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SMALLPOX AND THE DOUBLE DECREMENT TABLE A PIECE OF ACTUARIAL PRE-HISTORY

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1. INTRODUCTION

MORE than 200 years ago, on 30 April 1760, Daniel Bernoulli (1766) read a memoir to the Royal Academy of Sciences in Paris entitled *Essai d'une nouvelle analyse de la mortalité causée par la petite vérole, et des avantages de l'inoculation pour la prévenir* (see Bradley, 1971, for a translation). In this remarkable memoir Bernoulli produced the first double decrement life table and one of the related single decrement tables, as well as deriving a mathematical model of the behaviour of smallpox in a community. This model was the forerunner of considerable developments in the mathematical theory of infectious diseases, a description of which is given in N. T. J. Bailey (1975). During the half century following Bernoulli's memoir there were a number of papers by other authors on the subject of that memoir; these, and the original memoir, seem to be little known to actuaries and are the subject of the present paper. They could have been the starting point of the actuarial development of exposed-to-risk formulae, but in fact were not.

2. NOTATION

The various authors whose work is to be considered used different mathematical notations, although there are similarities. In this paper a uniform notation is used in which nearly every letter has been used by at least one author to denote the item here assigned to it. In the list below age has been denoted by the suffix x. Duvillar (1806) is the first of the authors to use this notation, although Trembley (1799) used a superfix in brackets, e.g. $y^{(i)}$. In an attempt to retain the flavour of these early works age has usually only been indicated where the author did so.

The notation adopted here is as follows:

x = age
y_x = number living at age x when all causes of death operate
s_x = number out of y_x who have still not had smallpox
w_x = number out of y_x who have had smallpox and recovered

thus $y_x = s_x + w_x$

g_x = number living at age x who have not yet had smallpox and who will eventually catch it

v_x = number living at age x who will eventually die of smallpox
u_x = number who have died of smallpox from birth to age x
$1/n_x$ = annual proportion getting smallpox of those aged x who have still not had it
$1/m_x$ = proportion of those aged x getting smallpox who die of it
z_x = number living at age x when all causes of death except smallpox operate
b_x = number living at age x when smallpox is the only cause of death.

The definitions of $1/n_x$ and $1/m_x$ have deliberately been left a little loose because different authors use them in slightly different ways.

Those authors who use the finite difference symbol Δ in relation to the above notation usually give it the opposite sign to current practice and the same is done in this paper. For example

$$\Delta y_x = y_x - y_{x+1} = \text{deaths between ages } x \text{ and } x+1.$$

This is convenient when dealing with decreasing functions like y_x, s_x, z_x, g_x as all differences are then positive. Care is needed with increasing functions like u_x or with w_x which first increases with increasing age up to perhaps age 20 and thereafter decreases.

3. DANIEL BERNOULLI (1700–82)

3.1. Daniel Bernoulli was a member of a Basle family which produced eight notable mathematicians in three generations. He qualified as a physician and then turned to mathematics; he held professorships of mathematics, of anatomy, of botany and of natural philosophy. He made important contributions to many subjects, such as hydrodynamics, astronomy, calculus, probability and statistics. He was the first to use differential equations systematically for deriving formulae; examples of this are given below. He was one of the first to raise problems of testing statistical hypotheses and produced the first published table of the normal curve. He is also credited with the second independent introduction of the principle of maximum likelihood—the first being by J. H. Lambert. For Daniel Bernoulli's work on probability and statistics see the articles by O. B. Sheynin in Kendall & Plackett (1977).

3.2. The background to Bernoulli's (1766) memoir was the practice of inoculation against smallpox. This consisted in inoculation with material taken from a smallpox pustule on a person suffering from the disease. The aim was to produce a mild attack of smallpox which was thought to give a permanent immunity in the future. This practice had been used in Asia in ancient times and was being adopted in Europe on an increasing scale in the first half of the eighteenth century. Inoculation was, however, not without risk. An inoculated person occasionally died from the smallpox so acquired and could give rise to smallpox outbreaks among those who had not already had it or been inoculated. The merits of inoculation were a subject of controversy among doctors and mathematicians during the greater part of the eighteenth century. It was eventually superseded by vaccination with cowpox virus following the work of Jenner in this country at the end of the eighteenth century. Karn (1931) describes the history of the rise and fall of inoculation. Those interested in the possible effect of inoculation, and later vaccination, on smallpox mortality might refer to Guy (1882) and Burridge (1903). Whether the observed reduction in mortality was actually caused by these measures would seem to be a controversial matter.

3.3. Bernoulli's object was to compare the state of the community without inoculation with that when inoculation was universally practised and was always effective. He says:

> I was above all concerned to display in a single table the two conditions of mankind, the one as it actually is and the other as it would be if we were able to rid the whole human race of smallpox. I had in mind that the comparison of these two conditions would explain the difference and the contrast between them better than the most ample commentary; but I had in mind, too, the difficulty of the enterprise and the defective nature of the Bills of Mortality, which do not give the age of those carried off by smallpox and were bound to be a serious obstacle to my purpose. I could see immediately that to carry out such a design demands two items of elementary information: what is the risk, at various ages of being caught by smallpox, for those who have not already had it, and what is the risk, for those who are attacked by it, of dying of it? It is true that we have no specific information on these two points, but other items of information seemed to me to make up for them with a high degree of probability. (Bradley, 1971, 22)

3.4. Bernoulli's starting point was the life table prepared by the astronomer Edmund Halley (1693) from statistics of the City of Breslau. This table is an interesting study in itself (e.g. see Greenwood, 1948, 40). Here it is necessary to mention only that the way in which Halley set out his life table was not altogether clear. Todhunter (1865, 42) commented: "We do not feel confident of the meaning of this table." The table gave the superficial impression that the figures were of l_x starting with $l_1 = 1{,}000$ and this was how Bernoulli interpreted it; he therefore inserted $l_0 = 1{,}300$, which he considered to be reasonable. In fact Halley's table showed the values of L_{x-1} starting with $L_0 = 1{,}000$, put against age 1. However, Bernoulli's misconception is not material to his purpose.

3.5. Bernoulli assumed that no person could have smallpox more than once and overcame the absence, at that time, of any statistics of the age distribution of smallpox cases and deaths by assuming that the annual proportions of those who had not had smallpox catching it, $1/n$, and the proportions of those catching it who died, $1/m$, did not vary with age. These assumptions were not at variance with the facts then known about smallpox. Bernoulli assumed $n = m = 8$, which gave a proportion of the total deaths due to smallpox in reasonable agreement with the available figures (i.e. about 1 in 13) and also resulted in there being few persons living over say age 25 who had not had smallpox; again in accordance with general experience at that time.

3.6. Bernoulli next obtained a formula for the number of persons at each age in the life table who had never had smallpox; he argued as follows. The survivors s who have not had smallpox decrease by

(i) those who catch smallpox (whether or not they die of it) and,
(ii) those who die of other causes without ever having had smallpox.

In an element of time dx the decrement of s is $-ds$ (ds, and dy used below, are inherently negative and the negative sign is therefore needed to convert them to positive numbers). The number attacked by smallpox is sdx/n and the number of these who die of smallpox is sdx/nm. The total number of deaths from all causes in time dx is $-dy$. Thus the number dying from other causes is $-dy-sdxnm$. But this number relates to y persons, whereas in forming an equation for s we are concerned with the number of deaths from other causes among s, i.e. with $(-dy-s\,.\,dx/nm)s/y$ deaths.

Therefore

$$-ds=s\,.\,dx/n+(-dy-s\,.\,dx/nm)s/y. \qquad (1)$$

Hence

$$(s\,.\,dy-y\,.\,ds)/s^2=y\,.\,dx/sn-dx/nm.$$

Put $y/s=r$ so that $dr=(s\,.\,dy-y\,.\,ds)/s^2$. Then $nm\,.\,dr=mr\,.\,dx-dx$ or $dx=nm\,.\,dr/(mr-1)$. Integrating gives $n\log(mr-1)=x+c$, where c is a constant to be determined, or $n\log(my/s-1)=x+c$. Now when $x=0$, $y=s$, which gives

$$c=n\log(m-1)$$

and hence

$$n\log\left(\frac{my/s-1}{m-1}\right)=x.$$

Therefore

$$my/s-1=(m-1)e^{x/n}.$$

Hence

$$s=\frac{m}{1+(m-1)e^{x/n}}\,.\,y. \qquad (2)$$

3.7. The way was now clear for Bernoulli to prepare his double decrement table (see Table 1 from the 1766 memoir). The second column gives the values of y according to Halley's (1693) table, the third column, the values of s at each age calculated by formula (2) and the fourth column the number living who had caught smallpox and recovered (i.e. $y-s$ or w). At this stage it must be realized that the various decremental figures shown in the succeeding columns are entered on the line below that which would be used at the present time. Column 5 gives the number catching smallpox in a year and is one-eighth (since $n=8$) of the mean of s on that line and the line above, i.e., the entry against the age x is $\frac{1}{8}\times\frac{1}{2}(s_{x-1}+s_x)$. Column 6, the number dying each year of smallpox, is one-eighth (since $m=8$) of the previous column, and column 7 is the sum of column 6 from the top downwards (i.e. u_x). Column 8 gives the number of deaths at each age from causes other than smallpox i.e., $y_{x-1}-y_x-$ col. 6. The table is taken only as far as age 24 because by then there are few persons living who have not had smallpox.

However, Bernoulli estimates that, of the 32 who have not had smallpox by age 24, 3 will eventually die of it.

3.8. Bernoulli goes on to consider what the position would be if every person were inoculated at birth and this resulted in eliminating smallpox as a cause of death. He therefore prepared a life table for the situation where there were no deaths from smallpox. His method is illustrated from the figures of Table 1:

(i) In the first year of life there were 17·1 deaths from smallpox, so that without smallpox the number who survived the year would be increased from 1,000 to 1,017·1.

(ii) If 133 (column 8) die during the second year from causes other than smallpox out of 1,000 alive at the beginning of that year, there will, by proportion, be 135·3 deaths out of the 1,017·1 alive at the beginning of the year in the smallpox-free state, leaving 881·8 living at the end of the second year; and so on for the remaining ages.

3.9. The results are set out in Bernoulli's Table 2 where the l_x columns for the

Table 1.

AGES par années	Survivans selon M. Halley	N'ayant pas eu la pet. vérole	Ayant eu la pet. vérole	Prenant la pet. vérole pendant ch. année	MORTS de la pet. vérole pendant chaq. ann.	SOMME des morts de la pet. vérole	MORTS par d'autres maladies pend. chaq. année
0	1300	1300	0				
1	1000	896	104	137	17,1	17,1	283
2	855	685	170	99	12,4	29,5	133
3	798	571	227	78	9,7	39,2	47
4	760	485	275	66	8,3	47,5	30
5	732	416	316	56	7,0	54,5	21
6	710	359	351	48	6,0	60,5	16
7	692	311	381	42	5,2	65,7	12,8
8	680	272	408	36	4,5	70,2	7,5
9	670	237	433	32	4,0	74,2	6
10	661	208	453	28	3,5	77,7	5,5
11	653	182	471	24,4	3,0	80,7	5
12	646	160	486	21,4	2,7	83,4	4,3
13	640	140	500	18,7	2,3	85,7	3,7
14	634	123	511	16,6	2,1	87,8	3,9
15	628	108	520	14,4	1,8	89,6	4,2
16	622	94	528	12,6	1,6	91,2	4,4
17	616	83	533	11,0	1,4	92,6	4,6
18	610	72	538	9,7	1,2	93,8	4,8
19	604	63	541	8,4	1,0	94,8	5
20	598	56	542	7,4	0,9	95,7	5,1
21	592	48,5	543	6,5	0,8	96,5	5,2
22	586	42,5	543	5,6	0,7	97,2	5,3
23	579	37	542	5,0	0,6	97,8	6,4
24	572	32,4	540	4,4	0,5	98,3	6,5

Table 2.

AGES par années	État naturel & variolique	ÉTAT non-varioliq.	Differ. ou gains	AGES par années	État naturel & variolique	ÉTAT non-varioliq.	Differ. ou gains
0	1300	1300	0	13	640	741,1	74,1
1	1000	1017,1	17,1	14	634	709,7	75,7
2	855	881,8	26,8	15	628	705,0	77,0
3	798	833,3	35,3	16	622	700,1	78,1
4	760	802,0	42,0	17	616	695,0	79,0
5	732	779,8	47,8	18	610	689,6	79,6
6	710	762,8	52,8	19	604	684,0	80,0
7	692	749,1	57,2	20	598	678,2	80,2
8	680	740,9	60,9	21	592	672,3	80,3
9	670	734,4	64,4	22	586	666,3	80,3
10	661	728,4	67,4	23	579	659,0	80,0
11	653	722,9	69,9	24	572	651,7	79,7
12	646	718,2	72,2	25	565	644,3	79,3

Cette Table fait voir d'un coup d'œil, combien sur 1300 enfans, supposés nés en même temps, il en resteroit de vivans d'année en année jusqu'à l'age de vingt-cinq ans, en les supposant tous sujets à la petite vérole; & combien il en resteroit s'ils étoient tous exempts de cette maladie, avec la comparaison & la difference des deux états.

(NOTE. The figure at the head of column 7 should be 7.4.1 (R.H.D.))

natural state (i.e. with smallpox deaths) and the non-smallpox state are compared. As an overall comparison of the two states, Bernoulli calculates the expectation of life at birth for each of his life tables. He finds an expectation of 26 years 7 months for the natural state and 29 years 9 months if smallpox were eliminated, a gain of 3·2 years. Karn (1931) points out that the method assumes smallpox mortality to be non-selective or, in other words, that those saved from death by smallpox are in future subject to the same rates of mortality as other persons. (Bernoulli does not mention this.)

3.10. However, Bernoulli points out that death sometimes occurs as a result of inoculation and he considers the effect if this happens in 1 out of every 200 inoculations at birth; this he thinks is taking the worst view. He finds that the expectation of life at birth in the non-smallpox state is reduced by less than 2 months, which still leaves a gain of 3 years.

3.11. It was only after having prepared his double decrement table and the corresponding single life table for the non-smallpox state that Bernoulli realized that it was possible to obtain a formula relating the l_x's of the two tables. As shown above the deaths in the period of time dx from causes other than smallpox are $-dy-sdx/\simeq m$ in respect of a population of y. Hence for a population of z (the l_x of the non-smallpox state)

$$-dz=-\frac{z}{y}(dy+s\,.\,dx/nm)$$

or

$$\frac{dz}{z} - \frac{dy}{y} = sdx/nmy. \tag{3}$$

Substituting (2) for s gives

$$\frac{dz}{z} - \frac{dy}{y} = \frac{1}{n}\left\{1 - \frac{(m-1)e^{x/n}}{1+(m-1)e^{x/n}}\right\}dx.$$

Integrating gives

$$\log\frac{z}{y} = \frac{x}{n} - \log\{1+(m-1)e^{x/n}\}dx + c.$$

Since $y = z$ when $x = 0$, therefore $c = \log m$, whence

$$z = \frac{me^{x/n}}{1+(m-1)e^{x/n}} . y. \tag{4}$$

3.12. Bernoulli gives only two numberical comparisons of z by his approximate method described above and by the exact formula (4); these are:

Values of z

x	Approximate (Table 2)	Exact (formula (4))
16	700·1	697·4
24	651·7	649·2

He considers these show reasonable agreement between his approximate method of constructing Table 2 and the exact formula (4). In each case the exact value is less than the approximate one. This would be expected, since the principal defect of Bernoulli's approximate method is that it makes no allowance for the deaths from other causes which would take place among the smallpox deaths saved in each particular year.

3.13. Bernoulli's memoir is workmanlike, scientific and realistic. He knew exactly what he wanted to do and proceeded to do it. Where his data were inadequate he made assumptions and then checked the effect with whatever information was available. Not only were his assumptions reasonable in the light of the knowledge at that time but they were such as to allow the mathematical problem to be solved.

4. JEAN LE ROND D'ALEMBERT (1717–83)

4.1. D'Alembert studied successively law, medicine and mathematics. His mathematical work showed many brilliant and original insights but he made mistakes. Todhunter (1865, 258) says that he "is known in the history of the Theory of Probability for his opposition to the opinions generally received".

4.2. Following Bernoulli's memoir read to the Academy of Sciences in Paris on 30 April 1760, d'Alembert read a memoir to the same body on 12 November 1760 criticizing Bernoulli's work. As appears to have been his custom d'Alembert had his memoir published in his *Opuscules Mathématiques* (1761); this included an additional section of Notes which was more than twice as long as the original memoir. Bradley (1971) contains a translation of d'Alembert's memoir; he has also translated the Notes and has kindly allowed a photocopy of this to be placed in the Institute Library. Bernoulli's memoir was not published until 1766 and included a short vindicatory introduction and a few additional sections commenting on d'Alembert's criticisms. He expressed the wish that his critics should have taken the trouble to make themselves familiar with the matters which they criticized. In later volumes of his *Opuscules* d'Alembert (1768a–d) returned to the subject but his work was largely repetitive or on questions of detail and added little to the matter under discussion.

4.3. Duvillard (1806) comments that d'Alembert (1761) showed more of the spirit of quibbling than of justice and included errors which he would certainly not have made had he taken the trouble to study the subject thoroughly. He suggests that d'Alembert was piqued that he had not himself thought of such a useful application of mathematics. However, a section of d'Alembert's (1761) Notes entitled 'Mathematical theory of Inoculation' makes a definite contribution to the mathematics of the problem. Having expressed doubts as to whether the risks of contracting, and of dying of, smallpox were constant at all ages, d'Alembert obtains a formula which avoids any assumption about these proportions. He represents by du the number of persons dying of smallpox in time dx and derives by a geometrical method (e.g. see Karn, 1931, 297) the formula

$$\frac{dz}{z}-\frac{dy}{y}=\frac{du}{y} \tag{5}$$

This is the same as Bernoulli's formula (3) above, if the number of deaths from smallpox (i.e., $s\,.\,dx/nm$) be replaced by du. Integration of (5) gives

$$z=cye\int_0^x\frac{du}{y}$$

The constant c is found to be unity since at $x=0$, $z=y$ and

$$\int_0^x\frac{du}{y}=0;$$

thus,

$$z=ye\int_0^x\frac{du}{y} \tag{6}$$

This formula is an exact general solution of the problem of deriving from a double decrement table one of the related single decrement tables and the credit for it must go to d'Alembert. However, it was subject to criticism. Trembley (1799) says that d'Alembert "has substituted for the elegant analysis of M. Bernoulli a mathematical theory so highly mathematical, that neither he nor anyone that I know has applied it". Todhunter (1865, 268) comments: "The result is not of practical use because the value of the integral $\int \frac{du}{y}$ is not known". This would have been a valid criticism at the time d'Alembert wrote but not in 1865, because, long before that, data of smallpox deaths by age had become available and Duvillard (1806) had pointed out that the integral could then be evaluated by the Euler–Maclaurin expansion.

5. JOHANN HEINRICH LAMBERT (1728–77)

5.1. Trembley (1799) and Duvillard (1806) refer in passing to a paper by Lambert (1772), but Todhunter (1865) does not mention it and it seems to be comparatively little known to later writers, except for the smallpox statistics by age which it contains. I came to this paper at a late stage in my studies and was quite unprepared for what I found. In this paper Lambert, a German mathematician, gives the earliest application known to me of what would now be called actuarial formulae for dealing with mortality data; for this reason the paper seems to me of considerable importance. Mr W. W. Mehlig has kindly made a translation of the original German and it is hoped to publish this, together with a fuller consideration of Lambert's work, in a later part of *J.I.A.*

5.2. Lambert deals with the same problem as Bernoulli but makes no assumptions regarding n and m, since he now has available statistics of smallpox deaths by age at the Hague and a small age-related experience collected in Switzerland of smallpox cases and the resulting deaths (72 cases with 15 deaths). Starting with a mortality table which he had prepared he uses (i) an estimate by Süssmilch (1761) that, overall, 2 out of 25 deaths are due to smallpox, together with (ii) the Hague smallpox statistics, to split the total deaths at each age into those due to smallpox and to other causes.

5.3. Next Lambert derives a formula for obtaining z_x, the number living at age x when smallpox is eliminated as a cause of death. Out of the survivors y at each age, Δv die of smallpox in a year and $\Delta y - \Delta v$ die of other causes. If the Δv do not die of smallpox, at least some of them will die of other causes during the year. Lambert considers the two extreme cases:

(i) Assuming that the Δv deaths from smallpox all take place right at the beginning of the year, there then remain $y - \Delta v$. Of these $\Delta y - \Delta v$ die of other causes during the year.

(ii) Assuming that the Δv deaths from smallpox all take place right at the end of the year, by that time $\Delta y - \Delta v$ out of the initial y will have already died of other causes.

Now taking the mean of the two extreme cases, we have that, out of $y-\frac{1}{2}\Delta v$ who start the year, $\Delta y-\Delta v$ die of other causes during the year. Therefore the rate of mortality when smallpox is excluded is given by

$$\frac{\Delta z_x}{z_x}=\frac{\Delta y_x-\Delta v_x}{y_x-\frac{1}{2}\Delta v_x}. \tag{7}$$

5.4. By taking account of the deaths from other causes which would occur among the smallpox deaths saved each year if smallpox were to be eliminated, this formula remedies the major defect of Bernoulli's approximate calculations. Had Bernoulli's approximate values of z_x been calculated by Lambert's formula (7) they would have been very close to the values given by Bernoulli's formula (4) (see Table 3). In fact Lambert's formula is the same as one given by Bailey & Haycocks (1946), who show that it does not quite satisfy one of the fundamental criteria for double decrement tables.

Table 3. *Values of* z_x *obtained from Bernoulli's Table 1*

Age	Correct value by formula (4)	Bernoulli's approximate figures (Table 2)	By Lambert's formula (7)
0	1300·0	1300·0	1300·0
1	1014·9	1017·1	1015·2
2	879·3	881·8	879·7
3	830·5	833·3	830·8
4	799·3	802·0	799·7
5	777·1	779·8	777·5
6	760·1	762·8	760·4
7	746·4	749·1	746·6
8	738·3	740·9	738·5
9	731·8	734·4	732·0
10	725·7	728·4	726·0
11	720·3	722·9	720·5
12	715·5	718·2	715·7
13	711·4	714·1	711·6
14	707·0	709·7	707·3
15	702·3	705·0	702·6
16	697·4	700·1	697·7
17	692·2	695·0	692·5
18	686·8	689·6	687·1
19	681·2	684·0	681·5
20	675·5	678·2	675·7
21	669·6	672·3	669·8
22	663·7	666·3	663·8
23	656·4	659·0	656·5
24	649·1	651·7	649·1

5.5. Lambert now uses his small experience of smallpox cases and the resulting deaths to derive from the values of Δv_x in his table a column of Δg_x, the numbers catching smallpox at each age. He then obtains as follows a formula relating w_{x+1} to w_x, the numbers living who had had smallpox and recovered from it:

$$w_{x+1} = w_x - \begin{pmatrix}\text{those of } w_x \text{ who die of}\\ \text{other causes in the year}\end{pmatrix} + \begin{pmatrix}\text{those who catch smallpox and recover,}\\ \text{less the subsequent deaths from}\\ \text{other causes during the year}\end{pmatrix}$$

or,

$$w_x + 1 = w_x - w_x\left(1 - \frac{\Delta z_x}{z_x}\right) + (\Delta g_x - \Delta v_x)\left(1 - \tfrac{1}{2}\frac{\Delta z_x}{z_x}\right) \tag{8}$$

Although Lambert says several times that his tables must be regarded only as examples of method, he does show that, according to his data, some rather scanty, it is not justifiable to take $n = m$, as did Bernoulli, and that neither n or m are independent of age.

5.6. Lambert's (1772) paper is severely practical. He shows how numerical data can be used to study Bernoulli's problem and points the way to what later became known as actuarial calculations. Thus it can be said that the practical and theoretical foundations of double decrement tables had been laid down three-quarters of a century before the Institute of Actuaries was founded.

6. JEAN TREMBLEY (1749–1811)

6.1 Trembley (1799) considers the same problem as Bernoulli but works in units of a year. He assumes that all smallpox cases and deaths occur at the beginning of each year of age and derives a formula relating s_{x+1} to s_x in which both n and m can vary with age. His formula can be written

$$s_{x+1} = \frac{y_{x+1} s_x (n_x - 1)}{(y_x - \Delta v_x) n_x} \tag{9}$$

and can be derived by equating s_{x+1} to s_x less smallpox cases (s_x/n_x, on the stated assumptions) less deaths from other causes

$$s_x\left(1 - \frac{1}{n_x}\right)\frac{\Delta y_x - \Delta v_x}{y_x - \Delta v_x}$$

using argument of § 5.3(i). Trembley's method does not show up clearly the assumptions which it involves regarding the smallpox cases and the deaths. If n and m are constant Trembley shows that, if the age intervals are made infinitely small, his formula (9) tends to Bernoulli's formula (2). Trembley's formula would seem of less practical use than Lambert's because his assumptions regarding the smallpox cases and deaths are less realistic.

6.2. Trembley then investigates Bernoulli's assumption that n and m do not vary with age, basing his test on the figures of smallpox deaths by age at the Hague given by Lambert (1772) and some larger figures for Berlin. However, he makes no use of Lambert's very small experience of smallpox cases and deaths by age. He concludes that n varies little with age but that m shows substantial variation. However, in a later note Trembley (1807) says that his "method is worth absolutely nothing and I owe some excuses to the public for having presented it to them". He then derives what he says are accurate formulae which involve equating two expressions for s_x which are based on different assumptions. From these he finds that the values of n vary enormously with age, much more than do those of m. He concludes: "I had drawn up tables of these variations but I have suppressed them because I have reflected that all this calculation rested basically on more or less arbitrary assumptions, and that the smallest change in these assumptions gave quite different results." Further he finds that if his two expression for s_x are based on the same assumptions they reduce to the same value and "the calculation falls down". I cannot recall ever having come across such a candid refutation by an author of his own work.

7. EMMANUEL ÉTIENNE DUVILLARD (1755–1832)

7.1. Duvillard has been described by Quiquet (1934) as "the first French actuary". Duvillard (1806) considers the same problem as Bernoulli (1766) but now the results have to be related, not to smallpox inoculation, but to the much less risky method of vaccination introduced by Jenner in 1796. Greenwood (1948, 66) says of Duvillard's book that it is " a monograph which, although seldom read, for it is scarce and 'practically' obsolete, has been rightly described by Farr as a classic of vital statistics . . . and this book of nearly 200 quarto pages may still be read with profit". While I am in complete agreement with these views, it seems to me that Duvillard's very extensive and detailed mathematical and numerical treatment owes a great debt to Lambert's comparatively short paper. Duvillard starts with 86 pages of mathematics in which he obtains d'Alembert's formula (6), in a slightly different form, and derives the formulae of Lambert for w_x and z_x and corresponding formulae based on each of Lambert's two extreme assumptions set out in § 5.3 above. He also gives various formulae relating n and m to the other variables, and shows how to calculate the expectations of life of various groups of lives involved in his formulae (e.g. y_x, s_x, w_x and v_x). He shows too how to determine the effect which the increasing population resulting from the introduction of vaccination from a particular date will have on the observed rates of mortality of the community, assuming that adequate food is available to feed the larger population.

7.2. Duvillard also considers the case when n and m vary with age and gives a formula

$$z_x = \frac{y_x}{1 - \int \frac{1}{m_x} \cdot \frac{1}{n_x} \cdot e^{-\int \frac{dx}{n_x}} dx} \tag{10}$$

which is also obtained by Laplace (1812, 414). Duvillard also gives formulae in finite difference form for this case. Much of Laplace (1812) had previously been published in his many memoirs and it is therefore likely that his derivation of formula (10) antedates that of Duvillard. Neither of the two authors mentions the other. For what it is worth I searched through Laplace's (1886–1912) complete works but found no mention of formula (10). It must, however, be remembered that a few of Laplace's memoirs are not included in his *Œuvres complètes* (Stigler, 1977).

7.3. Ten pages of observed data regarding smallpox (including those given by Lambert and Trembley) are followed by 44 pages on the application of the mathematical theory to the data and 38 pages of life tables setting out the results. The basis of the numerical part of Duvillard (1806) is a mortality table which he had presented to l'Institut national in 1796. He says little about this table except that it is based on a fairly large number of observations made in various places in France before the Revolution (i.e. 101,542 deaths and a population of 2,920,672 individuals). It appears to have been prepared by adding together the deaths by age. It seems curious that Duvillard should have used this method when his book contains a mathematical and numerical investigation of the effects of an increasing population on the observed mortality rates. However, he states that "at the time when the facts were being collected the relations between the annual marriages, births, deaths, the mortality of one age to another, the number living at each age . . . had every uniformity that we can expect . . . ". Presumably he considered this to justify him in adding the deaths to get the numbers living. Nevertheless French assurance companies only abandoned the use of this table in 1894 (Quiquet, 1934, 52).

7.4. Using the statistics of smallpox deaths and deaths from all causes by age in Berlin and the Hague, Duvillard prepared a series of age-related ratios of smallpox deaths to total deaths. The application of this series of ratios to the deaths from all causes according to his mortality table gave the smallpox deaths by age. Using these the values of z_x (the life table with smallpox excluded) were calculated by Lambert's formula (7).

7.5. By this stage Duvillard had prepared a double decrement table like that of Bernoulli's Table 1 (but without the third, fourth and fifth columns) and also the single decrement table with smallpox eliminated. He wished to insert the s_x column showing the numbers living who had still not had smallpox. The values of n_x and m_x at each age were to some extent indeterminate so long as the number of smallpox deaths was reproduced. However, there were many rather imprecise constraints. It was known that many of those catching smallpox recovered from it. The values of n_x and m_x must not show violent changes from one age to the next. The s_x column must show a reasonable progression; if it decreased too slowly the value of n or m needed to give the required number of smallpox deaths might become unreasonably high; too rapid a decrease might require a value of n or m which was obviously too low. Duvillard decided from his study of the available statistics that only some 3–4% of those living at age 30 should still not have had smallpox; thus the value obtained for s_{30}/y_{30} should lie in this range. Using the various formulae he had derived he obtained values of n_x and m_x and considered whether the results appeared reasonable. He came to the conclusion that n could reasonably be taken as a constant independent of age, but that m varied quite widely. He arrived at 8·173223 (compared with Bernoulli's 8) as a suitable value of n; from this the s_x column was calculated by the formula

$$s_x = z_x e^{-x/n} \tag{11}$$

a form obtained by substituting Bernoulli's formula (4) into (2). The values of m corresponding to the chosen value of n varied between 3 and 34 over the age range 0–25 with the minimum at age 1 and the maximum at age 10.

7.6. These values of n and m differ substantially from those given by Lambert (1772, § 155) where for ages 0–9, n starts at 39, decreases to 4 at age 5 and then increases to 13, while m starts at 3, increases to 10 at age 5 and then decreases to 5. Although the only age-related experience which Duvillard quotes of smallpox cases and deaths is that given by Lambert (1772, § 130), he appears to make little use of it. I have not come across any other age-related figures comparing smallpox cases and deaths, but some must surely exist somewhere. I think that Duvillard's method of studying n_x and m_x avoids the type of error made by Trembley (1799 and 1807) but have not completely convinced myself of this.

7.7. Duvillard calculated a large number of expectations of life and a few of these are given below:

Age	Natural state	Non-smallpox state	Increase
0	28·76	32·26	3·50
5	43·40	44·42	1·02
10	40·80	41·31	0·51

It is clear from these that, to have much effect, vaccination or inoculation would need to be done at as young an age as possible. It will be noticed that Duvillard's figure of 3·50 years for the increase in the expectation of life resulting from elimination of smallpox is little different from that of 3·2 years obtained by Bernoulli.

7.8. Some of Duvillard's figures for the expectation of life at birth for certain groups of lives in the natural state are of interest:

	Expectation (years)	Number out of 1,000 births
Those who will never have smallpox	2·00	333
Those who will be attacked by smallpox	42·13	667
Those who will be attacked by smallpox and recover	47·76	581
Those who will die of smallpox	3·94	86

7.9. At first sight some of these figures appear surprising. In order to appreciate their significance it must be realized that, like Bernoulli's Table 1, Duvillard's tables show that most smallpox cases occur at young ages. Duvillard's tables show that by age 10 about 80% of all births will have either had smallpox and recovered or died from it or other causes. Also by age 10 about 90% of the smallpox deaths have already taken place. This explains not only the small expectation of those who will die of smallpox, but the even smaller expectation of those who will never have smallpox, since nearly all of those will die at very young ages.

8. JOSHUA MILNE (1776–1851)

The last study to be mentioned of the effects of eliminating smallpox is that related to the Carlisle table prepared by Milne (1815) who was Actuary to the Sun Life Assurance Society. The data on which this table was based were published in a tract by Dr John Heysham (1797) and consisted of population enumerations (in age groups) of two parishes of Carlisle in 1780 and 1787, and the corresponding deaths during the 9 years 1779 to 1787 which were also in age groups apart from the first 5 years of life. Separate figures in 5-year age groups only were available for deaths from smallpox. The expectation of life at birth for the Carlisle table was 38·7 years and Milne estimated that the elimination of deaths from smallpox would increase this to about 43·0 years, an increase of about 4·3 years. His calculations would seem to be rather approximate as the smallpox deaths were in 5-year age groups, with 91% in the under-5 group. However, the smallpox deaths did correspond with those from all causes; they were not for some different country in a different period of time, as was the case with both Duvillard's and Lambert's figures.

9. DISCUSSION

9.1. As has already been shown, by 1772 we had, for the relation between a double decrement table and the corresponding single decrement tables, an exact theoretical formula (6), and an approximate practical formula (7) for numerical applications. There remained the problems of (i) deriving accurate practical formulae for application to numerical data, in other words changing from a force of mortality, or an integral, to finite intervals and, (ii) obtaining exact formulae in a form more convenient than d'Alembert's (6). As already mentioned, Duvillard (1806) indicated that formula (6) could be evaluated numerically by the Euler–Maclaurin expansion. In spite of the cumbersome nature of this method Karn (1931) used it to show the effect of eliminating cancer, tuberculosis and heart disease from the English Life Tables Nos. 8 and 9. She compared (Karn, 1933) these results with those obtained by other less laborious methods.

9.2. Cournot (1843, 317) and Makeham (1867), apparently independently, appear to have been the first to set out the two fundamental relations applying to double decrement tables; these are, in modern notation,

$$\mu_x = \mu_x^a + \mu_x^b \tag{12}$$

$$(ap)_x = p_x^a \times p_x^b \tag{13}$$

where μ_x and $(ap)_x$ apply to the double decrement table when the two decrements are combined and a and b denote the two decremental causes. Makeham (1875) shows that (13) is satisfied by Bernoulli's formula (4), d'Alembert's (6) and Duvillard's (or Laplace's) (10). The initial differential equations from which these equations were derived obviously satisfy (12). Bernoulli's (4) relates to the single decrement table excluding smallpox deaths and it may be of interest to derive the corresponding formula for the single decrement table for death only from smallpox and to show that the two formulae satisfy (13).

9.3. Let b_x denote the number living in the single decrement table subject to death from smallpox only. The argument deriving (1) showed that the deaths from smallpox among y persons in time dx are sdx/mn. Therefore

$$-db = \frac{s \cdot dx}{mn} \cdot \frac{b}{y}. \tag{14}$$

Substituting (2) into (14) and integrating gives

$$\log b = -x/n + \log\{1 + (m-1)e^{x/n}\} + c$$

and it is found that $c = \log(y_0/m)$. Whence

$$b_x = \frac{1+(m-1)e^{x/n}}{me^{x/n}} \cdot y_0. \tag{15}$$

Now (13) is satisfied if

$$\frac{y}{y_0} = \frac{z}{y_0} \cdot \frac{b}{y_0} \text{ or } zb = y_0 y$$

Substituting (1) and (15) for z and b respectively gives

$$zb = \left\{\frac{me^{x/n}y}{1+(m-1)e^{x/n}}\right\} \left\{\frac{1+(m-1)e^{x/n}}{me^{x/n}}\right\} y_0 = y_0 y$$

thus confirming that (13) is satisfied.

9.4. It is interesting to see that in the life table for smallpox deaths only, b_x never decreases to zero. In fact making x infinite in (15) gives

$$b_\infty = y_0\left(\frac{m-1}{m}\right).$$

Thus for Bernoulli's figures $b_0 = 1{,}300$, $m = n = 8$, the minimum value of b_x is therefore 1,137·5. Hence the total smallpox deaths are 162·5 or one-eighth of 1,300, as of course they should be, since seven-eighths of those catching smallpox recover from it and, as they cannot again have smallpox, there are no other causes from which they can die! This is an extreme example of the absurdity which can arise from the uncritical elimination of a cause of death. However, one is left wondering whether the elimination of, say, deaths from heart disease (Karn, 1931) may contain some degree of similar unrealism.

9.5. It is perhaps worth pointing out that formula (4) for z was obtained by regarding Bernoulli's Table 1 as a double decrement table with (i) death from smallpox and (ii) death from other causes. But a formula for z could just as well have been obtained by taking the decrements as (i) catching smallpox and (ii) dying from other causes without having had smallpox. The deaths from other causes among s in time dx are $-ds-s\,.\,dx/n$ (i.e. the total decrement of s less the number getting smallpox). Therefore

$$-dz = -\frac{z}{s}(ds+sdx/n).$$

Integrating and evaluating the constant of integration gives $z=e^{x/n}\,s$ which is equivalent to (4) expressed in the form already given in (11).

9.6. Lambert's (1772) formula (7) is one which is still used today in dealing with double decrement tables and is to be found in the current textbooks on life contingencies (Neill, 1977) and construction of tables (Benjamin & Haycocks, 1970). Bailey & Haycocks (1946, 25) show that it does not exactly satisfy the fundamental relation (13) and indicate how the fraction in the denominator can be modified to get approximate agreement. Seal (1977) deals also with the development of multiple decrement formulae subsequent to the papers discussed here.

9.7. It is perhaps appropriate to speculate a little as to why no use was made of the eighteenth- and early nineteenth-century work described above when, in the mid-nineteenth century, exposed to risk formulae were needed for mortality investigations of assured lives. This work was not unknown to actuaries, since Milne (1815) mentions Bernoulli (1766) and Duvillard (1806). Why then was no use made of this work and why were exposed to risk formulae developed independently?

9.8. I suggest the reason might be on the following lines. All the work that had been done commenced with a mortality table which had already been prepared, often on faulty premises, and the figures for smallpox deaths which had been grafted on were usually obtained from some unrelated source. The situation was very different in the mid-nineteenth century when actuaries were preparing mortality tables from data of assured lives. The data were, by comparison, of high quality, giving dates or years of birth, of becoming assured, of withdrawal, of death, etc. Thus the problem was to develop schedules for summarizing the data and formulae for finding the exposed to risk which exactly corresponded with the deaths. They might also be involved with life years, policy years and calendar years. No existing mortality table formed the basis of their work, their aim was to prepare such a table from their data. I am not surprised that they did not get much help from the way in which the double decrement tables had been developed, even if they did realize that the exclusion of mortality from smallpox was fundamentally the same as excluding withdrawals. In any case they also had similar sorts of complications arising from new entrants, and retirements and possibly from beginners and enders as well.

9.9. Exposed to risk formulae were soon based on the concept of time of exposure to risk. Seal (1977) says that this dates back at least to Woolhouse (1867) but not he thinks to Lambert (1772). I agree with this and would add that I do not think the time concept had appeared in Duvillard (1806) either. Both Lambert and Duvillard considered the two extreme cases and took some sort of a mean between them. The time concept can be found in Woolhouse (1839) on which his 1867 paper is based and it was used in the Seventeen Offices' Experience Tables (1843).

9.10. Whatever was the reason that the earlier work was not used, I am of the opinion that the constructors of mortality tables from assured lives data were right not to use it. The time of exposure is a much more acceptable principle on which to base exposed to risk formulae than to take the mean of the extreme cases, even if the end result is the same.

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The location of copies of some of the rarer of the above references is indicated as follows:

(A) Library of the Institute of Actuaries.
(S) Library of the Royal Statistical Society.
(U) Library of University College, Gower Street, London.
(B) The British Library, London.