

RISK THEORY AND THE DETERMINATION OF SOLVENCY1. Introduction

1.1 The use of Risk Theory in General Insurance goes back an extremely long way. Barrows (1835) distinguished between the probability of a building catching fire and the amount consumed by fire once it had started. The general principles were used in life assurance models by Dormoy (1878). In 1903, Lundberg, in a doctoral thesis at Uppsala University, introduced the general concepts which have become established under the general title of Risk Theory.

1.2 Lundberg, in 1909, introduced the idea of Spielfonds or probability of ruin in unlimited period. In modern literature this expression is $\pi(\omega)$ the probability that an insurance company, with initial capital ω will eventually become insolvent in unlimited time. The probability of survival is $U(\omega) = 1 - \pi(\omega)$ The probability of survival in time t , given initial capital ω is similarly designated by the symbol $U(\omega, t)$.

1.3 The risk theoretical model of an insurance company is essentially a naive one. No considerations are given to the precise level of reserves (or provisions). Premiums are received, and claims are paid immediately (or, alternatively, as adequate reserve set up for the provision of claims). The claims distribution is split into two distinct elements, namely:-

- (i) The probability distribution of the number of claims n in a specific time t . $P_n(t)$
- (ii) The probability distribution of the size of claims $B(y)$

1.4 Examples of the probability distribution used in respect of the number of claims are:-

- (a) Poisson (independence of claims)
- (b) Negative Binomial (accident prone claims with a fixed level of "accident proneness").
- (c) Generalised Waring (a series of distributions which represent varying degrees of accident proneness).

1.5 Examples of probability distribution used in respect of the size of claims are:

- (a) Normal
- (b) Lognormal
- (c) Inverse normal
- (d) Exponential
- (e) Pareto
- (f) Gamma

2. Probability of Survival in Unlimited Time

2.1 The formula in the Poisson case for the survival probability in infinite time, given initial capital of ω is given by:-

$$u(\omega) = 1 - C(\omega) e^{-R\omega}$$

$C(\omega)$ is a function that depends on $B(y)$ the claims size distribution.

R is the positive root satisfying the equation $M(s) = 1$ where $M(s)$ is the moment generating (Laplace transform) of $B(y)$

2.2 The method of proof of this formula appears in the Risk Theory text book. A similar proof, using the approach of random walks with one absorption barrier (representing insolvency) appears in Cox and Miller's book on Stochastic processes.

2.3 For a company with premium rates equal to $(1+\lambda)Kt$ where

K = claims paid in the period 1

t = period covered by premiums

λ = safety loading on risk premiums

$C(\omega)$ may be shown to be approximately equal to $\frac{1}{1+\lambda}$

2.4 A slightly more refined approach is to let $\beta(y)$ represent the Laplace-Stieltjes transform of $B(y)$ given by

$$\beta(s) = \int_0^\infty e^{-sy} dB(y)$$

$$\text{Then } u(\omega) \sim 1 - \frac{\lambda}{-\beta'(\lambda) - 1 - \lambda} e^{-R\omega}$$

where R is the smallest positive root of $\beta(-R) = 1 + R(1+\lambda)$

2.5 Another approximation is that of Beekman and Bowers. This

gives
$$U(\omega) = \frac{\lambda}{1+\lambda} + \frac{1}{1+\lambda} P(\alpha, \omega/\beta)$$

where
$$\beta = \frac{2}{3} \frac{E(Y^3)}{E(Y^2)} + \frac{E(Y^2)}{2\lambda} (1+\lambda)$$

$$\alpha = E(Y^2)(1+\lambda)/2\lambda\beta$$

E = expected value

$$P(\alpha, \alpha + z\sqrt{\alpha}) = \frac{1}{\Gamma(\alpha)} \int_0^{\alpha + z\sqrt{\alpha}} e^{-y} y^{\alpha-1} dy$$

i.e. standard incomplete gamma

This approximation relies on the fact that distributions may be approximated by gamma or normal type distributions.

2.6 Examples.

(i) If $B(y)$ = exponential

$$U(\omega) = 1 - \frac{1}{1+\lambda} \exp\left(\frac{-\lambda\omega}{1+\lambda}\right)$$

(ii) For the Poisson case; if $U(0) = \lambda/(1+\lambda)$

$$U(\omega) = U(0) + \int_0^\omega U(\omega-y) \cdot \left(1 - \frac{B(y)}{1+\lambda}\right) dy$$

This is solvable by approximate integration techniques, given that $U(0)$ is known.

(iii) Let us consider an insurance company with initial capital X_0 and designate the capital at time n as X_n . Assume the premiums paid each year are constant $Y_r = \delta, r=1,2,\dots$. The total claims W_i for $i=1,2,\dots$ can be regarded as a large number of independent claims, and thus W_i can be considered (assuming the Central Limit theorem) as being normally distributed with mean μ and variance σ^2 .

(Note. This will not occur in practice unless for example, all claims are fixed).

$$Z_r = \text{Change in capital in period } r \\ = Y_r - W_r$$

Which is normally distribution with mean $\delta - \mu$ and variance σ^2

$$\text{Let } \pi(X_0) = 1 - U(X_0) \simeq e^{-RX_0}$$

Since our distribution is normal

$$R = 2(\delta - \mu) / \sigma^2$$

$$\text{and hence } \pi(X_0) \simeq \text{Exp} \left(- \frac{2(\delta - \mu) \cdot X_0}{\sigma^2} \right)$$

- (iv) Let Company A have m prospective policyholders. The yearly claims per policy has mean δ and standard deviation σ . The expected total claims is thus $m\delta$. Let the running expense be $\bar{\delta}$ per policy. The shareholder borrows the capital at a rate of 100α per cent per annum and a ruin probability of β is acceptable. Using example (iii) the minimum capital required (assuming the average annual premium per policy is minimised) is given by the expression.

$$X_0 = \sigma \left(\frac{m}{2\alpha} \log \left(\frac{1}{\beta} \right) \right)^{1/2}$$

Note; This function depends mainly on the standard deviation and the number of policies issued.

The average minimum premium is

$$\left(\delta + \frac{\alpha X_0}{m} \right) + \bar{\delta}$$

For a Company with

$$\alpha = 0.15$$

$$\bar{\delta} = 20\% \text{ of premiums}$$

$$\beta = .01$$

$$\delta = £20$$

$$\sigma = £20$$

$$m = 100,000$$

The minimum capital is £25,000. This is 1.25% of the sum assured at risk for $\beta = .001$, the capital increase to £30,000.

These results are not inconsistent with the type of additional reserves required for life assurance. If σ increases to £200, the level of capital required increases by 10. Thus the major influence on the level of capital required is the standard deviation of the claims distribution.

3. Probability of Survival in Finite Time.

3.1 There appear to be only four approximation methods of solving the probability of finite solvency $U(\omega, t)$. These use approximation of $F(x, t)$, which is the standard form of distribution function for aggregate claims.

$$F(x, t) = \sum_{n=0}^{\infty} P_n(t) B^{n*}(x)$$

3.2 Basically, the initial working formula is

$$U(\omega, t) = F(\omega + (1+\lambda)t, t) - (1+\lambda) \int_0^t U(\omega, t-\tau) f(\omega + (1+\lambda)\tau, \tau) d\tau$$

Where $f(\cdot, t)$ = frequency function of aggregate claims in epoch t

$$F(x, t) = \int_0^x f(y, t) dy + f(0, t)$$

3.3 Arfwedson's Formula

(i) Assume the aggregate claims in the interval $(0, t)$ amounts to $Z < (1+\lambda)t$. The probability of this is $\int f(z, t) dz$

(ii) If $X(t)$ = aggregate claims in time interval $(0, t)$. Then, the probability that $X(\tau)$ $0 \leq \tau < t$ has never exceeded $(1+\lambda)\tau$ (i.e. never been ruined at any specific time) = $1 - \frac{Z}{(1+\lambda)t}$

$$\begin{aligned} \text{(iii)} \quad U(0, t) &= F((1+\lambda)t, t) \\ &= \frac{(1+\lambda)}{t} \int_0^t z f((1+\lambda)z, t) dz \end{aligned}$$

(iv) In the Poisson exponential case with $\lambda = \mu = 1$

$$f(x, t) = \frac{t}{x} f(t, \lambda)$$

and hence

$$U(0, t) = F((1+\lambda)t, t) - (1+\lambda) \int_0^t e^{-(t-z)} f((1+\lambda)t, z) dz$$

The monetary value on the right hand side is constant and the addition of U , gives the standard Arfwedsons formula

$$U(u, t) = F(u + (1+\lambda)t, t) - (1+\lambda) \int_0^t e^{-(t-z)} f(u + (1+\lambda)t, z) dz$$

This is now solvable by using queuing theory.

The above approach uses the approximation of distribution by incomplete gammas.

3.4 Monte Carlo Techniques

There are standard techniques and have been used with reasonable success. The number of simulations required is usually large.

3.5 Esscher Approximation

This gives, using the gamma distribution notation in the previous section,

$$U(0, t) = \frac{1}{(1+\lambda)t\beta} \left[A P(\alpha, A) - \alpha P(\alpha+1, A) \right] - B P(\alpha, B) + \alpha P(\alpha+1, B)$$

Where $A = \alpha + \lambda t\beta$
 $B = \alpha - t\beta$

$$\alpha = 4K_2^3 / K_3^2$$

$$\beta = \sqrt{\alpha} / K_2$$

Where K_2 and K_3 are the second and third cumulants of the distribution function F . F is found by the approximation

$$F(t + z\sqrt{K_2}, t) = P(\alpha, \alpha + z/\alpha)$$

and f by differentiation

$$f(t + z\sqrt{K_2}, t) \approx \frac{\beta}{\Gamma(\alpha)} e^{-\alpha - z/\alpha} (\alpha + z/\alpha)^{\alpha-1}$$

Thus $U(0, t)$ may be calculated, and the final solution is derived by using a computer programme.

3.6 The other methods of approximation use incomplete gamma ratio's or approximations using two variable. The results tend to differ substantially from other methods. However, the Esscher approximation does tend to agree with the Arfwedson method.

4. Summary

4.1 With the advent of computers, it is becoming increasingly possible to obtain approximation of $U(u,t)$. A more detailed summary of the results may be found in Seals expensive monograph entitled "Survival Probabilities; The Goal of Risk Theory". This includes calculations of $U(u,t)$ using the various formula. What is surprising from this is that the level of capital appears to be substantially higher than one would anticipate for a reasonable survival probability, and the mix and type of business and the effects of reinsurance have obviously not been considered. For example

$$U(5,1) = 98616 \text{ for } \lambda = 0.1$$

that is then a 1½% probability of ruin in 1 year if capital is 5 times expected claims!

4.2 It is clear that survival probabilities may be of use in determining adequacy of capital, or give indications of "high risk" portfolio. As a measure of predicting insolvency for real companies they are only usable if that company has a large portfolio of one specific class of business. If this is not the case, then the estimates are, at best, reasonable first approximations where doubt must arise, in some cases, in the significance of the first figure! What is clear is that little work has been done on this mathematical problem and, for the main part, what has been done is of academic interest only.