigma_2^2}\right)\right\}$$

Then if $Y_1 = \exp(X_1)$ and $Y_2 = \exp(X_2)$, we describe Y_1 and Y_2 as being jointly lognormally distributed with the same parameters. Their density function is:

$$f_Y(y_1, y_2) = \frac{1}{2\pi y_1 y_2 \sigma_1\sigma_2(1-\rho^2)^{1/2}}$$
$$\exp\left\{-\frac{1}{2(1-\rho^2)}\left(\frac{(x_1-\mu_1)^2}{\sigma_1^2} - \frac{2\rho(x_1-\mu_1)(x_2-\mu_2)}{\sigma_1\sigma_2} + \frac{(x_2-\mu_2)^2}{\sigma_2^2}\right)\right\}$$

where $x_1 = \log y_1$, $x_2 = \log y_2$.

C.9.8 Putting $Y_1 = F^*(T)$ and $Y_2 = H^*(T)$ we calculate the value of the maxi option, expressed in units of the numeraire, as:

$$E[H^*(T).\max(g.F^*(T), 1)] = \int_{1/g}^{\infty}\int_0^{\infty} H^*(T).g.F^*(T).f(H^*, F^*).dH^*.dF^*$$
$$+ \int_0^{1/g}\int_0^{\infty} H^*(T).f(H^*, F^*).dH^*.dF^*$$

We can first calculate what we shall denote:

$$E[Y_1^{r1}.Y_2^{r2}; a, b; 0, \infty] = \int_a^b \int_0^{\infty} y_1^{r1}.y_2^{r2}.f_Y(y_1, y_2).dy_2.dy_1$$

i.e. y_1 runs from a to b and y_2 runs from 0 to ∞.

C.9.9 Putting $x_1 = \log y_1$ and $x_2 = \log y_2$ we rewrite the integral as

$$I = \int_{\log a}^{\log b}\int_0^{\infty} \exp(r_1x_1).\exp(r_2x_2).f_X(x_1, x_2).dx_2.dx_1$$
$$= \int_{\log a}^{\log b}\int_{-\infty}^{\infty} \exp(r_1x_1).\exp(r_2x_2)\frac{1}{2\pi\sigma_1\sigma_2(1-\rho^2)^{1/2}}$$
$$\exp\left\{-\frac{1}{2(1-\rho^2)}\left(\frac{(x_1-\mu_1)^2}{\sigma_1^2} - \frac{2\rho(x_1-\mu_1)(x_2-\mu_2)}{\sigma_1\sigma_2} + \frac{(x_2-\mu_2)^2}{\sigma_2^2}\right)\right\}dx_2.dx_1$$

and after a great deal of straightforward but tedious manipulation we get:

$$\mathrm{I} = \exp(r_1\mu_1 + r_1^2\sigma_1^2/2 + r_1 r_2\rho\sigma_1\sigma_2 + r_2\mu_2 + r_2^2\sigma_2^2/2).$$
$$.[\mathrm{N}\{(\log b - \mu_1 - r_2\rho\sigma_1\sigma_2 - r_1\sigma_2^2)/\sigma_1\} - \mathrm{N}\{(\log a - \mu_1 - r_2\rho\sigma_1\sigma_2 - r_1\sigma_2^2)/\sigma_1\}]$$

C.9.10 So with $r_1 = r_2 = 1$, we have:

$$\begin{aligned} \mathrm{E}[\mathrm{Y}_1\mathrm{Y}_2; a, b; 0, \infty] = \exp(\mu_1 + \mu_2 + \tfrac{1}{2}\sigma_1^2 + \rho\sigma_1\sigma_2 + \tfrac{1}{2}\sigma_2^2).& \\ .[\mathrm{N}\{(\log b - \mu_1 - \rho\sigma_1\sigma_2 - \sigma_1^2)/\sigma_1\}& \\ - \mathrm{N}\{((\log a - \mu_1 - \rho\sigma_1\sigma_2 - \sigma_1^2)/\sigma_1\}]& \end{aligned}$$

and putting $b = \infty$ we have:

$$\begin{aligned} \mathrm{E}[\mathrm{Y}_1\mathrm{Y}_2; a, \infty; 0, \infty] = \exp(\mu_1 + \mu_2 + \tfrac{1}{2}\sigma_1^2 + \rho\sigma_1\sigma_2 + \tfrac{1}{2}\sigma_2^2).& \\ .[1 - \mathrm{N}\{(\log a - \mu_1 - \rho\sigma_1\sigma_2 - \sigma_1^2)/\sigma_1\}]& \end{aligned}$$

Then with $r_1 = 0$, $r_2 = 1$

$$\begin{aligned} \mathrm{E}[\mathrm{Y}_1; a, b; 0, \infty] = \exp(\mu_2 + \tfrac{1}{2}\sigma_2^2).& \\ .[\mathrm{N}\{(\log b - \mu_1 - \rho\sigma_1\sigma_2)/\sigma_1\} - \mathrm{N}\{(\log a - \mu_1 - \rho\sigma_1\sigma_2)/\sigma_1\}]& \end{aligned}$$

and with $a = 0$ we have

$$\mathrm{E}[\mathrm{Y}_1; 0, b; 0, \infty] = \exp(\mu_2 + \tfrac{1}{2}\sigma_2^2).[\mathrm{N}\{(\log b - \mu_1 - \rho\sigma_1\sigma_2)/\sigma_1\} - 0]$$

C.9.11 We now get the value of the maxi option, V(0), as:

$$\begin{aligned} \mathrm{V}(0) &= \mathrm{B}(0).\mathrm{E}[\mathrm{H}^*(\mathrm{T}).\max(g.\mathrm{F}^*(\mathrm{T}), 1)] \\ &= \mathrm{B}(0).\left[\int_{1/g}^{\infty}\int_0^{\infty} \mathrm{H}^*(\mathrm{T}).g.\mathrm{F}^*(\mathrm{T}).f(\mathrm{H}^*, \mathrm{F}^*).d\mathrm{H}^*.d\mathrm{F}^*\right. \\ &\quad \left. + \int_0^{1/g}\int_0^{\infty} \mathrm{H}^*(\mathrm{T}).f(\mathrm{H}^*, \mathrm{F}^*).d\mathrm{H}^*.d\mathrm{F}^*\right] \\ &= \mathrm{B}(0).g.\exp(\mu_1 + \mu_2 + \tfrac{1}{2}\sigma_1^2 + \rho\sigma_1\sigma_2 + \tfrac{1}{2}\sigma_2^2). \\ &\quad .[1 - \mathrm{N}\{(\log(1/g) - \mu_1 - \rho\sigma_1\sigma_2 - \sigma_1^2)/\sigma_1\}] \\ &\quad + \mathrm{B}(0).\exp(\mu_2 + \tfrac{1}{2}\sigma_2^2).\mathrm{N}\{(\log(1/g) - \mu_1 - \rho\sigma_1\sigma_2)/\sigma_1\} \end{aligned}$$

Now $\mu_1 = \mathrm{E}[\mathrm{K}(\mathrm{T})] = \mathrm{K}(0) - \tfrac{1}{2}\sigma_\mathrm{F}^2.\mathrm{T}$ and $\sigma_1^2 = \mathrm{Var}[\mathrm{K}(\mathrm{T})] = \sigma_\mathrm{F}^2.\mathrm{T}$, so $\mu_1 + \tfrac{1}{2}\sigma_1^2 = \mathrm{K}(0) = \log \mathrm{F}^*(0) = \log \mathrm{F}(0)$, since $\mathrm{F}^*(0) = \mathrm{F}(0)$.

C.9.12 Further

$$\mu_2 = \mathrm{E[L(T)]} = \mathrm{L}(0) - \tfrac{1}{2}[\mathrm{T}^3.\sigma_\mathrm{R}^2/3 + \mathrm{T}^2.\rho_{23}.\sigma_\mathrm{R}.\sigma_\mathrm{S} + \mathrm{T}.\sigma_\mathrm{S}^2]$$

and

$$\sigma_2^2 = \mathrm{Var[L(T)]} = [\mathrm{T}^3.\sigma_\mathrm{R}^2/3 + \mathrm{T}^2.\rho_{23}.\sigma_\mathrm{R}.\sigma_\mathrm{S} + \mathrm{T}.\sigma_\mathrm{S}^2],$$

so

$$\mu_2 = \tfrac{1}{2}\sigma_2^2 = \mathrm{L}(0) = \log \mathrm{H}^*(0) = \log \mathrm{H}(0).$$

Also

$$\rho\sigma_1\sigma_2 = \mathrm{Cov} = \mathrm{Covariance[K(T), L(T)]} = \mathrm{T}^2.\rho_{12}.\sigma_\mathrm{R}.\sigma_\mathrm{F}/2 + \mathrm{T}.\rho_{13}.\sigma_\mathrm{S}.\sigma_\mathrm{F}$$

C.9.13 So

$$\begin{aligned}
\mathrm{V}(0) = {} & \mathrm{B}(0).g.\exp(\mathrm{K}(0) + \mathrm{L}(0) + \mathrm{Cov})). \\
& .[1 - \mathrm{N}\{(\log(1/g) - \mathrm{K}(0) + \tfrac{1}{2}\sigma_\mathrm{F}^2.\mathrm{T} - \mathrm{Cov} - \sigma_\mathrm{F}^2.\mathrm{T})/\sigma_\mathrm{F}\sqrt{}\mathrm{T}\}] \\
& + \mathrm{B}(0).\exp(\mathrm{L}(0)).\mathrm{N}\{(\log(1/g) - \mathrm{K}(0) + \tfrac{1}{2}\sigma_\mathrm{F}^2.\mathrm{T} - \mathrm{Cov})/\sigma_\mathrm{F}\sqrt{}\mathrm{T}\} \\
= {} & \mathrm{B}(0).g.\mathrm{F}(0).\mathrm{H}(0).\exp(\mathrm{Cov}). \\
& .[1 - \mathrm{N}\{(-\log(g) - \log \mathrm{F}(0) - \tfrac{1}{2}\sigma_\mathrm{F}^2.\mathrm{T} - \mathrm{Cov})/\sigma_\mathrm{F}\sqrt{}\mathrm{T}\}] \\
& + \mathrm{B}(0).\mathrm{H}(0).\mathrm{N}\{(-\log(g) - \log \mathrm{F}(0) + \tfrac{1}{2}\sigma_\mathrm{F}^2.\mathrm{T} - \mathrm{Cov})/\sigma_\mathrm{F}\sqrt{}\mathrm{T}\} \\
= {} & \mathrm{B}(0).g.\mathrm{F}(0).\mathrm{S}(0)/\mathrm{B}(0).\exp(\mathrm{Cov}). \\
& .\mathrm{N}\{\log(g.\mathrm{F}(0)) + \tfrac{1}{2}\sigma_\mathrm{F}^2.\mathrm{T} + \mathrm{Cov})/\sigma_\mathrm{F}\sqrt{}\mathrm{T}\} \\
& + \mathrm{B}(0).\mathrm{S}(0)/\mathrm{B}(0).\mathrm{N}\{(-\log(g.\mathrm{F}(0)) + \tfrac{1}{2}\sigma_\mathrm{F}^2.\mathrm{T} - \mathrm{Cov})/\sigma_\mathrm{F}\sqrt{}\mathrm{T}\} \\
= {} & \mathrm{F}(0).\mathrm{S}(0).g.\exp(\mathrm{Cov}).\mathrm{N}\{(\log(g.\mathrm{F}(0)) + \mathrm{Cov})/\sigma_\mathrm{F}\sqrt{}\mathrm{T} + \tfrac{1}{2}\sigma_\mathrm{F}\sqrt{}\mathrm{T}\} \\
& + \mathrm{S}(0).\mathrm{N}\{-(\log(g.\mathrm{F}(0)) + \mathrm{Cov})/\sigma_\mathrm{F}\sqrt{}\mathrm{T} + \tfrac{1}{2}\sigma_\mathrm{F}\sqrt{}\mathrm{T}\}
\end{aligned}$$

C.9.14 We assumed that the starting date of the option was time $t = 0$, and the expiry date was time T. But the above formula applies equally at general time t if F(0) and S(0) are replaced by F(t) and S(t), and T is replaced by $\mathrm{T} - t$, giving:

$$\begin{aligned}
\mathrm{V}(t) = {} & \mathrm{F}(t).\mathrm{S}(t).g.\exp(\mathrm{Cov}).\mathrm{N}\{(\log(g.\mathrm{F}(t)) + \mathrm{Cov})/\sigma_\mathrm{F}\sqrt{}(\mathrm{T} - t) + \tfrac{1}{2}\sigma_\mathrm{F}\sqrt{}(\mathrm{T} - t)\} \\
& + \mathrm{S}(t).\mathrm{N}\{-(\log(g.\mathrm{F}(t)) + \mathrm{Cov})/\sigma_\mathrm{F}\sqrt{}(\mathrm{T} - t) + \tfrac{1}{2}\sigma_\mathrm{F}\sqrt{}(\mathrm{T} - t)\} \\
= {} & \mathrm{S}(t).[g.\mathrm{F}(t).\exp(\mathrm{Cov}).\mathrm{N}\{(\log(g.\mathrm{F}(t)) + \mathrm{Cov})/\sigma_\mathrm{F}\sqrt{}(\mathrm{T} - t) + \tfrac{1}{2}\sigma_\mathrm{F}\sqrt{}(\mathrm{T} - t)\} \\
& + \mathrm{N}\{-(\log(g.\mathrm{F}(t)) + \mathrm{Cov})/\sigma_\mathrm{F}\sqrt{}(\mathrm{T} - t) + \tfrac{1}{2}\sigma_\mathrm{F}\sqrt{}(\mathrm{T} - t)\}] \\
= {} & \mathrm{S}(t).[\mathrm{G}.\mathrm{N}\{\log(\mathrm{G})/\Sigma + \tfrac{1}{2}\Sigma\} + \mathrm{N}\{-\log(\mathrm{G})/\Sigma + \tfrac{1}{2}\Sigma\}] \\
= {} & \mathrm{S}(t).[\mathrm{G}.\mathrm{N}(d_1) + \mathrm{N}(d_2)]
\end{aligned}$$

with

$$\begin{aligned} d_1 &= \log(\mathrm{G})/\Sigma + \tfrac{1}{2}\Sigma \\ d_2 &= -\log(\mathrm{G})/\Sigma + \tfrac{1}{2}\Sigma \\ \mathrm{G} &= g.\mathrm{F}(t).\exp(\mathrm{Cov}) \\ \mathrm{Cov} &= (\mathrm{T}-t)^2.\rho_{12}.\sigma_\mathrm{R}.\sigma_\mathrm{F}/2 + (\mathrm{T}-t).\rho_{13}.\sigma_\mathrm{S}.\sigma_\mathrm{F} \end{aligned}$$

and

$$\Sigma = \sigma_\mathrm{F}\sqrt{(\mathrm{T}-t)}$$

C.9.15 Note that $\mu_\mathrm{S}()$, $\mu_\mathrm{R}()$ and $\mu_\mathrm{F}()$ do not enter this formula, so they can be any suitable functions that we like. This is similar to what is found in the Black-Scholes result for an ordinary option on a share. Note also that ρ_{23} does not come in either; any correlation between the zcb and the share does not affect the value of the option. However, ρ_{12} and ρ_{13} remain relevant.

C.9.16 In order for our logic to be valid, there are technical conditions that must be met, mainly that the integral of the variance over any finite time is itself finite. The conditions are met in our model, effectively because we assume constant values of σ_S, σ_R and σ_F.

C.9.17 We used the zcb as the numeraire. We could have taken either the share or the dla as the numeraire, or just left the expressions in pounds, and in each case we should have obtained the same results.

C.10 *The Hedging Proportions*

C.10.1 We must now find the hedging proportions. The value of the option calculated above is a deterministic value, but it is only so because of the possibility of hedging with a self-financing portfolio, in such a way that the proceeds of the investment always exactly meet the required option value. This is possible only in a world in which hedging can be carried out continuously and costlessly, and of course only if the true 'real world' model is the same as the hypothesised real world model we have defined in order to calculate the option value.

C.10.2 Imagine that at each time t we hold a portfolio consisting of amounts $\varphi_\mathrm{S}(t)$ invested in the share, $\varphi_\mathrm{D}(t)$ invested in the deferred annuity and $\varphi_\mathrm{B}(t)$ invested in the zero coupon bond. We wish the value of this portfolio to equal the value of the option at all times. Thus we would like:

$$\varphi_\mathrm{S}(t) + \varphi_\mathrm{D}(t) + \varphi_\mathrm{B}(t) = \mathrm{V}(t)$$

for all t. For this to occur, the initial values must be equal:

$$\varphi_S(0) + \varphi_D(0) + \varphi_B(0) = V(0)$$

and also the derivative of the portfolio must equal the derivative of the option, so we would have:

$$dV(t) = d\varphi_S(t) + d\varphi_D(t) + d\varphi_B(t)$$

for all t.

C.10.3 It is again easier to work in terms of the numeraire, so we proceed by defining:

$$U(t) = V(t)/B(t)$$
$$\varphi_H(t) = \varphi_S(t)/B(t)$$
$$\varphi_F(t) = \varphi_D(t)/B(t)$$
$$\varphi_A(t) = \varphi_B(t)/B(t)$$

and we require:

$$U(t) = \varphi_H(t) + \varphi_F(t) + \varphi_A(t)$$

and:

$$\begin{aligned} dU(t) &= d\varphi_H(t) + d\varphi_F(t) + d\varphi_A(t) \\ &= \varphi_H(t)/H(t).dH(t) + \varphi_F(t)/F(t).dF(t) + \varphi_A(t)/A(t).dA(t) \end{aligned}$$

where $A(t)$ is the value of the zcb in terms of the zcb, so $A(t) = 1$ and $dA(t) = 0$ for all t.

C.10.4 To calculate $dU(t)$ we need Ito's formula again. We have:

$$\begin{aligned} U(t) &= V(t)/B(t) \\ &= S(t)/B(t).[G.N(d_1) + N(d_2)] \\ &= H(t).[G.N(d_1) + N(d_2)] \end{aligned}$$

Now $dU(t) = \partial U/\partial H.dH(t) + \partial U/\partial F.dF(t) +$ non-stochastic terms, so if we set:

$$\varphi_H(t)/H(t) = \partial U/\partial H$$
$$\varphi_F(t)/F(t) = \partial U/\partial F$$

and:

$$\varphi_A(t) = U(t) - \varphi_H(t) - \varphi_F(t),$$

we shall achieve what we require.

C.10.5 We therefore need $\partial U/\partial H$ and $\partial U/\partial F$.

$$U(t) = H(t).[G.N(d_1) + N(d_2)]$$

so

$$\partial U/\partial H = [G.N(d_1) + N(d_2)]$$

and

$$\varphi_H(t) = H(t).\partial U/\partial H = U(t).$$

C.10.6 Then

$$\begin{aligned}\partial U/\partial F &= \partial\{H(t).[G.N(d_1) + N(d_2)]\}/\partial F\\ &= H(t).\partial[G.N(d_1) + N(d_2)]/\partial F\end{aligned}$$

Now

$$G = g.F(t).\exp(\mathrm{Cov})$$

so

$$\begin{aligned}&\partial G/\partial F = g.\exp(\mathrm{Cov})\\ &\partial N(x)/\partial x = \exp(-x^2/2)/\surd(2\pi)\\ &d_1 = \log(G)/\Sigma + \tfrac{1}{2}\Sigma = \log(g.F(t).\exp(\mathrm{Cov}))/\Sigma + \tfrac{1}{2}\Sigma\end{aligned}$$

so

$$\begin{aligned}&\partial d_1/\partial F = 1/(F(t).\Sigma)\\ &d_2 = -\log(G)/\Sigma + \tfrac{1}{2}\Sigma = -\log(g.F(t).\exp(\mathrm{Cov}))/\Sigma + \tfrac{1}{2}\Sigma\end{aligned}$$

and

$$\partial d_2/\partial F = -1/(F(t).\Sigma)$$

So:

$$\begin{aligned}\partial U/\partial F = H(t).[&g.\exp(\mathrm{Cov}).N(d_1) + G.\exp(-d_1^2/2)/(\surd(2\pi).F(t).\Sigma))\\ &+ \exp(-d_2^2/2)/(\surd(2\pi).(-F(t)).\Sigma)]\\ = H(t).[&g.\exp(\mathrm{Cov}).N(d_1) + [G.\exp(-d_1^2/2) - \exp(-d_2^2/2)]/\surd(2\pi).F(t).\Sigma]\end{aligned}$$

C.10.7 Then:

$$\begin{aligned}\varphi_F(t) &= F(t).\partial U/\partial F\\ &= F(t).H(t).[g.\exp(\mathrm{Cov}).N(d_1) + [G.\exp(-d_1^2/2)\\ &\quad - \exp(-d_2^2/2)]/\{\sqrt{(2\pi)}.F(t).\Sigma\}]\\ &= H(t).G.N(d_1) + H(t).[G.\exp(-d_1^2/2) - \exp(-d_2^2/2)]/\{\sqrt{(2\pi)}.\Sigma\}\end{aligned}$$

But

$$\begin{aligned}d_1 &= \log(G)/\Sigma + \tfrac{1}{2}\Sigma\\ d_2 &= -\log(G)/\Sigma + \tfrac{1}{2}\Sigma\\ -d_1^2/2 &= -\tfrac{1}{2}(\log(G)/\Sigma + \tfrac{1}{2}\Sigma)^2\\ &= -\tfrac{1}{2}(\log(G)^2/\Sigma^2 + 2\log(G)/\Sigma.\tfrac{1}{2}\Sigma + \tfrac{1}{4}\Sigma^2)\\ &= -\tfrac{1}{2}(\log(G)^2/\Sigma^2 + \log(G) + \tfrac{1}{4}\Sigma^2)\\ -d_2^2/2 &= -\tfrac{1}{2}(-\log(G)/\Sigma + \tfrac{1}{2}\Sigma)^2\\ &= -\tfrac{1}{2}(\log(G)^2/\Sigma^2 - \log(G) + \tfrac{1}{4}\Sigma^2)\\ G.\exp(-d_1^2/2) &= \exp(\log(G) - d_1^2/2)\\ &= \exp(\log(G) - \tfrac{1}{2}(\log(G)^2/\Sigma^2 + \log(G) + \tfrac{1}{4}\Sigma^2))\\ &= \exp(-\tfrac{1}{2}(\log(G)^2/\Sigma^2 - 2\log(G) + \log(G) + \tfrac{1}{4}\Sigma^2))\\ &= \exp(-\tfrac{1}{2}(\log(G)^2/\Sigma^2 - \log(G) + \tfrac{1}{4}\Sigma^2))\\ &= \exp(-d_2^2/2)\end{aligned}$$

so

$$G.\exp(-d_1^2/2) - \exp(-d_2^2/2) = 0$$

and

$$\varphi_F(t) = H(t).G.N(d_1)$$

C.10.8 Then we calculate

$$\varphi_A(t) = U(t) - \varphi_H(t) - \varphi_F(t),$$

and then:

$$\begin{aligned}\varphi_S(t) &= \varphi_H(t).B(t)\\ \varphi_D(t) &= \varphi_F(t).B(t)\\ \varphi_B(t) &= \varphi_A(t).B(t)\end{aligned}$$

or, more directly:

$$\begin{aligned}
\varphi_S(t) &= \varphi_H(t).B(t) \\
&= U(t).B(t) \\
&= V(t) \\
\varphi_D(t) &= \varphi_F(t).B(t) \\
&= H(t).B(t).G.N(d_1) \\
&= S(t).G.N(d_1) \\
\varphi_B(t) &= V(t) - \varphi_S(t) - \varphi_D(t) = -\varphi_D(t).
\end{aligned}$$

The amount invested in units is the full value of the Maxi option. The amounts invested in the dla and the zcb are equal but of opposite signs, the former positive (implying a 'long' position) and the latter negative (or 'short', equivalent to borrowing). The more the option is in the money the larger are these offsetting amounts, approaching in the limit the full value of the option. If the option is very far out of the money, these amounts are both small.

C.11 *Validation*

C.11.1 We noted that the value of the option that we have calculated is only what we have calculated because of the possibility of a self-financing hedge strategy. To demonstrate this we use simulation over time steps of length h. We noted in C.9.8 that $\mu_S()$, $\mu_R()$ and $\mu_F()$ do not enter the option pricing formula, so we can make them any simple functions that we like. We choose $\mu_S() = \mu_S$, a constant, $\mu_R() = \alpha_R(\theta_R - R(t))$ and $\mu_F() = \alpha_F(\theta_F - \log F(t)) + \frac{1}{2}\sigma_F^2$, both autoregressive functions, the latter with a bias whose use will become apparent.

C.11.2 The differential equation for $S(t)$ is:

$$dS(t) = \mu_S.S(t).dt + \sigma_S.S(t).dW_3$$

from which we can derive (using Ito again):

$$d\log S(t) = (\mu_S - \tfrac{1}{2}\sigma_S^2).dt + \sigma_S.dW_3$$

whence:

$$\log S(t+h) = \log S(t) + \mu_{S,h} + s_{S,h}.W_3(t+h),$$

where $\mu_{S,h} = (\mu_S - \frac{1}{2}\sigma_S^2)h$ and $s_{S,h} = \sigma_S\sqrt{h}$, and W_3 is a unit normal random variable. This is, of course, a random walk model with drift for $\log S(t)$.

C.11.3 The differential equation for R(t) is:

$$\mathrm{dR}(t) = \alpha_{\mathrm{R}}(\theta_{\mathrm{R}} - \mathrm{R}(t)).\mathrm{d}t + \sigma_{\mathrm{R}}.\mathrm{dW}_2,$$

which is an Orenstein-Uhlenbeck process, whence:

$$\mathrm{R}(t+h) = \mathrm{R}(t) + a_{\mathrm{R},h}.(\mathrm{R}(t) - \theta_{\mathrm{R}}) + s_{\mathrm{R},h}.\mathrm{W}_2(t+h)$$

with $\alpha_{\mathrm{R},h} = \exp(-\alpha_{\mathrm{R}}h)$, $s_{\mathrm{R},h} = \sigma_{\mathrm{R}}\surd\{(1-\alpha^2_{\mathrm{R},h})/(2\alpha_{\mathrm{R}})\}$ and W_2 is a unit normal random variable. This is a first order autoregressive, or AR(1), time series model for R(t).

C.11.4 The differential equation for F(t) is:

$$\mathrm{dF}(t) = \{\alpha_F(\theta_{\mathrm{F}} - \log \mathrm{F}(t)) - \tfrac{1}{2}\sigma^2_{\mathrm{S}}\}.\mathrm{F}(t).\mathrm{d}t + \sigma_{\mathrm{F}}.\mathrm{F}(t).\mathrm{dW}_1,$$

from which we can derive, using Ito again:

$$\mathrm{d}\log \mathrm{F}(t) = \alpha_{\mathrm{F}}(\theta_{\mathrm{F}} - \log \mathrm{F}(t)).\mathrm{d}t + \sigma_{\mathrm{F}}.\mathrm{dW}_1$$

which is another Orenstein-Uhlenbeck process, whence

$$\log \mathrm{F}(t+h) = \log \mathrm{F}(t) + \alpha_{\mathrm{R},h}.(\log \mathrm{F}(t) - \theta_{\mathrm{R}}) + s_{\mathrm{R},h}.\mathrm{W}_1(t+h)$$

with $\alpha_{\mathrm{F},h} = \exp(-\alpha_{\mathrm{F}}h)$, $s_{\mathrm{F},h} = \sigma_{\mathrm{F}}\surd\{(1-\alpha^2_{\mathrm{F},h})/(2\alpha_{\mathrm{F}})\}$ and W_1 is a unit normal random variable.

C.11.5 Note that W_1, W_2 and W_3 are related through:

$$\begin{aligned}\mathrm{W}_1 &= \mathrm{Z}_1\\ \mathrm{W}_2 &= c_{21}.\mathrm{Z}_1 + c_{22}.\mathrm{Z}_2\\ \mathrm{W}_3 &= c_{31}.\mathrm{Z}_1 + c_{32}.\mathrm{Z}_2 + c_{33}.\mathrm{Z}_3\end{aligned}$$

where Z_1, Z_2 and Z_3 are independent unit normal variates and the cs are defined as in C.7.

C.11.6 To perform the simulation we need to select a horizon, T, a guaranteed rate, g, values of the parameters, μ_{S}, σ_{S}, etc, and values of S(0), R(0) and F(0). From these, we calculate the derived values, $\mathrm{B}(0) = \exp(-\mathrm{T.R}(0))$ and $\mathrm{D}(0) = \mathrm{F}(0).\mathrm{B}(0)$. In one simulation, we simulate the 'real world' values of S(t), R(t) and F(t), for each step $t = h, 2h, 3h$, ..., as just described, and calculate the derived values, B(t) and D(t). This gives us one 'real world' scenario. We then consider the option. We start the simulation with an amount invested of $\mathrm{Y}(0) = \mathrm{V}(0)$, invested in amounts $\varphi_{\mathrm{S}}(0)$, $\varphi_{\mathrm{D}}(0)$ and $\varphi_{\mathrm{B}}(0)$ as calculated in C.10. At time h the values of these investments will have altered to

$\varphi_S(0).S(h)/S(0)$, etc. The sum of these, denoted $Y(h)$, will almost certainly not exactly match the value of $V(h)$. We then consider five possible investment strategies:

(i) invest the correct amounts in the share and in the dla, $\varphi_S(h)$ and $\varphi_D(h)$ respectively, based on $V(h)$ and the formula in C.10, and let the balance be invested in the zcb;

(ii) invest the correct amounts in the share and in the zcb, $\varphi_S(h)$ and $\varphi_B(h)$ respectively, and let the balance be invested in the dla;

(iii) invest the correct amounts in the zcb and in the dla, $\varphi_B(h)$ and $\varphi_D(h)$ respectively, and let the balance be invested in the share;

(iv) invest the correct proportions in each of the three tradeables;

(v) invest the correct proportions at time 0, and leave them unchanged until the option expires (the only purpose of this being to show that this is not a matching strategy).

We choose one of these strategies, initially strategy (i).

C.11.7 Following the chosen strategy, by time T we shall have an amount $Y(T)$. This will almost certainly not exactly match the required payoff, $P(T)$, which we can calculate as $P(T) = S(T).\max(g.F(T), 1)$. The discrepancy we call the hedging error, denoted as $E(T) = P(T) - Y(T)$. In the first place we wish to investigate the distribution of $E(T)$.

C.11.8 We have chosen parameters:

$$\begin{aligned}
\mu_S &= 0.1 \\
\sigma_S &= 0.2 \\
\theta_R &= 0.065 \\
\alpha_R &= 0.125 \\
\sigma_R &= 0.0125 \\
\theta_F &= 2.2 \\
\alpha_F &= 0.1 \\
\sigma_F &= 0.065 \\
\rho_{SR} &= -0.3 \\
\rho_{SF} &= 0.3 \\
\rho_{RF} &= -0.99
\end{aligned}$$

These are very loosely based on the actual experience in the U.K., at monthly intervals, from December 1950 to August 2002. The value of $F(t)$ depends in principle on the mortality table used. For this experiment we have used PA(90), with an annuity due at age 65. We then assume:

$T = 20$ years

$g = 0.111$

$S(0) = 100$

$R(0) = 0.05$

$F(0) = 9.0$

The exact values are not important for this experiment; but the option is almost at the money.

C.11.9 We first show the results of 1,000 simulations with $h = 1$ year, and the hedging being carried out according to strategy (i), in which, at each rebalancing point, the correct amounts are invested in the share and the dla, with the bond taking the balance. Figure C.1 shows the values of Y(T), the investment proceeds, plotted against the values of P(T), the amount required to pay off the option. One can see that in general the investment proceeds correspond with the amounts required, but by no means perfectly.

C.11.10 The results depend so strongly on the value of S(T) that it seems more informative to work in units of the share. The investment proceeds are $Y^*(T) = Y(T)/S(T)$, and the required option is $P^*(T) = P(T)/S(T) = \max(g.F(T), 1)$. Figure C.2 shows the values of $Y^*(T)$ and $P^*(T)$ plotted against $g.F(T)$. One can now see that, although the investment results cluster around the desired target, they are, in some cases, very far away from it. In one case the investment proceeds are negative. But in other cases the investment proceeds provide a proportionately very large profit.

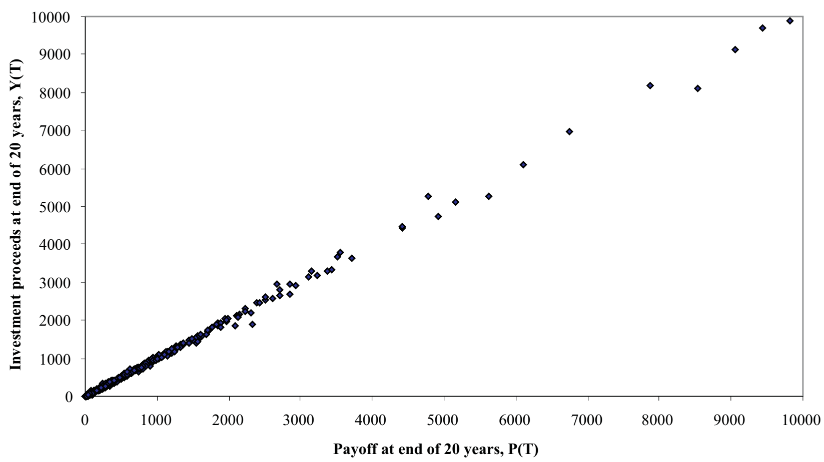

Figure C.1. Y(T) versus P(T), hedging yearly.

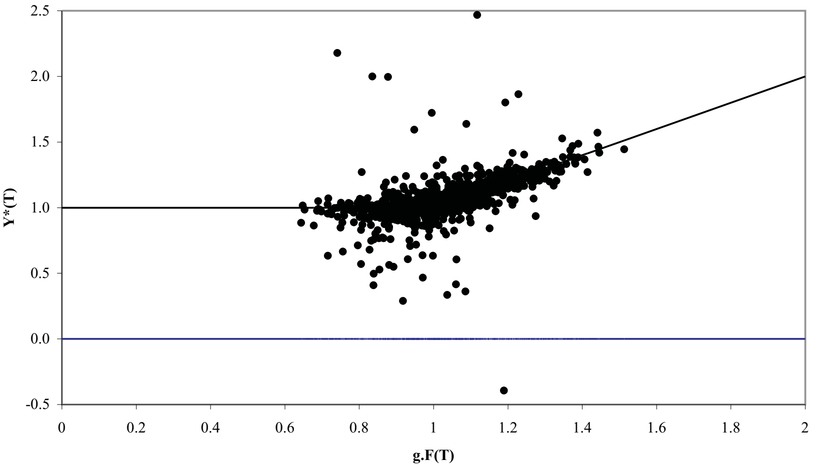

Figure C.2. $Y^*(T)$ versus g.F(T), hedging yearly.

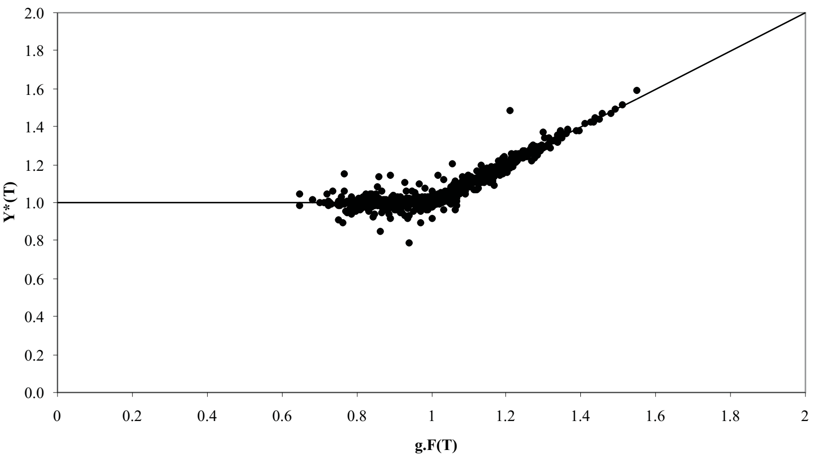

Figure C.3. $Y^*(T)$ versus g.F(T), hedging twice per month.

C.11.11 We next use $h = \frac{1}{2}$ month. Figure C.3 shows the same variables as Figure C.2, but with hedging twice per month. One can see that the correspondence is rather closer. The extreme values are very much less extreme than with yearly hedging, but there are still some proportionately large deficits and profits.

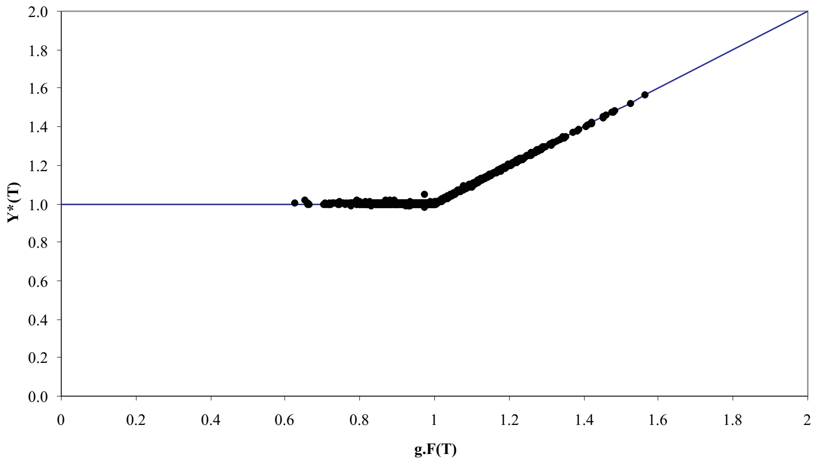

Figure C.4. Y*(T) versus g.F(T), hedging 128 times per month.

C.11.12 Now we use $h = 1/128$ of a month, which implies rebalancing the hedge about every six hours, day and night, every day. Figure C.4 shows the results. One can see that the results now correspond very closely, but still not perfectly.

C.11.13 The results demonstrate that the investment strategy we have described, if it is applied according to the option formulae, and if hedging is sufficiently frequent, does indeed give results that correspond with the required payoff. This validates the option and hedging formulae that we have developed. However, it also demonstrates that, with any practical schedule of hedging, the results do not correspond perfectly, so that additional initial contingency reserves might be thought prudent. In addition, we have assumed so far that hedging can be carried out at no cost. To be more realistic, we need to take transaction costs into account. Finally, we have assumed that the real world investment in fact behaves in accordance with the stochastic processes we have defined and have used for the calculation of option values and hedging quantities. The true behaviour of the real world may actually be very different. If we take into account all of these, the distribution of hedging errors will, almost always, widen considerably, and the investment proceeds may well not even centre on the desired target.

C.12 *Numerical Results*

C.12.1 We now give some numerical results. With the parameters we have used, the value of the Maxi option is 108.14. The initial hedging strategy is to invest 108.14 in the share, 43.20 in the dla and −43.20 in the zcb. Of course the value of the option, and the initial hedging quantities vary enormously, depending on how far in or out of the money it is, the term, and the chosen parameters. It can be seen from Figure C.2 that in one simulation, with yearly hedging, the value of the proceeds, $Y^*(T)$, is negative. This occurred because the option was well in the money, so the proportions in zcb and dla were large; then in one year share prices fell, and interest rates moved so that the value of the zcb reduced by less than the value of the dla. Such a result is possible if hedging is sufficiently infrequent.

C.12.2 The most useful statistic to measure the success of hedging is the deficit at maturity, defined as $D^*(T) = (P(T) - Y(T))/S(T)$. Some statistics of $D^*(T)$ with the different hedging frequencies, using investment strategy (i), are shown in Table C.12.1.

Table C.12.1. Statistics of $D^*(T)$ with different frequencies of hedging.

Frequency	Mean	Standard deviation	Minimum	Maximum
yearly	0.43	14.04	−135.10	158.19
twice per month	0.02	2.63	−27.53	20.99
128 times pm	−0.01	0.33	−4.62	1.93

It can be seen that more frequent hedging reduces the hedging error, but not to zero. Note that these are based on only 1000 simulations, so do not give reliable extreme quantile measures.

C.12.3 The results with other investment strategies, (ii to v) above, with hedging twice per month, are shown in Table C.12.2.

Table C.12.2. Statistics of $D^*(T)$ with different hedging strategies.

Strategy	Mean	Standard deviation	Minimum	Maximum
(i)	0.018	2.63	−27.53	20.99
(ii)	0.018	2.60	−34.30	21.72
(iii)	0.016	1.10	−4.76	5.01
(iv)	0.011	1.06	−5.81	4.58
(v)	−0.463	39.24	−421.45	492.00

We see that the last strategy (v), which involves no hedging at all after the initial portfolio is set up, is quite unsatisfactory. Strategies (i) and (ii), where the right amount is put into the share with either the zcb or the dla getting the correct amount and the other getting the balance, give similar results. Strategies (iii), with the share taking the balance, and (iv), with the hedging proportions being maintained, give similar results and are distinctly better than either (i) or (ii). Note that this is in contrast with results we have found elsewhere for other types of option.

C.12.4 We next consider the present values of the deficit, when discounted at different rates, corresponding to the share, the zcb, the dla, or the option mix. We again assume hedging twice a month, now with strategy (iv). The results are shown in Table C.12.3.

Table C.12.3. Statistics of $D^*(T)$ discounted with different discount methods.

Discount as:	Mean	Standard deviation	Minimum	Maximum
Share	0.011	1.06	−5.81	4.58
Zcb	0.012	2.46	−19.22	14.53
Dla	0.014	2.29	−19.16	12.53
Option	0.025	1.04	−3.91	4.29

From this table we see that the discounted value has the least dispersion when the deficit is assumed to be discounted as if it had been invested in the same proportions as the option. Thus if a contingency reserve is set up, sufficient to meet say the 99% quantile or CTE, and that reserve is invested in the same proportions as the option, it will meet 99% of deficits. Further, the contingency reserve required will probably be smaller than if other investment strategies, either for the total investment, or for the contingency reserve had been adopted. Again, this is in contrast with what we have found elsewhere.

C.12.5 We now consider hedging over different periods, 10, 20, 30 and 40 years. We fix our 'standard' as hedging twice a month, with strategy (iv) and discounting by the option proceeds. The results are shown in Table C.12.4, in which we show the option price and the amount to be invested initially in the dla; the amount to go into the share is the same as the option price, and the amount to go into the zcb is the negative of the dla amount. It is interesting that the option price decreases as the term lengthens. We comment further on this in Section C.13. In spite of these low option prices,

Table C.12.4. Statistics of the option and of $D^*(T)$ for different terms.

Term (years)	Option price	Initial dla hedge	Mean	Standard deviation	Minimum	Maximum
10	108.26	54.45	0.002	0.93	−2.78	3.62
20	108.14	43.20	0.025	1.04	−3.91	4.29
30	105.37	27.24	0.062	1.15	−4.24	4.91
40	102.49	13.03	0.049	1.14	−4.42	4.25

the hedging process still meets the required payoff, which in money terms varies enormously, with quite small errors, which do not increase much with term.

C.12.6 We now consider various starting values for the fla, viz. 7, 8, 9, 10 and 11, from well out of the money to well into the money, all for term 20, with the standard assumptions. The results are shown in Table C.12.5. Naturally the option price increases with the value of F(0). The hedging errors also increase somewhat with F(0).

Table C.12.5. Statistics of the option and of $D^*(T)$ for different values of F(0).

F(0)	Option price	Initial dla hedge	Mean	Standard deviation	Minimum	Maximum
7	101.68	12.28	0.025	0.84	−3.41	3.35
8	104.13	25.65	0.025	0.93	−3.67	3.77
9	108.14	43.20	0.025	1.04	−3.91	4.29
10	113.70	62.85	0.036	1.19	−4.22	4.89
11	120.63	82.67	0.049	1.35	−4.91	5.54

C.12.7 We could go on with this exercise, showing for example the results of allowing for transaction costs. It is obvious that allowing for transaction costs on purchases or sales of investments increases the deficit, and that this increase is generally greater the more frequent is the hedging. In other experiments we have found that hedging twice per month achieves the optimum balance between keeping the hedging error down, and not incurring too great costs, but this would depend very much on the level of transaction costs assumed. We have assumed hedging at regular time intervals. Another strategy, described by Boyle and Hardy (1997) as a 'move-based' strategy, is to look at the position at frequent time intervals, but to rebalance the portfolio only when it diverges from the desired position by more than some threshold amount. This, too, is worth investigating.

C.12.8 We could also show the results of setting the values of the 'real world' parameters different from those assumed for calculating the option prices. If the 'wrong' parameter values are used for the option pricing and hedging, in general the mean value of the hedging error is altered, possibly in either direction, and sometimes severely.

C.13 *Properties of the Option Pricing Formula*

C.13.1 The formula for the option price, shown in C.9.14, bears some resemblance to the formula for a maxi option on an ordinary share based on the usual Black-Scholes formula, but it has an important difference, the covariance term that enters into the calculation of G, and hence into the calculation of d_1 and d_2. This is affected by two correlation coefficients, that between shares and the fla, ρ_{13}, and that between the fla and the zcb rate, ρ_{12}. The latter has a bigger effect the longer the term, because of the $(T - t)^2$ factor. It is likely that the value of the fla and the zcb rate are very negatively correlated, because both depend on interest rates, though at different parts of the term structure. A high negative value for ρ_{12} reduces the value of the option, especially at longer terms. In effect, during the hedging process, the values of the holdings of the zcb and the dla, one short and one long, offset one another to a great extent over each short interval. On the other hand, the value of the correlation coefficient between shares and the fla may well be positive, though probably smaller in absolute value than ρ_{12}, and acts to increase the value of the option.

C.13.2 Thus, with the other parameters as shown in C.11.8, if we set the two correlation coefficients to zero, the option price increases with term, from about £104 per £100 invested for term 1 to about £125 for term 40. If we set the value of ρ_{13} to 0.5, it increases from about £105 to about £160. If, instead, we set the value of ρ_{12} to -0.5 the option price is still about £104 for term 1, increases to over £107 at term 7, and then reduces to very close to £100 by term 40. If both changes are made, the maximum rises to over £111 by term 9, but thereafter the value falls to close to £100 with longer terms.

C.13.3 Obviously, if the guarantee amount, g, increases, the option price increases. Also, if the option is more in the money the price increases, and it reduces if it is more out of the money. However, with the other parameters as in C.9.14, when the option is in the money, the option price still rises to a maximum and then falls for longer terms. It has to be very far into the money before the price at term 40 increases much. If the value of ρ_{12} is altered to -0.9, as we use in experiments with the Wilkie model, the long term option value remains close to £100 even if we increase the value of the fla to the ridiculous value of 50. Note that, if interest rates remain positive, an annuity value is never greater than the expectation of life at the commencement age. Even with the lightest mortality table we use, PMA92B1977, the value of an annuity due at age 65 at 0% does not go above 22.1.

C.13.4 Provided ρ_{13} is positive, an increase in the standard deviation of the share price, σ_S, increases the value of the option. If ρ_{13} were negative, it would have the opposite effect, and if ρ_{13} is zero, σ_S has no effect at all. Because ρ_{12} is almost certainly negative, an increase in the value of the standard deviation of the zcb rate, σ_R, has the apparently perverse effect of reducing the option price.

C.13.5 The effect of the standard deviation of the fla, σ_F, seems to be more complicated. If both correlation coefficients are zero, then an increase in σ_F results in an increase in the option price. However, with ρ_{12} negative and ρ_{13} positive, a small increase in σ_F can result in a decrease in the option price at all durations, whereas a large increase in σ_F can produce an increase in the option price at short durations but a decrease at longer durations.

The pattern deserves further investigation, but with the relatively large number of parameters, generalisations may be misleading.

C.14 *Deflators*

C.14.1 A number of actuarial papers recently have advocated the use of deflators as a way of calculating option prices through simulation with the 'real world' model. The deflator for any model is readily calculated as what is technically known as the 'Radon-Nikodym derivative', expressed in terms of the numeraire. In our model the deflator is therefore (see Baxter & Rennie, 1996, page 186):

$$\text{Deflator}(0) = \text{B}(0).\exp\left[-\Sigma_{i=1,n}\int_0^{\text{T}}\gamma_i(t)\text{dZ}_i(t) - \int_0^{\text{T}}|\gamma(t)|^2\text{d}t\right]$$

C.14.2 We had only two γs, viz:

$$\gamma_1(t) = \mu_{\text{F}}(\,)/\sigma_{\text{F}} = [\alpha_{\text{F}}(\theta_{\text{F}} - \log \text{F}(t)) + \tfrac{1}{2}\sigma_{\text{F}}^2]/\sigma_{\text{F}}$$

$$\begin{aligned}\gamma_4(t) &= [(\text{T}-t).\mu_{\text{R}}(\,) + \mu_{\text{S}}(\,) + \tfrac{1}{2}\sigma_{\text{R}}^2(\text{T}-t)^2 - \text{R}(t) + (\text{T}-t).\sigma_{\text{S}}.\sigma_{\text{R}}.\rho_{23} \\ &\quad - \{(\text{T}-t).\sigma_{\text{R}}.c_{21} + \sigma_{\text{S}}.c_{31}\}.\mu_{\text{F}}(\,)/\sigma_{\text{F}}]/\sigma_4(t) \\ &= [(\text{T}-t).\alpha_R(\theta_{\text{R}} - \text{R}(t)) + \mu_{\text{S}} + \tfrac{1}{2}\sigma_{\text{R}}^2(\text{T}-t)^2 - \text{R}(t) + (\text{T}-t).\sigma_{\text{S}}.\sigma_{\text{R}}.\rho_{23} \\ &\quad - \{(\text{T}-t).\sigma_{\text{R}}.c_{21} + \sigma_{\text{S}}.c_{31}\}.\mu_{\text{F}}(\,)/\sigma_{\text{F}}]/\sigma_4(t)\end{aligned}$$

so:

$$\text{Deflator}(0) = \text{B}(0).\exp\left[-\int_0^{\text{T}}\gamma_1(t)\text{dZ}_1(t) - \int_0^{\text{T}}\gamma_4(t)\text{dZ}_4(t) - \int_0^{\text{T}}(\gamma_1(t)^2 + \gamma_4(t)^2).\text{d}t\right]$$

But

$$\text{dZ}_4 = \{(\text{T}-t).\sigma_{\text{R}}.c_{22} + \sigma_{\text{S}}.c_{32}\}/\sigma_4(t).\text{dZ}_2 + \sigma_{\text{S}}.c_{33}/\sigma_4(t).\text{dZ}_3$$

with

$$\sigma_4(t)^2 = (\text{T}-t)^2.c_{22}^2.\sigma_{\text{R}}^2 + 2(\text{T}-t).c_{22}.c_{32}.\sigma_{\text{R}}.\sigma_{\text{S}} + (c_{32}^2 + c_{33}^2).\sigma_{\text{S}}^2$$

C.14.3 This does not seem to be analytically tractable. However, within any simulation, it can be calculated approximately numerically, by simulating with suitably small step sizes and summing. In our view the deflator method has no advantages over the much more direct analytical method which we have used. Indeed to use the deflator method to carry out the empirical hedging calculations as we have done in Section C.11 would be quite impracticable.

C.15 *Final Comments*

C.15.1 This Appendix has been lengthy, and we have intentionally shown almost all our working. We wished to demonstrate to actuaries that discovering option pricing formulae and the corresponding hedging strategies is, in many cases, not difficult, although it may be tedious and it always needs care. One requires to be familiar with the practical methodology, and also to be able to use certain mathematical techniques.

C.15.2 The practical steps are generally straightforward:
— decide on the tradeable assets (C.5.2 and C.5.3);
— define stochastic derivatives for those assets, or for suitable functions of them (C.5 and C.6);
— choose a suitable numeraire, and discount the tradeables in relation to that numeraire (C.7);
— define the 'shadow' martingale functions, corresponding to the discounted tradeables (C.8);
— express the option payoff in terms of the discounted tradeables (C.9.1);
— calculate the expected value of the payoff, usually as functions of lognormally distributed variables, to get the value of the option (C.9);
— differentiate the option price with respect to time and equate the derivatives to get the hedging proportions (C.10).

C.15.3 The mathematical techniques required are also straightforward. Besides ordinary calculus and the properties of normal and lognormal distributions, one needs to know:
— Ito's formula (C.6.1);
— the product rule for differentiating $X(t).Y(t)$ (C.7.2);
— that if $X(T) = \int_0^T x(t).dZ_1$, and $x(t)$ is deterministic, then $X(T)$ is normally distributed with mean zero and variance $\int_0^T x(t)^2.dt$ (C9.4);
— if further $Y(T) = \int_0^T y(t).dZ_1$, and $y(t)$ is deterministic, then the covariance of $X(T)$ and $Y(T)$ is $\int_0^T x(t).y(t).dt$ (C9.6).

It is also useful to have at hand the result in relation to bivariate lognormal distributions quoted in C9.6, which corresponds to a similar result for a univariate lognormal that can easily be derived from it.

C.15.4 It is desirable to arrange matters so that functions are normally or lognormally distributed, and also so that the $x()$ and $y()$ functions noted above are deterministic. The deflator methodology (C.13) may break down because of this latter point.

APPENDIX D

STOCHASTIC BRIDGES

D.1 If a stochastic model, such as the Wilkie model, is defined only for steps at annual (or other discrete) intervals, and it is desired to use the model to simulate values of the variables at more frequent intervals, such as monthly, one method of doing this is to use stochastic bridges, as a method of stochastic interpolation. The 'Brownian bridge' is familiar in the literature. We describe it, and introduce also the 'Ornstein-Uhlenbeck (OU) bridge', which we have not seen referred to elsewhere, though the principles are not difficult, and we claim no originality for this method.

D.2 Imagine that we have simulated values of some variable X_t at integral values of t. We now wish to interpolate over shorter intervals, with say n short intervals per unit. We consider only the period from t to $t+1$. All other longer intervals are dealt with in the same way. It is convenient to denote the new values as x_0 to x_n, with $x_0 = X_t$ and $x_n = X_{t+1}$. We first show how a Brownian bridge can be constructed. The principle of a Brownian bridge is that out of all possible paths from t to $t+1$ starting at X_t, we select only the subset that start at X_t and end at X_{t+1}; this is a subset consisting of all possible bridges; we then select one of those bridges at random.

D.3 We use the terms 'years' and 'months' to denote the longer and shorter periods, with n months per year. Of course these can be any suitable periods, and n can have any integral value, not necessarily 12. We start by choosing a value for the monthly standard deviation, say σ_m. If the original model for X_t is a discrete random walk with standard deviation σ_y then a sensible and consistent choice for σ_m is to put $\sigma_m = \sigma_y/\sqrt{n}$, but this is not essential. We then generate n random unit normal variables, z_1 to z_n, multiply each by σ_m to get e_1 to e_n, with $e_j = \sigma_m \times z_j$, $j = 1$ to n, and calculate the sum of the e_js:

$$S = \Sigma_{j=1,n} e_j.$$

We then compare $x_0 + S$ with the 'target' X_{t+1}, calculate the difference, and divide by n to give an adjustment, d:

$$d = \{X_{t+1} - (x_0 + S)\}/n.$$

D.4 We then add the adjustment to each value of e_j to give $f_j = e_j + d$, and calculate each x_j as:

$$x_j = x_{j-1} + f_j.$$

We can see that now $x_n = x_0 + \Sigma_{j=1,n} f_j = x_0 + \Sigma_{j=1,n}(e_j + d) = x_0 + \Sigma_{j=1,n} e_j + nd = x_0 + S + nd = X_{t+1}$, as desired.

D.5 An OU Bridge is useful if the annual series has been generated from a first order autoregressive, AR(1) model, or can be treated as if it has. Assume that the annual model is:

$$X_t = \mu_y + \alpha_y(X_{t-1} - \mu_y) + \sigma_y.z_t.$$

This would be equivalent to a monthly model of the same type:

$$x_j = \mu_m + \alpha_m(x_{j-1} - \mu_m) + \sigma_m.z_j,$$

with $\mu_m = \mu_y = \mu$, $\alpha_m = \alpha_y^{1/n}$ and $\sigma_m = \sigma_y.\sqrt{\{(1 - \alpha_m^2)/(1 - \alpha_y^2)\}}$. We shall therefore simulate the values of x_1 to x_{n-1} using this monthly AR(1) model.

D.6 We start as before by generating n random unit normal variables, z_1 to z_n. We then multiply each by σ_m to get e_1 to e_n, and calculate:

$$S_1 = \Sigma_{j=1,n} \alpha_m^{n-j} e_j$$
$$S_2 = \Sigma_{j=1,n} \alpha_m^{n-j} = (1 - \alpha_y)/(1 - \alpha_m)$$

We then calculate the adjustment, d, as:

$$d = \{(X_{t+1} - \mu) - \alpha_y(x_0 - \mu) - S_1)\}/S_2.$$

D.7 We then add the adjustment, as before, to each value of e_j to give $f_j = e_j + d$, and now calculate each x_j as:

$$x_j = \mu_m + \alpha_m(x_{j-1} - \mu_m) + \sigma_m.z_j + f_j.$$

We shall find, as before, that $x_n = X_{t+1}$.

D.8 Although it is consistent to relate the monthly parameters, σ_m in the case of the Brownian bridge, and μ_m, α_m and σ_m for the OU bridge, to the corresponding yearly parameters, especially if the annual model has been generated as a random walk or as an AR(1) model, it is not essential to do so. Either model may be useful for interpolating when the annual model is a more complicated one that cannot easily be replicated by a corresponding monthly model, but which nevertheless can be treated over the course of the year either as a local random walk or as a local AR(1) model, as we have done in practice for the Wilkie model.