Presented to the Staple Inn Actuarial Society

on 14th March 1995

A REVIEW OF WILKIE'S STOCHASTIC INVESTMENT MODEL

by

Paul Huber

Presented at a joint meeting with the
Royal Statistical Society, General Applications Section

ABSTRACT

This paper reviews the stochastic investment model developed by Wilkie (1984, 1986). This model's four component models are described and analysed from a statistical perspective. The distributions of their predicted values are derived and potential problems with the model's structure are discussed. Suggestions are made for the future development of actuarial stochastic investment models.

The paper shows that Wilkie's model does not provide a particularly good description of the data. The retail prices index model does not appear to correctly allow for the apparent non-stationarity and shocks in the inflation data. These features seem to contribute towards a spurious autoregressive effect in the retail prices index model, and inappropriate transfer functions with the retail prices index in the share dividend yield and share dividend index models. Both the share dividend index and Consols yield models appear to be over-parameterised.

KEYWORDS

Stochastic Investment Models; Wilkie's Model; Financial Time Series

1. INTRODUCTION

Although Wilkie's model (Wilkie, 1984, 1986) appears to have become the standard actuarial stochastic investment model in the United Kingdom (e.g. Daykin and Hey, 1990; Ross, 1989), only a small part of it (the retail prices index model) has been reviewed from a statistical perspective (e.g. Geoghegan *et al.*, 1992; Kitts, 1988, 1990). Geoghegan *et al.* (1992: 179) concluded that "... there was little evidence to suggest that a better fitting parsimonious model could be estimated using standard Box-Jenkins methodology." This paper reviews Wilkie's entire model from a statistical perspective and provides evidence that challenges this conclusion.

Wilkie's model is composed of four connected models, a retail prices index model, a share dividend yield model, a share dividend index model, and a Consols yield model. These models are analysed consecutively by assessing their structure, the data on which they were based and their predicted values over the period 1983-93.

Each model's structure is first examined by describing the distribution of the predicted values, and by assessing the appropriateness of the transformations made to the raw data and the significance of the parameters. The criteria used to examine the transformations include whether they result in a stationary series, whether they are compatible with each other, whether they have a meaningful interpretation, and whether they prevent inadmissible values from occurring. The significance of each parameter is assessed by analysing its sensitivity to various features in the data.

The data on which the model was based is then described. Although the parameter estimates are conditional on the data over the period 1661-1919, the model was only fitted to the data over the period 1919-82. Therefore, this section will concentrate on the data after 1919. A number of problems with this data set are reported and each model is refitted to a corrected data set.

Each model is then used to calculate one year ahead predicted values over the period 1983-93 and goodness-of-fit tests are carried out on the resulting residuals.

Finally, the overall stability and suitability of Wilkie's model is discussed and suggestions for an alternative model are given.

The SAS computer package was used to perform all the calculations.

2. RETAIL PRICES INDEX MODEL

The retail prices index model is defined by the following equation (for $t > 0$):

$$\nabla \log_e Q(t) = QMU + QA \times \left(\nabla \log_e Q(t-1) - QMU\right) + QSD.QZ(t)$$

where $Q(t)$ is a retail prices index, ∇ represents the backwards difference operator, and $QZ(t)$ is a sequence of independently distributed unit normal random variables.

Parameter values for the full standard basis and "neutral" initial conditions are:

$QMU = 0.05$, $QA = 0.6$, $QSD = 0.05$, $\nabla \log_e Q(0) = QMU$.

This model has attracted all the attention in the other reviews of Wilkie's model (Kitts, 1988, 1990; Geoghegan *et al.*, 1992). These reviews generally criticised this model for failing to explicitly take into account "... the existence of bursts of inflation

... the existence of large, irregular shocks ... the possible non-normality of residuals ..." (Geoghegan *et al.*, 1992: 179). This section illustrates the extent to which these features were taken into account by the retail prices index model. The conclusions reached are similar to those in the above-mentioned reviews, but are arrived at using a different approach.

2.1 THE STRUCTURE OF THE MODEL

2.1.1 The distribution of the predicted values

As shown by Kitts (1988) and Hürlimann (1992), the distribution of the predicted values of the retail prices index model is (for $t, k > 0$ and $QA \neq \pm 1$):

$$\nabla \log_e Q(t+k|t) \sim \mathrm{N}\left(QMU + QA^k \times \left(\nabla \log_e Q(t) - QMU\right), \frac{QSD^2 \times \left(1 - QA^{2k}\right)}{\left(1 - QA^2\right)} \right)$$

For $t, k > 0$ and $QA = 1$:

$$\nabla \log_e Q(t+k|t) \sim \mathrm{N}\left(\nabla \log_e Q(t), k \times QSD^2\right)$$

Therefore, the predicted values of the force of inflation have a mean and variance that tend to QMU and $QSD^2/(1-QA^2)$ respectively (for $-1 < QA < 1$) at a rate determined by the value of QA. The fluctuations about the mean are equally likely to be positive or negative. The predicted rate of inflation has a lognormal distribution. From a neutral starting position a 95 percent prediction interval for the force of inflation in the following year, using the full standard basis, is (–0.05, 0.15). As this interval is relatively wide, the model provides little information on the level of future inflation rates.

2.1.2 The transformation

The retail prices index was transformed into a series of the force of inflation. This transformation has a meaningful interpretation and prevents inadmissible values from occurring, but does not appear to result in a stationary series. The mean and standard deviation of the force of inflation changes from –0.025 and 0.076 over the period 1919-33, to 0.039 and 0.035 over the period 1934-73, and to 0.138 and 0.052 over the period 1974-82 (see figure 1, that plots Wilkie's original data). This observation is supported by Wilkie's finding (Appendix B of Geoghegan *et al.*, 1992) that the force of inflation series is heteroskedastic and various other econometric papers (Franses and Paap, 1994; Osborn, 1990). The following section shows how this important feature influenced the retail prices index model.

2.1.3 The significance of the parameters

The retail prices index model expresses the force of inflation in any year as a linear function of the force of inflation in the previous year. The slope of this function is QA. Figure 2 illustrates the significance of this relationship by plotting the force of inflation in each year (minus QMU) against the force of inflation in the previous year (minus QMU), and the retail prices index model. The residuals in every year are equal to the vertical distance between these points and the line.

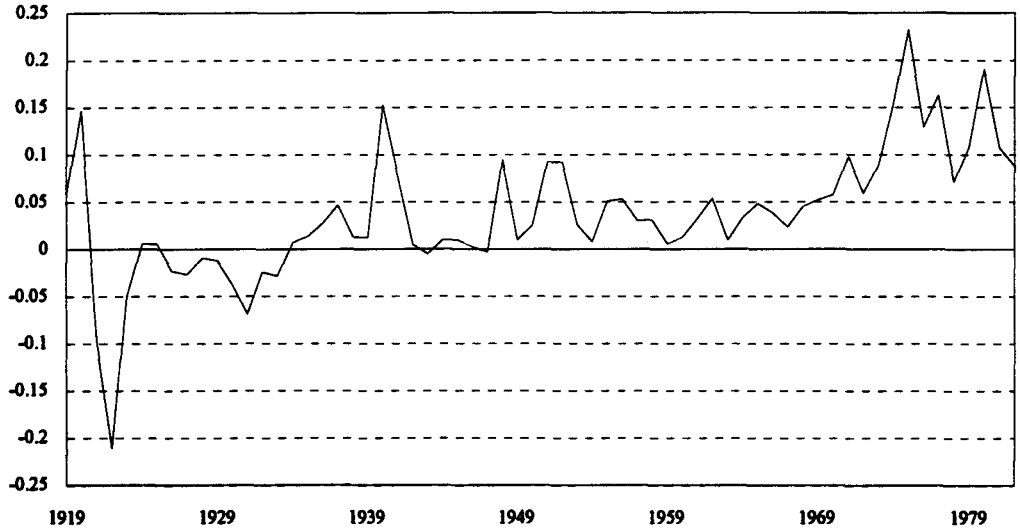

Figure 1. The force of inflation, $\nabla \log_e Q(t)$.

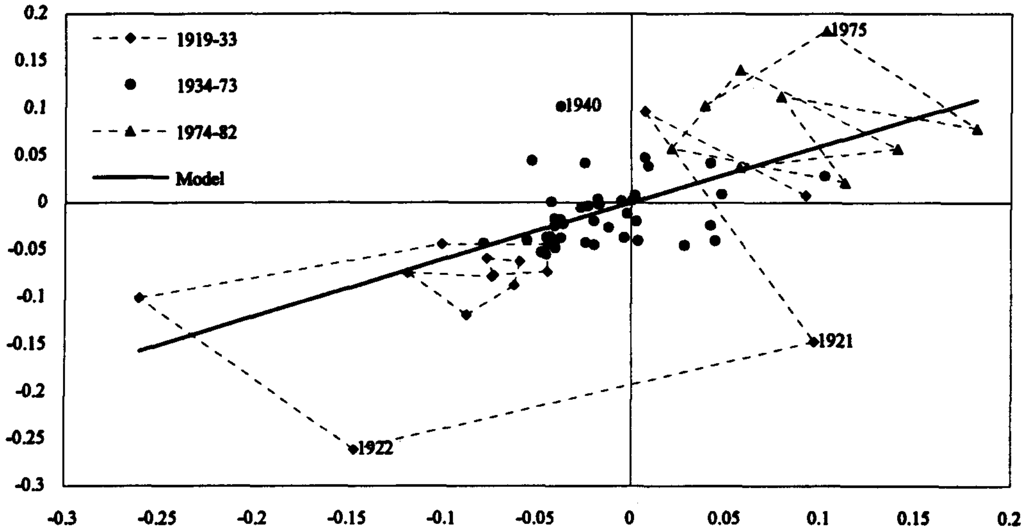

Figure 2. $\nabla \log_e Q(t) - QMU$ plotted against $\nabla \log_e Q(t-1) - QMU$.

Figure 2 shows that the model provides a reasonable fit to the data over the period 1919-82, but not over the three sub-periods (which are represented separately in figure 2). The parameter estimates for the period 1934-73 are presented in table 1. (The other two sub-periods are too short for any meaningful parameter estimates to be calculated.) This shows that *QA* is not significantly different from zero over this central sub-period, which contradicts Wilkie's remark that "[t]here is fairly little uncertainty about the appropriate value[s] for *QA* ..." (Wilkie, 1986: 346).

Table 1. *Estimated parameters for the retail prices index model, 1934-73*

QA	*QMU*	*QSD*
0.2545	0.0395	0.0331
(0.1541)	(0.0073)	

Therefore, the significance of *QA* appears to be dependent on the changes in mean that occurred around 1934 and 1974; probably referred to as "... the existence of bursts of inflation ...", by Geoghegan *et al.* (1992).

If a non-stationary mean was the only feature of the data, then *QA* would be expected to equal one. However, assuming the residuals are independent and normally distributed, *QA* is significantly less than one (Wilkie (1984) estimates *QA* as 0.5976, with a standard error of 0.0985). As the residuals are clearly not normally distributed (The skewness coefficient, $\beta_1^{1/2} = -0.49$ and the kurtosis coefficient, $\beta_2 = 6.07$), further analysis is required before the hypothesis, that the force of inflation series has a non-stationary mean, can be rejected.

An important feature contributing to the non-normality of the residuals is the sharp changes in the force of inflation; referred to as "... the existence of large, irregular shocks ...", by Geoghegan *et al.* (1992). These shocks have a relatively large influence on the regression and, taken on their own, would be expected to result in a value of zero for *QA*. Therefore, it is likely that these shocks mask the non-stationarity in the data. Excluding the shocks in the years 1920-23, 1940-41, 1948, and 1951-52 from the regression, over the period 1919-82, results in an estimate of 0.8 for *QA* with a standard error of 0.09. This supports the hypothesis that the force of inflation series has a non-stationary mean. It is difficult to arrive at any definite conclusions because the shocks over the period 1974-82 cannot be excluded without eliminating the entire period, which removes the major source of the non-stationarity.

Therefore, the retail prices index model appears to "average" the effects of a non-stationary mean and shocks in the data. This results in a model that does not produce either future changes in the mean or future shocks, rather it produces a spurious tendency for the series to revert to the mean.

Using a stationary model to describe non-stationary data makes it very difficult to determine appropriate values for *QMU* and *QSD* because the data provides a number of alternatives depending on the period considered. According to Wilkie (1986: 346), there is "... considerable uncertainty about the value to use for *QMU*, where anything between 0.04 and 0.10 might be justifiable ..."

2.2 THE DATA

2.2.1 Description of the data

The inflation index, used in Wilkie (1984), was constructed by linking the Schumpeter-Gilboy Consumers' Goods Index A and B (1661-1790), the Gayer, Rostow and Schwarz Domestic and Imported Commodities Index (1790-1850), the Rousseaux Overall Price Index (1850-1871), the Board of Trade Wholesale Price Index (1871-1914), the Cost of Living Index (1914-1947), the Interim Index of Retail Prices (1947-1956), and the General Index of Retail Prices (1956-1982).

The data for the earlier indices, the Cost of Living Index and the Retail Prices Index over the period 1947-61, was obtained from Mitchell (1962) and Mitchell and Jones (1971). This data represents the annual average values of these indices (not June values as was intended), which may have the effect of inducing a spurious moving average effect, see: Working (1960). From 1962 the inflation index was constructed from the June Retail Prices Index values.

Most of the earlier indices are of doubtful relevance to the modelling of future inflation rates because they do not measure changes in the general level of retail prices. The Gayer, Rostow and Schwarz index is based on the prices of commodities and the Board of Trade index is based on the prices of wholesale goods. These indices also tend to have a very narrow coverage of goods.

The Cost of Living Index used constant weights that were based on a working class family budget enquiry made in 1904. This index was dominated by the food category that made up 60 percent of the index. (Comparable weights for the Interim Index in 1947 and the General Index in 1994 are 35 percent and 14 percent respectively.) Allen (1948) questioned the appropriateness of this index, because it had a very narrow coverage and tended to concentrate on items that were subsidised during World War II. Using weights based on a working class family budget enquiry made in 1937-38, Allen estimated that the index would have increased by approximately 60 percent over the period 1938-47 compared with an increase of approximately 30 percent in the official figures. For the above reasons, it is probably not suitable to use this index in the construction of a retail prices index model.

The Interim Index of Retail Prices used weights that were based on a family budget enquiry made in 1937-38 (covering workers with incomes not exceeding £250 per annum). Minor modifications were made to these weights in January 1952. As this index did not cover all types of households, it is not strictly compatible with the General Index of Retail Prices.

The results of a comprehensive family budget enquiry made in 1953-54, covering all types of households except for pensioners and the extremely wealthy, were used to determine the initial weights of the General Index of Retail Prices. This index's weights have been updated annually since 1962 based on the results of family expenditure surveys for the three years ending in June in the previous year.

2.2.2 Refitting the model to the corrected data

A corrected data set was constructed using month-end index values rather than annual averages. The inflation rates before 1947 were calculated using July index values (rather than June values) because the Cost of Living index was calculated at the beginning of every month. Unconditional maximum likelihood parameter estimates for this corrected data are not significantly different from those in Wilkie (1984), but QA is less stable than Wilkie's estimates imply. The revised estimates of QA are: 0.5031, 0.5040, and 0.6023, with standard errors of 0.1095, 0.1241, and 0.1327, over the periods 1919-82, 1933-82, and 1946-82, respectively. These estimates are also dependent on the particular month used (Franses and Paap, 1994). Over the period 1930-82, revised estimates of QA are: 0.5716 and 0.7416, with standard errors of 0.1142 and 0.0912, using June and December values respectively. The effect of using the maximum likelihood estimation technique as opposed to Wilkie's method (least squares conditional on all the available historical data), was negligible.

2.3 THE RESIDUALS OVER THE PERIOD 1983-93

Table 2 presents the one-step ahead residuals for Wilkie's retail prices index model over the period 1983-93. There are too few residuals for a detailed statistical analysis, but some general observations can be made. The standard deviation of the residuals is 0.0247, which is greater than the standard deviation of the actual data: 0.0237. Therefore, over the period 1983-93, a simpler model (with QA equal to zero) would have provided a better fit than the retail prices index model.

Table 2. *Residuals for the retail prices index model, 1983-93*

Year	*Actual*	*Predicted*	*Residual*
1983	0.0359	0.0726	-0.0367
1984	0.0501	0.0415	0.0086
1985	0.0673	0.0501	0.0172
1986	0.0247	0.0604	-0.0357
1987	0.0411	0.0348	0.0063
1988	0.0451	0.0447	0.0004
1989	0.0793	0.0471	0.0323
1990	0.0934	0.0676	0.0258
1991	0.0568	0.0761	-0.0193
1992	0.0380	0.0541	-0.0160
1993	0.0121	0.0428	-0.0307

The standard deviation of the residuals is also significantly less than QSD, which is to be expected because there were no major shocks over this period.

3. SHARE DIVIDEND YIELDS MODEL

The share dividend yields model is defined by the following equations (for $t > 0$):

$$\log_e Y(t) = YW.\nabla \log_e Q(t) + YN(t)$$

where: $$(1 - YA.\mathrm{B}) \times (YN(t) - \log_e YMU) = YE(t) = YSD.YZ(t)$$

where $Q(t)$ is a retail prices index, $Y(t)$ is a share dividend yield, ∇ represents the backwards difference operator, B represents the backwards step operator, and $YZ(t)$ is a sequence of independently distributed unit normal random variables.

Parameter values for the full standard basis and "neutral" initial conditions are:

$YW = 1.35$, $YMU = 0.04$, $YA = 0.6$, $YSD = 0.175$, $\nabla \log_e Q(0) = QMU$, $Y(0) = YMU.e^{YW.QMU}$.

3.1 THE STRUCTURE OF THE MODEL

3.1.1 The distribution of the predicted values

The distribution of the predicted values of the share dividend yields model is:

$$\log_e Y(t+k|t) \sim \mathrm{N}(\mu_y(t+k|t), \sigma_y^2(t+k|t)), \quad \text{for } t, k > 0$$

where (for $YA \neq \pm 1$ and $QA \neq \pm 1$):

$$\begin{aligned}\mu_y(t+k|t) = \log_e YMU + YW.QMU + QA^k.YW \times (\nabla \log_e Q(t) - QMU) \\ + YA^k \times (\log_e Y(t) - \log_e YMU - YW.\nabla \log_e Q(t))\end{aligned}$$

$$\sigma_y^2(t+k|t) = \frac{YSD^2 \times \left(1 - YA^{2k}\right)}{\left(1 - YA^2\right)} + \frac{(YW.QSD)^2 \times \left(1 - QA^{2k}\right)}{\left(1 - QA^2\right)}$$

The predicted share dividend yields have a lognormal distribution. Note that if $YA = QA$, then the force of inflation in year t has no influence on the mean, $\mu_y(t+k|t)$. From a neutral starting position a 95 percent prediction interval for the logarithm of the share dividend yield in the following year, using the full standard basis, is $(-3.52 = \log_e(0.03), -2.78 = \log_e(0.06))$. This interval is fairly wide, suggesting that this model is dominated by the error term.

3.1.2 The transformation

The share dividend yield data was transformed by taking logarithms. This transformation prevents inadmissible values from occurring and appears to produce a stationary time series, but its mean increases substantially over the period 1974-82 (see figure 3). The transformation does not have any meaningful interpretation and is incompatible with the retail prices index transformation. Taking logarithms of the dividend yield causes a change in yields from y_1 to y_2 to be as significant as a change from y_t to $y_t \times y_2 / y_1$, for any initial yield y_t. On the other hand, the inflation transformation causes a change in the inflation rate from e_1 to e_2 to be as significant as a change from e_t to $(1+e_t) \times (1+e_2)/(1+e_1) - 1$, for any initial inflation rate e_t. This reduces the significance (relative to the retail prices index transformation) of the high yields in 1920-21, 1940, and 1974-75, and increases the relative significance of the low yields in 1933-37 and 1943-47.

3.1.3 The significance of the parameters

The share dividend yield model can be represented as follows (for $t > 0$):

$$\begin{aligned} &\log_e Y(t) - \log_e YMU - YW.\nabla \log_e Q(t) \\ &\qquad = YA \times \left(\log_e Y(t-1) - \log_e YMU - YW.\nabla \log_e Q(t-1)\right) + YE(t) \end{aligned}$$

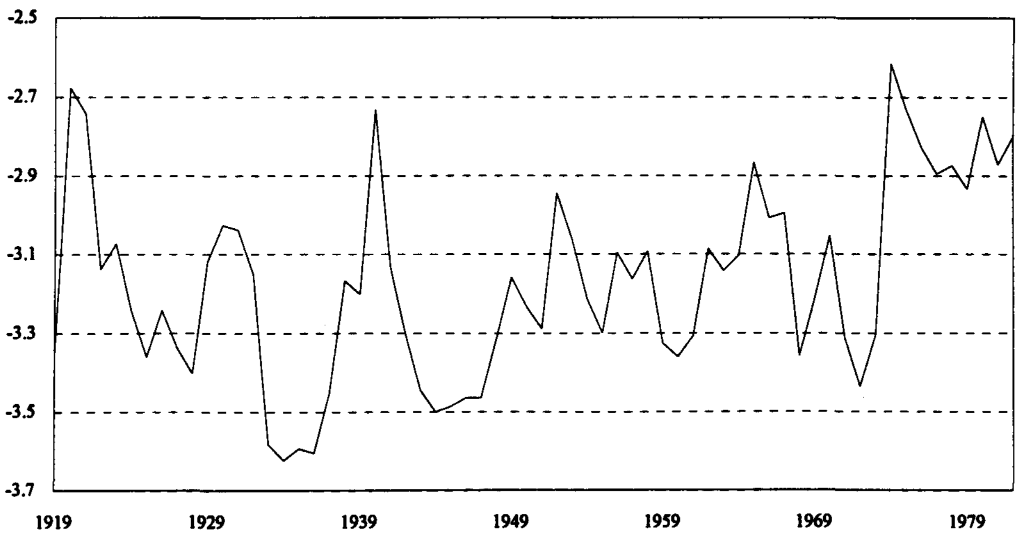

Figure 3. The log of the share dividend yield, $\log_e Y(t)$.

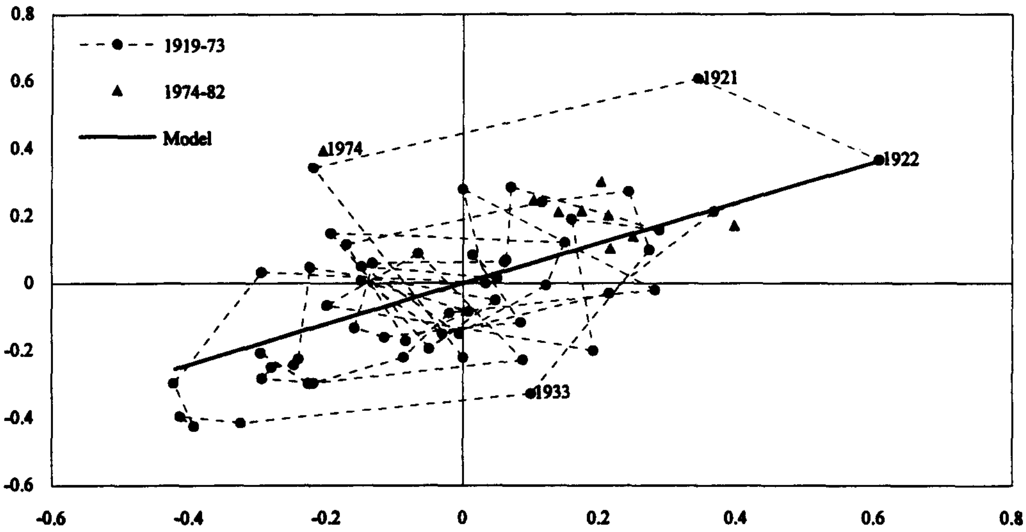

Figure 4. $\log_e Y(t) - \log_e YMU - YW . \nabla \log_e Q(t)$ plotted against $\log_e Y(t-1) - \log_e YMU - YW . \nabla \log_e Q(t-1)$

Using this representation, figure 4 shows that the autoregressive nature of the model is not related to the increase in yields and inflation over the period 1974-82. A negative value of *YA* would have been obtained over this period.

The share dividend yield model can also be represented as follows (for $t > 0$):

$$(1 - YA.\mathrm{B}) \times (\log_e Y(t) - \log_e YMU) = YW \times (1 - YA.\mathrm{B}) \times \nabla \log_e Q(t) + YE(t)$$

Using this representation, figure 5 illustrates the sensitivity of *YW* to the outliers in 1920-22, 1940, 1974-75, and 1980. This explains why "[t]he values of *YW* vary considerably according to the period chosen ..." (Wilkie, 1984: 58). Over the periods 1919-82, 1933-82, and 1946-82 Wilkie (1984) estimated *YW* as 1.35, 2.41, and 1.77 respectively.

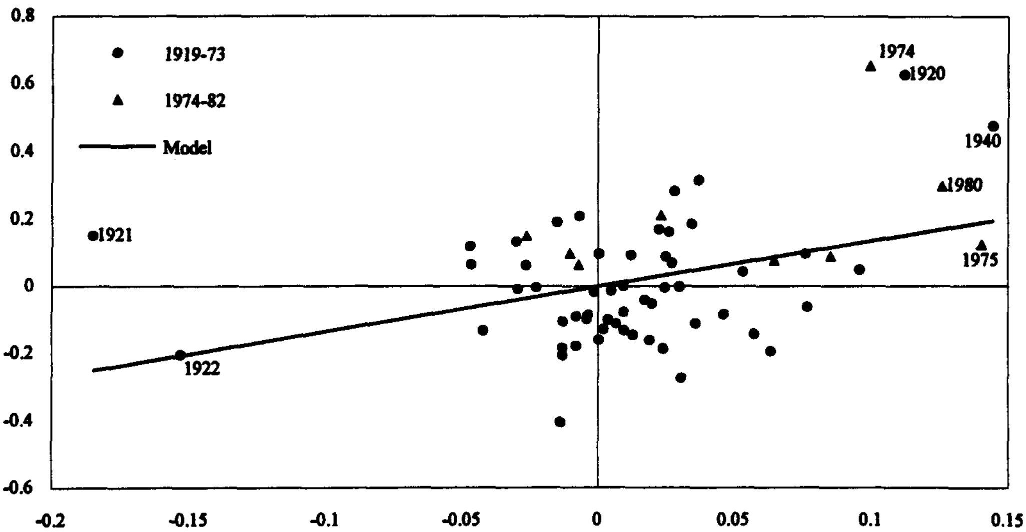

Figure 5. $(1 - YA . \mathrm{B}) \times (\log_e Y(t) - \log_e YMU)$ plotted against $(1 - YA . \mathrm{B}) \times \nabla \log_e Q(t)$.

These outliers all correspond to years in which inflation shocks occurred and the outliers in 1920, 1940, and 1974 correspond to years in which the greatest increases in yields occurred. If these outliers are excluded from the regression then *YW* becomes insignificantly different from zero. Therefore, *YW* does not describe a general tendency for changes in yields to be correlated with changes in inflation, but describes the tendency for large increases in yields to be correlated with inflation shocks. As the retail prices index model does not allow for these shocks (see section 2), *YW* should be set to zero for modelling purposes.

3.2 THE DATA

3.2.1 Description of the data

The share dividend yield data, used in Wilkie (1984), was obtained from the BZW equity index (1919-30), the Actuaries Industrials (All Classes Combined) Index (1931-53), the Second Series Actuaries Industrials (All Classes Combined) Index (1954-61), and the FT-Actuaries All Share Index (1962-82).

The data from the BZW index is only calculated at the end of every year (not in June as was assumed). If the BZW data is to be included then end of December values should be used.

There are a number of significant differences between these indices that may distort the true underlying relationships in the data. The FT-Actuaries index includes shares from all types of companies, whereas the others exclude financial company shares. The Actuaries indices are geometrically averaged indices (Haycocks and Plymen, 1956), whereas the others are arithmetically averaged indices. The BZW index was based on 30 shares; the Actuaries indices are based on roughly 150 shares; the FT-Actuaries index is currently based on roughly 850 shares (594 in 1962).

The Actuaries price indices have generally underperformed other equity price indices over similar time periods. Over the periods 1930-49, 1940-50, and 1950-60, the Actuaries price index increased by –48, 96, and 152 percent respectively, compared to increases of –35, 115, and 224 percent respectively in the *Investors Chronicle* equity price index (Haycocks and Plymen, 1964). (The *Investors Chronicle* index was an arithmetically averaged index based on roughly 100 shares.)

A significant event that should be allowed for when modelling share indices is that in November 1972, the government froze dividend payments. This initial freeze was changed to a maximum increase in dividends of 5 percent in March 1973, 12.5 percent in July 1974, and 10 percent in July 1975. The controls expired in July 1979.

3.2.2 Refitting the model to the corrected data

A corrected data set was constructed using both December and June values. Unconditional maximum likelihood parameter estimates for this data, using June values, are not significantly different from those reported in Wilkie (1984). When December values are used, *YW* is not significant. A revised estimate of *YW* is: 0.4017, with a standard error of 0.3937, over the period 1919-82 using December values.

3.3 THE RESIDUALS OVER THE PERIOD 1983-93

Table 3 presents the one-step ahead residuals for Wilkie's share dividend yield model over the period 1983-93.

Table 3. *Residuals for the share dividend yield model, 1983-93*

Year	*Actual*	*Predicted*	*Residual*
1983	-3.0879	-2.9892	-0.0986
1984	-3.0221	-3.1017	0.0796
1985	-3.0366	-3.0505	0.0140
1986	-3.2545	-3.1307	-0.1238
1987	-3.4933	-3.2047	-0.2886
1988	-3.1749	-3.3560	0.1811
1989	-3.1442	-3.1219	-0.0223
1990	-3.0534	-3.1122	0.0589
1991	-2.9838	-3.1186	0.1348
1992	-3.0241	-3.0725	0.0483
1993	-3.2493	-3.1165	-0.1329

The standard deviation of the residuals is 0.1370 which is not significantly less than the standard deviation of the actual data, 0.1496, at the 39 percent significance level using an F-test. Therefore, over the period 1983-93, a simpler model (with *YA* and *YW* equal to zero) would not have provided a significantly worse fit.

4. SHARE DIVIDEND INDEX MODEL

The share dividend index model is defined by the following equations (for $t > 0$):

$$\nabla \log_e D(t) = DM(t) + DX.\nabla \log_e Q(t) + DMU + DY.YE(t-1)$$
$$+ (1 + DB.\mathrm{B}) \times DSD.DZ(t)$$

where: $$DM(t) = DW \times \left(\frac{DD}{1-(1-DD)\times \mathrm{B}} \right) \times \nabla \log_e Q(t)$$

where $Q(t)$ is a retail prices index, $D(t)$ is a share dividend index, $YE(t)$ is obtained from the share dividend yield model, ∇ represents the backwards difference operator, B represents the backwards step operator, and $DZ(t)$ is a sequence of independently distributed unit normal random variables.

Parameter values for the full standard basis (reduced standard basis in brackets where these values differ) and "neutral" initial conditions are:

$DW = 0.8$, $DD = 0.2$, $DX = 0.2$, $DY = -0.2$ (-0.3), $DMU = 0$, $DB = 0.375$ (0), $DSD = 0.075$ (0.1), $YE(0) = 0$, $DM(0) = DW.QMU$, $DZ(0) = 0$.

4.1 THE STRUCTURE OF THE MODEL

4.1.1 The distribution of the predicted values

The distribution of the predicted values of the share dividend index model is:

$$\nabla \log_e D(t+k|t) \sim \mathrm{N}\left(\mu_d(t+k|t), \sigma_d^2(t+k|t)\right), \quad \text{for } t, k > 0$$

where, for $k = 1$:

$$\mu_d(t+1|t) = (DW.DD + DX) \times (QMU + QA \times (\nabla \log_e Q(t) - QMU))$$
$$+ DM(t) \times (1 - DD) + DMU + DY.YE(t) + DB.DSD.DZ(t)$$
$$\sigma_d^2(t+1|t) = DSD^2 + QSD^2 \times (DW.DD + DX)^2$$

For $k > 1$, $(1 - DD) \neq \pm 1$, $QA.(1 - DD) \neq 1$, $QA - (1 - DD) \neq 0$, and $QA \neq \pm 1$:

$$\mu_d(t+k|t) = DMU + QMU \times (DX + DW) + (DM(t) - DW.QMU) \times (1 - DD)^k$$
$$+ (\nabla \log_e Q(t) - QMU) \times (DX.QA^k + (\alpha - DX) \times (QA^k - (1 - DD)^k))$$

$$\sigma_d^2(t+k|t) = DSD^2 \times (1 + DB^2) + YSD^2.DY^2$$
$$+ QSD^2 \times \left(\alpha^2 \times \left(\frac{1 - QA^{2k}}{1 - QA^2} \right) - 2.\alpha.\beta \times \left(\frac{1 - (QA.(1 - DD))^k}{1 - QA.(1 - DD)} \right) + \beta^2 \times \left(\frac{1 - (1 - DD)^{2k}}{1 - (1 - DD)^2} \right) \right)$$

where: $\alpha = \dfrac{DW.DD.QA}{QA - (1 - DD)} + DX$, and $\beta = \dfrac{DW.DD.(1 - DD)}{QA - (1 - DD)}$

From a neutral starting position a 95 percent prediction interval for the force of growth of dividends in the following year, using the full standard basis, is $(-0.10, 0.20)$. The predicted growth of share dividends has a lognormal distribution.

4.1.2 The transformation

The share dividend index was transformed into a series of the force of change in the share dividend index (see figure 6). This transformation appears to be appropriate, but it is incompatible with the dividend yield transformation (see section 3.1.2).

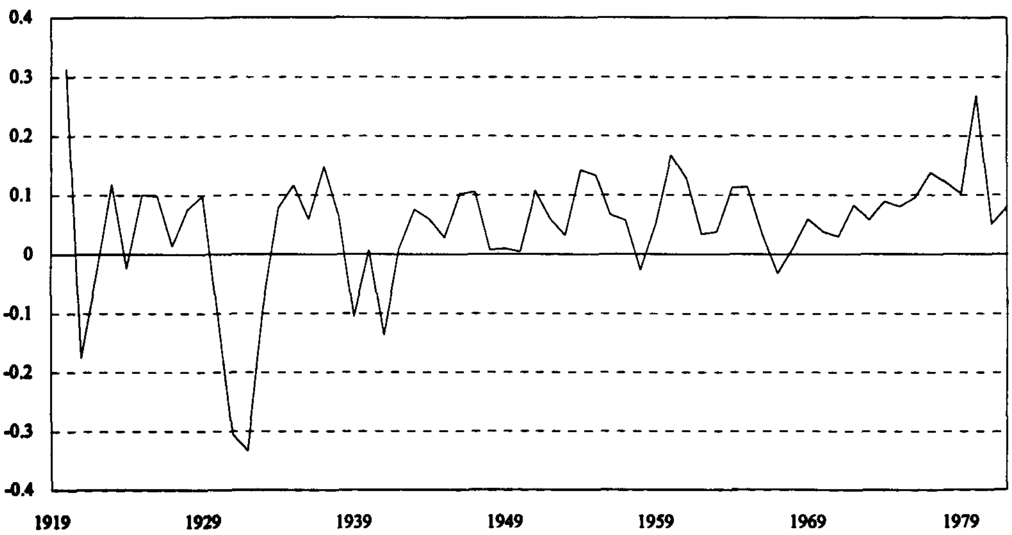

Figure 6. The force of increase in the share dividend index, $\nabla \log_e D(t)$.

4.1.3 The significance of the parameters

The share dividend index model is over-parameterised, even based on Wilkie's estimates (see table 4). Virtually all the parameters are not significantly different from zero over the periods 1933-82 and 1946-82. Over the period 1920-82, *DX* is not significantly different from zero and *DD*, *DW*, *DX*, and *DMU* are highly correlated with one another (see table 5). Surprisingly, Wilkie (1984, 1986) did not comment on this extremely poor fit. In addition, the standard errors of *DD* and *DW* were grossly under-estimated by Wilkie (1984). Revised estimates of these standard errors, over the period 1920-82, are: 0.1415 and 0.9503, respectively. These estimates imply that *DD*, *DW*, and *DX* are not significantly different from zero over all the periods considered.

Table 6 shows the parameter estimates obtained when excluding *DD*, *DW*, and *DMU*. As *DSD* does not increase significantly, these parameters appear to be superfluous. This suggests that lagged terms of $\nabla \log_e Q(t)$ contribute little additional information to the share dividend index model. The high correlation between *DMU* and, *DD* and *DW*, suggests that the term *DM(t)* is a measure of the mean of $\nabla \log_e D(t)$. After *DD* and *DW* were excluded, the term involving *DX* was found to provide a better measure of the mean than the constant *DMU*.

Table 4. *Estimated parameters for the dividend index model*

Period	*DD*	*DW*	*DX*	*DMU*	*DY*	*DB*	*DSD*
1920-82	0.1151	1.3240	0.3721	-0.0104	-0.2667	0.3931	0.0702
	(0.0582)	(0.5458)	(0.2145)	(0.0193)	(0.0527)	(0.1301)	
1933-82	0.0669	1.1529	0.3227	0.0037	-0.1766	0.3612	0.0543
	(0.1373)	(1.3968)	(0.2300)	(0.0250)	(0.0453)	(0.1662)	
1946-82	0.1167	0.6221	0.2258	0.0316	-0.1056	0.2846	0.0490
	(0.1634)	(0.6305)	(0.2465)	(0.0222)	(0.0494)	(0.1865)	

Source: Table 7.2 of Wilkie (1984)

Table 5. *Correlation matrix of the parameter estimates, 1920-82*

Parameter	*DD*	*DW*	*DX*	*DMU*	*DY*	*DB*
DD	1	-0.8220	-0.4738	0.7103	0.0026	0.0257
DW	-	1	0.0762	-0.8592	-0.1487	-0.0309
DX	-	-	1	-0.2249	0.2203	0.0160
DMU	-	-	-	1	0.1076	0.0317
DY	-	-	-	-	1	0.1981
DB	-	-	-	-	-	1

Table 6. *Estimated parameters for the dividend index model*

Period	*DD*	*DW*	*DX*	*DMU*	*DY*	*DB*	*DSD*
1920-82	-	-	0.8162	-	-0.2267	0.4778	0.0749
			(0.1550)		(0.0502)	(0.1156)	
1934-73	0.0214	1.9327	-0.2018	0.0278	-0.1855	0.3102	0.0503
	(0.1975)	(10.9230)	(0.2870)	(0.0357)	(0.0549)	(0.1643)	
1934-73	-	-	-	0.0460	-0.1707	0.3654	0.0511
				(0.0116)	(0.0508)	(0.1541)	

To assess whether the retail prices index transfer function was only included because the force of inflation has a non-stationary mean, the model was fitted both including and excluding this transfer function over the period 1934-73 (see table 6). These estimates illustrate that this transfer function is of no significance over this period in which the force of inflation is relatively stationary (see section 2.1.3). Therefore, the retail prices index transfer function appears to only measure the mean of $\nabla \log_e D(t)$. As Wilkie's model assumes a stationary mean force of inflation, the parameters *DW*, *DD*, and *DX* should be set to zero for modelling purposes.

The parameters *DY* and *DB* both appear to have a fairly meaningful role in the share dividend index model, but their values are influenced by outliers. The parameter *DB* is mainly affected by the outliers in the period 1920-42 and becomes only marginally significant once these outliers are taken into account (see table 4). (Note that setting *DB* to zero corresponds to the reduced standard basis.) Parameter estimates of $DY=-0.2$ and $DB=0.3$ appear to be appropriate after taking the outliers into account. The value and significance of *DY* are hardly affected if the errors from the share dividend yield model are calculated with *YW* set to zero.

4.2 THE DATA

4.2.1 Description of the data

The share dividend index data was obtained from the same sources as the share dividend yield data. Therefore, the comments made in section 3.2.1 are equally applicable to this section.

4.2.2 Refitting the model to the corrected data

A corrected data set was constructed using a share dividend index linked when the underlying dividend indices first overlap. Unconditional maximum likelihood parameter estimates for this corrected data were found to be not significantly different from those presented in Wilkie (1984).

4.3 THE RESIDUALS OVER THE PERIOD 1983-93

Table 7 presents the one-step ahead residuals for Wilkie's share dividend index model over the period 1983-93. The standard deviation of the residuals is 0.0590 which is not significantly less than the standard deviation of the actual data, 0.0735, at the 25 percent significance level using an F-test. Therefore, over the period 1983-93, Wilkie's share dividend index model does not provide a significantly better fit than a model that simply predicts the force of dividend growth by the mean force of dividend growth.

There is a significant cross-correlation between these residuals and the retail prices index model's residuals at a lag of zero and the share dividend yield model's residuals at a lag of 2. This suggests that the transfer function with the retail prices index model was incorrectly specified over this period. The significant cross-correlation with the yield residuals is mainly caused by the fall in yields in 1987. The average of the residuals is 0.0188, which is high but not significantly different from zero.

Table 7. *Residuals for the dividend index model, 1983-93*

Year	*Actual*	*Predicted*	*Residual*
1983	0.0625	0.0577	0.0048
1984	0.1267	0.1038	0.0229
1985	0.1852	0.0747	0.1105
1986	0.0966	0.1024	-0.0058
1987	0.1074	0.0844	0.0229
1988	0.1383	0.1255	0.0128
1989	0.1652	0.0372	0.1279
1990	0.1521	0.1283	0.0238
1991	0.0609	0.0633	-0.0025
1992	0.0063	0.0297	-0.0234
1993	-0.0620	0.0259	-0.0879

5. CONSOLS YIELD MODEL

The Consols yield model is defined by the following equations (for $t > 0$):

$$C(t) = CM(t) + CN(t)$$

where:
$$CM(t) = CW \times \left(\frac{CD}{1-(1-CD)\times \mathrm{B}} \right) \times \nabla \log_e Q(t)$$

$$\log_e CN(t) = \left(CA1.\mathrm{B} + CA2.\mathrm{B}^2 + CA3.\mathrm{B}^3\right) \times \left(\log_e CN(t) - \log_e CMU\right)$$
$$+ \log_e CMU + CY.YE(t) + CB.CI(t) + CSD.CZ(t)$$

where $Q(t)$ is a retail prices index, $C(t)$ is the Consols yield, $YE(t)$ is obtained from the share dividend yield model, ∇ represents the backwards difference operator, B represents the backwards step operator, $CZ(t)$ is a sequence of independently distributed unit normal random variables, and $CI(t)$ is an intervention variable for 1974.

A minimum value of 0.005 is postulated for $C(t)$.

Parameter values for the full standard basis (reduced standard basis in brackets where these values differ) and "neutral" initial conditions are:

$CW = 1$, $CD = 0.045$ (0.05), $CY = 0.06$ (0), $CMU = 0.035$, $CA1 = 1.2$ (0.91), $CA2 = -0.48$ (0), $CA3 = 0.2$ (0), $CB = 0$, $CSD = 0.14$ (0.165), $CM(0) = CW.QMU$, $CN(0) = CN(-1) = CN(-2) = CMU$.

5.1 THE STRUCTURE OF THE MODEL

5.1.1 The distribution of the predicted values

The distribution of the predicted values of the allowance for future inflation is:

$$CM(t+k|t) \sim \mathrm{N}\left(\mu_{cm}(t+k|t), \sigma^2_{cm}(t+k|t)\right), \quad \text{for } t, k > 0$$

where (for $(1-CD)\neq\pm1$, $QA.(1-CD)\neq1$, $QA-(1-CD)\neq0$, and $QA\neq\pm1$):

$$\mu_{cm}(t+k|t)=CW.QMU+\left(CM(t)-CW.QMU\right)\times(1-CD)^k$$
$$+\left(\nabla\log_e Q(t)-QMU\right)\times CD.CW.QA\times\left(\frac{QA^k-(1-CD)^k}{QA-(1-CD)}\right)$$

$$\sigma_{cm}^2(t+k|t)=QSD^2\times\left(\frac{CW.CD}{QA-(1-CD)}\right)^2\times\left(QA^2\times\left(\frac{1-QA^{2k}}{1-QA^2}\right)\right.$$
$$\left.-2.QA.(1-CD)\times\left(\frac{1-(QA.(1-CD))^k}{1-QA.(1-CD)}\right)+(1-CD)^2\times\left(\frac{1-(1-CD)^{2k}}{1-(1-CD)^2}\right)\right)$$

From a neutral starting position a 95 percent prediction interval for the allowance for expected future inflation in the following year, using the full standard basis, is $(0.0456, 0.0544)$. This interval is very narrow, suggesting that there is little uncertainty about the allowance for future inflation.

The distribution of the predicted values of the Consols real yield model is:

$$\log_e CN(t+k|t)\sim \mathrm{N}\left(\mu_{cn}(t+k|t),\sigma_{cn}^2(t+k|t)\right),\quad \text{for } t,k>0$$

where:

$$\mu_{cn}(t+k|t)=\phi_{1,k}\times\left(\log_e CN(t)-\log_e CMU\right)+\phi_{2,k}\times\left(\log_e CN(t-1)-\log_e CMU\right)$$
$$+\phi_{3,k}\times\left(\log_e CN(t-2)-\log_e CMU\right)+\log_e CMU$$

$$\sigma_{cn}^2(t+k|t)=\left((CY.YSD)^2+CSD^2\right)\times\sum_{n=0}^{k-1}\left(\phi_{1,n}\right)^2$$

where (for $k>0$, and $i=1$, 2, and 3):

$$\phi_{i,k}=CA1.\phi_{i,k-1}+CA2.\phi_{i,k-2}+CA3.\phi_{i,k-3}$$

$$\phi_{1,0}=1,\ \phi_{1,-1}=0,\ \phi_{1,-2}=0,\ \phi_{2,0}=0,\ \phi_{2,-1}=1,\ \phi_{2,-2}=0,\ \phi_{3,0}=0,\ \phi_{3,-1}=0,\ \phi_{3,-2}=1$$

For the full standard basis (for $k>0$, and $i=1$, 2, and 3):

$$\phi_{i,k}=\alpha_i.\lambda^k+\gamma^k\times\left(\beta_i.\cos(\theta.(k-1))+\delta_i.\sin(\theta.(k-1))\right)$$

where:

$\lambda=0.9143$, $\gamma=0.4677$, $\theta=1.2604$, $\alpha_1=1.0536$, $\beta_1=0.5062$, $\delta_1=0.2185$, $\alpha_2=-0.3010$, $\beta_2=-0.4378$, $\delta_2=-0.4566$, $\alpha_3=0.2305$, $\beta_3=-0.0229$, $\delta_3=0.2347$.

For the reduced standard basis (for $t,k>0$ and $CA1\neq\pm1$):

$$\phi_{1,k}=CA1^k,\text{ and }\phi_{2,k}=\phi_{3,k}=0$$

From a neutral starting position a 95 percent prediction interval for the logarithm of the Consols real yield in the following year, using the full standard basis, is $(-3.6276=\log_e(0.0266),-3.0772=\log_e(0.0461))$. The predicted Consols real yield has a lognormal distribution.

The predicted values of the Consols yield model have the following mean and variance (for $t,k>0$):

$$\mu_c(t+k|t)=\mu_{cm}(t+k|t)+e^{\mu_{cn}(t+k|t)+0.5\times\sigma^2_{cn}(t+k|t)}$$

$$\sigma^2_c(t+k|t)=\sigma^2_{cm}(t+k|t)+\left(e^{2\times\mu_{cn}(t+k|t)+\sigma^2_{cn}(t+k|t)}\right)\times\left(e^{\sigma^2_{cn}(t+k|t)}-1\right)$$

From a neutral starting position a 95 percent prediction interval for the Consols yield in the following year, using the full standard basis, is roughly (0.0746, 0.0961). This interval is also very narrow.

5.1.2 The transformation

The inflation component of the Consols model was modelled without any transformation and the logarithm of the real yield was modelled. This results in a non-linear model that cannot be put into an ARIMA format. These transformations do not satisfy any of the requirements for suitable transformations. Wilkie's model prevents negative real yields from occurring, but allows negative nominal yields to occur, whereas negative real yields are possible but negative nominal yields are not. The logarithm of the Consols yield series has a highly non-stationary mean (see figure 7).

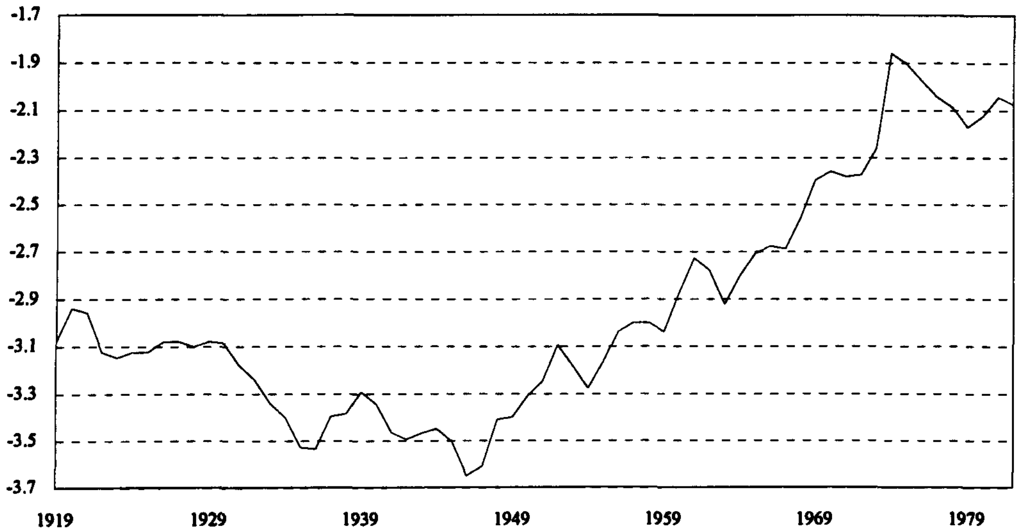

Figure 7. The log of the Consols yield, $\log_e C(t)$.

5.1.3 The significance of the parameters

Wilkie (1984) estimated the Consols yield model's parameters by setting *CW* to 1 and *CD* to a "plausible" value, estimating the other parameters so as to minimise *CSD*, and repeating this process, after adjusting *CD*, until *CSD* was minimised. To check whether Wilkie's estimates are optimal and to obtain an estimate of the standard error of *CD*, the Consols model was refitted with the parameter *CD* included in the fitting procedure (see table 8). (To prevent negative real yields, it was necessary to include the restriction: $0.0025 < CD < 0.0623$.) Table 8 shows that Wilkie's estimates are not optimal because they result in a higher residual standard deviation than the alternative estimates, and that *CD* is not significant. The parameter estimates for the model, excluding the term *CM(t)*, are also presented in table 8. These estimates provide an even better fit than those obtained including *CM(t)*, which was forced into the Consols yield model because of the constraints: $CW = 1$ and $0.0025 < CD < 0.0623$.

The parameter *CY* is not significant (see table 8) and consequently should be set to zero (as in the reduced standard basis).

The parameters *CA1* and *CA2*, and, *CA2* and *CA3* are highly correlated suggesting that the model is over-parameterised. This was noted by Wilkie (1984: 112), and allowed for in the reduced standard basis by setting *CA2* and *CA3* to zero. These parameters were not set to zero in the full standard basis because, according to Wilkie, this did not provide "... a satisfactory representation of the past ..." (Wilkie, 1984: 111). Surprisingly, this discrepancy was not pursued further.

Omitting *CA2* from the model, significantly increases *CSD* and decreases the significance of *CA3*. This implies that *CA1*, *CA2*, and *CA3* are functions of one another rather than that some of them are superfluous. An ARIMA model of the Consols yield series is likely to contain a unit root because it has a non-stationary mean. This implies that: $CA1 = 1 - CA2 - CA3$. Using this parameterisation, *CA2* and *CA3* still appear to be functions of one another. A suitable parameterisation for *CA2* was found to be: $CA2 = -2 \times CA3$.

Table 8. *Estimated parameters for the Consols model, 1919-82*

Parameter	*Estimate*	*Wilkie's Estimates*	*Setting CM(t) = 0*	
CMU	0.0363 (0.0035)	0.0355 (0.0085)	0.1471 (0.3439)	-
CA1	1.1882 (0.1163)	1.2019 (0.1064)	1.1943 (0.1155)	$1 + CA3$
CA2	-0.5048 (0.1744)	-0.4806 (0.1716)	-0.5014 (0.1727)	$-2 \times CA3$
CA3	0.2987 (0.1125)	0.2045 (0.1115)	0.2954 (0.1104)	0.2580 (0.0852)
CD	0.0124 (0.0082)	0.0450	-	-
CY	0.0600 (0.0792)	0.0649 (0.0633)	0.0475 (0.0671)	0.0258 (0.0629)
CB	0.4132 (0.1091)	0.4968 (0.1409)	0.3522 (0.0898)	0.3614 (0.0876)
CSD	0.0926	0.1288	0.0784	0.0796

Source: Table 8.6 of Wilkie (1984)

Assuming $CA1 = 1 + CA3$ and $CA2 = -2 \times CA3$, the Consols yield model can be represented as:

$$\nabla \log_e C(t) = CA3 \times \nabla^2 \log_e C(t) + CY.YE(t) + CB.CI(t) + CE(t), \quad \text{for } t > 0$$

This parameterisation represents the change in the Consols yield as a function of the acceleration of the Consols yield. Table 8 presents this model's parameter estimates (*CMU* is irrelevant because $CA1 + CA2 + CA3 = 1$). This parameterisation appears to be appropriate because it does not significantly increase *CSD*. The value of *CA3* (given in table 8) is highly influenced by the years 1974-76. An estimate of $CA3 = 0.35$ is appropriate after taking these outliers into account.

Therefore, it appears that the Consols yield model attempts to represent the above relationship rather than the more elaborate relationships in the actual model. As far as the reduced standard basis is concerned, it seems appropriate to set *CY* to zero but it does not appear to be appropriate to set *CA2* and *CA3* to zero.

5.2 THE DATA

5.2.1 Description of the data

The Consols yield series used in Wilkie (1984) was obtained from Mitchell (1962) over the period 1756-29, *The Actuaries' Investment Index* over the period 1935-61, the *FT-SE Actuaries Share Indices* over the period 1962-80, and the *Financial Times* over the period 1981-82. Over the period 1930-34, the Consols yields were supposedly obtained from *The Actuaries' Investment Index*, but the Consols yield was only reported in *The Actuaries' Investment Index* from the December 1933. The yield used for 1934 is not equal to the yield reported in *The Actuaries' Investment Index.*

The yields in Mitchell (1962) appear to represent the coupon divided by the annual average of the daily prices of the stock (not the running yield at the end of June as was intended).

5.2.2 Refitting the model to the corrected data

A corrected data set was constructed from the prices of 2.5% Consolidated Stock reported in the *Financial Times*. It was not possible to obtain exact maximum likelihood estimates for the Consols yield model because of the non-linear transformation. Therefore, the estimation method used in Wilkie (1984) (conditional least squares) was used to try to obtain parameter estimates for the corrected data. It was not possible to obtain a reasonable set of estimates because they were found to be highly dependent on *CM*(0), and to be highly correlated with one another. This confirms that the model is ill-conditioned and over-parameterised (see section 5.1).

5.3 THE RESIDUALS OVER THE PERIOD 1983-93

Table 9 presents the one-step ahead residuals for Wilkie's Consols yield model over the period 1983-93. The standard deviation of the residuals is 0.2973 which is greater than the standard deviation of the actual data, 0.2519. Therefore, over this period, the Consols yield model provides a worse fit than a model that simply predicts the Consols real yield by the mean Consols real yield. The standard deviation of the residuals is also significantly greater than *CSD.*

Table 9. *Residuals for the Consols yield model, 1983-93*

Year	*Actual*	*Predicted*	*Residual*
1983	-3.4985	-2.9894	-0.5092
1984	-3.2603	-3.6345	0.3742
1985	-3.3639	-3.0791	-0.2848
1986	-3.8058	-3.4471	-0.3588
1987	-3.6553	-3.8899	0.2346
1988	-3.5357	-3.4897	-0.0460
1989	-3.5152	-3.5190	0.0038
1990	-3.2303	-3.5168	0.2864
1991	-3.2714	-3.1564	-0.1150
1992	-3.5832	-3.3435	-0.2398
1993	-4.0611	-3.6518	-0.4093

There is a significantly large positive correlation between these residuals and the retail prices index model's residuals at a lag of zero. This suggests that the allowance of future inflation model was incorrectly specified over this period. The average of the residuals is –0.0967, which is low but not significantly different from zero.

6. CONCLUSION AND AREAS FOR FUTURE RESEARCH

Wilkie's stochastic investment model does not provide a particularly good description of the data and does not appear to be any better than a model that simply uses the means as predictors for the four transformed series.

The transformations used by Wilkie (1986) are incompatible with one another, do not all have meaningful interpretations, and permit negative Consols yields. These problems can be overcome in future models by transforming each asset class (including inflation) into a series of the force of growth of total returns and, for assets with non-negative, non-constant cash-flows, a series of the force of growth of cash-flows. These transformations will not always result in a stationary series. Non-stationarity will need to be taken into account by using alternatives to standard ARIMA models, such as: cointegrated models, ARCH models and threshold models.

Wilkie's model is over-parameterised as *DD*, *DW*, *DB*, *CD*, and *CY* all appear to be insignificant and *CA1* and *CA2* can be replaced by $1+CA3$ and $-2\times CA3$, respectively. The inflation data contained shocks and was non-stationary. These features cannot be explicitly taken into account in standard ARIMA models and appear to have caused *QA* and *YW* to be incorrectly included in the model. Considerable further research is required to determine more appropriate relationships between the retail prices index model and the other models. The relationship represented by *YA* appears to be far less significant after 1974. Since 1982, *QSD* is significantly less than, and *CSD* is significantly greater than, their respective estimates.

There are numerous problems with the data on which Wilkie's model was based. In particular, the data was compiled from a number of different sources that are not entirely compatible. The effects of these differences should have been examined before the combined data was used. The effects of seasonality in the retail prices index also need to be examined.

ACKNOWLEDGEMENTS

Thanks are due to my supervisors Steven Haberman and Richard Verrall for their comments on earlier drafts of this paper.

REFERENCES

The Actuaries' Investment Index. The Institute of Actuaries, London, and the Faculty of Actuaries, Edinburgh, December 1928 - December 1962. (Also known as: *The Actuaries' Investment Index (Second Series)*, August 1953 - December 1957; *The Actuaries' Investment Index (Second Series-Revised)*, January 1958 - December 1962.)

ALLEN, R.G.D. (1948). Retail prices. *London & Cambridge Economic Service Bulletin*, **26**: 18.

BZW equity & gilt study. Barclays de Zoete Wedd Research Limited, London, 38th edition, 1992.

DAYKIN, C.D. & HEY, G.B. (1990). Managing uncertainty in a general insurance company. *Journal of the Institute of Actuaries*, **117**: 173-277.

Employment Gazette. H.M. Stationary Office, London, July 1914 - March 1994. (Previously known as: *The Board of Trade Labour Gazette*, July 1914 - June 1917; *The Labour Gazette*, July 1917 - May 1922; *The Ministry of Labour Gazette*, June 1922 - March 1949; *Ministry of Labour Gazette*, April 1949 - May 1968; *Employment and Productivity Gazette*, June 1968 - December 1970; *Department of Employment Gazette*, January 1971 - December 1978.)

Financial Times. The Financial Times Limited, London, various issues 30 July 1914 - 30 April 1994.

FT-Actuaries Share Indices. The Financial Times Limited, London, 1976 - 1990.

FT-SE Actuaries Share Indices. The Financial Times Limited, London, January 1969 - February 1994. (Previously known as: *FT-Actuaries Share Indices-Monthly UK Stock Indices*, January 1969 - December 1989; *FT-Actuaries Share Indices-Finstat*, January 1990 - December 1993.)

FRANSES, P.H. & PAAP, R. (1994). Model selection in periodic autoregressions. *Oxford Bulletin of Economics and Statistics*, **56**, 4: 421-439.

GEOGHEGAN, T.J., CLARKSON, R.S., FELDMAN, K.S., GREEN, S.J., KITTS, A., LAVECKY, J.P., ROSS, F.J.M., SMITH, W.J. & TOUTOUNCHI, A. (1992). Report on the Wilkie stochastic investment model. *Journal of the Institute of Actuaries*, **119**: 173-228.

HAYCOCKS, H.W. & PLYMEN, J. (1956). Investment policy and index numbers. *Journal of the Institute of Actuaries*, **82**: 333-390.

HAYCOCKS, H.W. & PLYMEN, J. (1964). The design, application and future development of the *Financial Times*-Actuaries Index. *Journal of the Institute of Actuaries*, **90**: 267-324.

HÜRLIMANN, W. (1992). Numerical evaluation of the Wilkie inflation model. *Insurance: Mathematics and Economics*, **11**: 311-314.

KITTS, A. (1988). Applications of stochastic financial models: a review. Department of Social Statistics, University of Southampton.

KITTS, A. (1990). Comments on a model of retail price inflation. *Journal of the Institute of Actuaries*, **117**: 407-412.

MITCHELL, B.R. (1962). *Abstract of British historical statistics*. Cambridge University Press, Cambridge.

MITCHELL, B.R. & JONES, H.G. (1971). *Second abstract of British historical statistics*. Cambridge University Press, Cambridge.

OSBORN, D.R. (1990). A survey of seasonality in UK macroeconomic variables. *International Journal of Forecasting*, **6**: 327-336.

ROSS, M.D. (1989). Modelling a with-profits life office. *Journal of the Institute of Actuaries*, **116**: 691-715.

Statistical Abstract for the United Kingdom. In each of the last fifteen years from 1910-1924. H.M. Stationary Office, London, **69**. 1926.

WILKIE, A.D. (1984). Steps towards a comprehensive stochastic model. *Occasional Actuarial Research Discussion Paper*, The Institute of Actuaries, London, **36**: 1-231.

WILKIE, A.D. (1986). A stochastic investment model for actuarial use. *Transactions of the Faculty of Actuaries*, **39**: 341-403.

WORKING, H. (1960). Note on the correlation of first differences of averages in a random chain. *Econometrica*, **28**: 916-918.

ISBN 1 874 770 70 0