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SOME APPLICATIONS OF THE POISSON DISTRIBUTION IN MORTALITY STUDIES

by

W. F. SCOTT, M.A., Ph.D., F.F.A.

SUMMARY

Let θ_x denote the number of deaths (from all causes or a certain cause) at age x last birthday arising from the central exposed to risk E^c_x. We show that, under certain conditions, θ_x has approximately a Poisson distribution with parameter

$$E^c_x \int_x^{x+1} \mu_y \, dy$$

where μ_y denotes the force of mortality at age y. This result is applied to find confidence intervals for the true central death rate, to estimate the distribution of the cost of claims and the expected death strain.

The frequently-used actuarial formula

$$\frac{A-E}{\sqrt{E}} \dot{\sim} N(0, 1)$$

where A is the actual deaths and E the expected deaths, is found to hold in large investigations, as is the alternative (but generally less accurate) formula

$$\frac{A-E}{\sqrt{A}} \dot{\sim} N(0, 1).$$

1. *The Poisson process*

Consider a life aged x subject to one or more modes of decrement μ^α_y, μ^β_y, ... $(x \leqslant y \leqslant x+T)$, and let $\mu_y = \mu^\alpha_y + \mu^\beta_y + \ldots$. We shall assume that μ^α_y and μ_y are continuous on $[x, x+T]$.

Suppose that if the life leaves the population by any mode of decrement before age $x+T$, he will be replaced immediately by another life of the same attained age. This " replacement " life will himself be replaced if he leaves before age $x+T$, and so on.

We shall now prove two lemmas which are required in the proof of theorem 1.1.

Lemma 1.1. For $h>0$ and y, $y+h$ in $[x, x+T]$, define

$$r_y(h) = \frac{{}_h(aq)^\alpha_y - h\mu^\alpha_y}{h}.$$

Then if $\varepsilon>0$ there is h_0 such that

$$| r_y(h) | \leqslant \varepsilon$$

for $0<h \leqslant h_0$ and y, $y+h$ in $[x, x+T]$.

Proof. Let $\varepsilon>0$. We have, by the theory of the multiple-decrement table and the mean value theorem,

$$\begin{aligned} {}_h(aq)^\alpha_y &= \int_0^h {}_t(ap)_y \, \mu^\alpha_{y+t} \, dt \\ &= \int_0^h e^{-\int_0^t \mu_{y+s} ds} \, \mu^\alpha_{y+t} \, dt \\ &= h\mu^\alpha_{y+\xi} e^{-\int_0^\xi \mu_{y+s} ds} \quad (0<\xi<h;\ \xi \text{ depends on } y) \\ &= h\mu^\alpha_y + h(\mu^\alpha_{y+\xi} - \mu^\alpha_y) e^{-\int_0^\xi \mu_{y+s} ds} + h\mu^\alpha_y (e^{-\int_0^\xi \mu_{y+s} ds} - 1). \end{aligned}$$

The fact that μ^α_y is continuous, and hence uniformly continuous, on $[x, x+T]$ shows that the second term of the expression on the r.h.s. is bounded in modulus by $h\varepsilon$ when h is small enough. The final term equals

$$-h\mu^\alpha_y \xi \mu_{y+\eta} \, e^{-\int_0^\eta \mu_{y+s} ds} \quad (0<\eta<\xi;\ \eta \text{ depends on } y)$$

which is bounded in modulus by Kh^2, where K is an upper bound for $\mu^\alpha_y \mu_y$ on $[x, x+T]$. This proves lemma 1.1.

Lemma 1.2. There is $M \geqslant 0$ such that

$${}_h(aq)^\alpha_y \leqslant Mh$$

for all y, $y+h$ in $[x, x+T]$, $h \geqslant 0$.

Proof. Let $h \geqslant 0$ and let y, $y+h$ be in $[x, x+T]$, and let K^1 be an upper bound for μ^α_y on $[x, x+T]$. It follows that

$$\begin{aligned} {}_h(aq)^\alpha_y &= \int_0^h {}_t(ap)_y \, \mu^\alpha_{y+t} \, dt \\ &\leqslant K^1 h \end{aligned}$$

so we may let $M = K^1$ to conclude the proof of lemma 1.2.

Theorem 1.1. Let ${}_T\theta^\alpha_x$ be the number of exits by mode α between ages x and $x+T$. Then

$${}_T\theta^\alpha_x \sim \text{Poisson}\left(\int_x^{x+T} \mu^\alpha_y \, dy\right) \qquad (1.1)$$

Proof. We shall show that the postulates of a Poisson process, extended to cover variable "λ", hold (cf. [1; XVII 2] or [2; §62]). Let $x \leqslant y < y+h \leqslant x+\mathrm{T}$. By lemma 1.1, the chance that at least one exit by mode α occurs between ages y and $y+h$, ${}_h(aq)^\alpha_y$, is $\mu^\alpha_y h + o(h)$.[1] We also note that the chance of two or more such exits is

$$\int_0^h {}_t(ap)_y \, \mu^\alpha_{y+t} \{{}_{h-t}(aq)^\alpha_{y+t}\} \, dt$$

$$\leqslant \mathrm{M}h \int_0^h {}_t(ap)_y \, \mu^\alpha_{y+t} \, dt \leqslant \mathrm{M}^2 h^2 = o(h), \text{ by lemma 1.2.}$$

This shows that the postulates of the Poisson process hold, and theorem 1.1 is proved.

If we consider N independent "replaceable" lives and let ${}_\mathrm{T}\theta^\alpha_x$ be the number of exits by mode α between ages x and $x+\mathrm{T}$, we obtain

$${}_\mathrm{T}\theta^\alpha_x \sim \text{Poisson}\left(\mathrm{N}\int_x^{x+\mathrm{T}} \mu^\alpha_y \, dy\right) \tag{1.2}$$

by the additive property of the Poisson distribution. (*Remark*: if $\mathrm{T} = 1$ we may write ${}_\mathrm{T}\theta^\alpha_x$ as θ^α_x).

Now let us consider a real investigation in which the central exposed to risk ${}_\mathrm{T}\mathrm{E}^c_x$ at ages x to $x+\mathrm{T}$ has been calculated by a formula (e.g., a census formula) in which the actual numbers of exits (${}_\mathrm{T}\theta^\alpha_x$, etc.) do not appear. Let us assume that the number of exits by mode α in the actual investigation is approximately the same as would occur if $\frac{{}_\mathrm{T}\mathrm{E}^c_x}{\mathrm{T}}$ (or, strictly, the nearest integer to this number) "replaceable" lives were followed from age x to age $x+\mathrm{T}$. Under this assumption

$${}_\mathrm{T}\theta^\alpha_x \mathrel{\dot{\sim}} \text{Poisson}\left({}_\mathrm{T}\mathrm{E}^c_x . \frac{\int_x^{x+\mathrm{T}} \mu^\alpha_y \, dy}{\mathrm{T}}\right)$$

Hence

$${}_\mathrm{T}\theta^\alpha_x \mathrel{\dot{\sim}} \text{Poisson}\left({}_\mathrm{T}\mathrm{E}^c_x . {}_\mathrm{T}m^\alpha_x\right) \tag{1.3}$$

where we write ${}_\mathrm{T}m^\alpha_x = \frac{\int_x^{x+\mathrm{T}} \mu^\alpha_y \, dy}{\mathrm{T}}$, this being approximately equal to the central rate of decrement by mode α between ages x and $x+\mathrm{T}$ (if T is reasonably small).

The *crude central rate of decrement by mode* α from age x to age $x+\mathrm{T}$ is defined to be

$${}_\mathrm{T}\hat{m}^\alpha_x = \frac{{}_\mathrm{T}\theta^\alpha_x}{{}_\mathrm{T}\mathrm{E}^c_x} \tag{1.4}$$

[1] $o(h)$ denotes a quantity such that $h^{-1}o(h) \to 0$ as $h \to 0$, uniformly in y.

where we again omit " T " if T = 1. It follows from (1.3) that the mean and variance of ${}_{\mathrm{T}}\hat{m}_x^\alpha$ are approximately equal to ${}_{\mathrm{T}}m_x^\alpha$ and ${}_{\mathrm{T}}m_x^\alpha/{}_{\mathrm{T}}\mathrm{E}_x^c$ respectively. It also follows that, if (1.3) holds exactly, the crude central rate of decrement ${}_{\mathrm{T}}\hat{m}_x^\alpha$ is an *unbiased* estimate of the true rate ${}_{\mathrm{T}}m_x^\alpha$. It is also " *efficient* " (i.e. of minimum variance) since a straightforward calculation shows that $\mathrm{var}({}_{\mathrm{T}}\hat{m}_x^\alpha) = {}_{\mathrm{T}}m_x^\alpha/{}_{\mathrm{T}}\mathrm{E}_x^c$ $= \{\text{Fisher's amount of information}\}^{-1}$ (see [3; section 2.9]).

2. *Confidence intervals for the true rate of decrement*

We shall find approximate confidence intervals for the true central rate of decrement by mode α between ages x and $x+\mathrm{T}$. We first require the following result from statistical theory.

Theorem 2.1. Let $\{\lambda_n\}$ be a sequence of positive numbers which tends to infinity, and let $x_n \sim \mathrm{Poisson}(\lambda_n)$. Define the variable

$$\eta_n = \begin{cases} \dfrac{x_n-\lambda_n}{\sqrt{x_n}} & \text{for } x_n \neq 0, \\ -\sqrt{\lambda_n} & \text{for } x_n = 0. \end{cases}$$

As n tends to infinity, the distribution of η_n converges to the standard normal distribution, N(0, 1).

Proof. Let $\varepsilon, \delta > 0$. By Tchebycheff's inequality [4; (15.7.2)] the probability that

$$\left|\frac{x_n-\lambda_n}{\lambda_n}\right| \leqslant \frac{1}{\sqrt{\varepsilon\lambda_n}}$$

is at least $1-\varepsilon$. It follows that, if $\lambda_n > \dfrac{1}{\varepsilon\delta^2}$, $\left|\dfrac{x_n-\lambda_n}{\lambda_n}\right| < \delta$, so $\dfrac{x_n}{\lambda_n}$ converges to 1 in probability. By [5; exercise 7.11], so does $\sqrt{\dfrac{x_n}{\lambda_n}}$. Define $\zeta_n = \sqrt{\dfrac{x_n}{\lambda_n}}$ for $x_n \neq 0$, and $\zeta_n = 1$ for $x_n = 0$. It follows that ζ_n converges to 1 in probability and $\zeta_n \neq 0$ for all n, so by [4; 20.2 and p. 254] the distribution of $\eta_n = \left(\dfrac{x_n-\lambda_n}{\sqrt{\lambda_n}}\right)/\zeta_n$ converges to N(0, 1).

Applying this result to ${}_{\mathrm{T}}\theta_x^\alpha$ we obtain, when the central exposed to risk is large,

$$\frac{{}_{\mathrm{T}}\theta_x^\alpha - {}_{\mathrm{T}}\mathrm{E}_x^c \cdot {}_{\mathrm{T}}m_x^\alpha}{\sqrt{{}_{\mathrm{T}}\theta_x^\alpha}} \dot{\sim} \mathrm{N}(0, 1) \qquad (2.1)$$

and hence

$$\frac{{}_{\mathrm{T}}\hat{m}_x^\alpha - {}_{\mathrm{T}}m_x^\alpha}{{}_{\mathrm{T}}\hat{m}_x^\alpha/\sqrt{{}_{\mathrm{T}}\theta_x^\alpha}} \dot{\sim} \mathrm{N}(0, 1) \qquad (2.2)$$

(If ${}_{\mathrm{T}}\theta_x^\alpha = 0$ the expressions on the l. h. s.'s of (2.1) and (2.2) are each taken as $-\sqrt{{}_{\mathrm{T}}\mathrm{E}_x^c \cdot {}_{\mathrm{T}}m_x^\alpha}$; when ${}_{\mathrm{T}}\mathrm{E}_x^c$ is large the probability that ${}_{\mathrm{T}}\theta_x^\alpha = 0$ is small.) It follows that, in a large investigation, an approximate P% confidence interval for the true central rate of decrement ${}_{\mathrm{T}}m_x^\alpha$ is

$$ {}_{\mathrm{T}}\hat{m}_x^\alpha \, (1 \pm d_p/\sqrt{{}_{\mathrm{T}}\theta_x^\alpha}) \tag{2.3} $$

where $\Phi(d_p) = \frac{1}{2}(1+\frac{\mathrm{P}}{100})$.

Example 2.1

The number of deaths from respiratory cancer (I.C.D. numbers 160-163) in Scotland in 1978 among males aged 55 to 59 last birthday was 306, and the corresponding estimate of the central exposed to risk (the mid-year estimate of population) was 140,000 [6; tables C2.1, A2.3]. Using formula (2.3) with α = death from respiratory cancer, an approximate 95% confidence interval for the true central death rate from this cause for men aged 55-59 last birthday is

$$ 0{\cdot}002179 \left(1 \pm \frac{1{\cdot}96}{\sqrt{306}}\right) $$

$= 2{\cdot}179 \pm 0{\cdot}244$ per thousand.

(There may also of course be sources of non-random variation in death rates, e.g. age misstatements, epidemics and changes in medical practice regarding the certification of cause of death.)

3. *Mortality by " lives " and " amounts "*

From now on we suppose that α represents mortality in a life office, pension fund or other study in which the sum assured (including any bonuses) is not fixed. One is naturally interested in the cost of claims as well as their number.

To simplify the notation we shall assume T = 1 and, for $i = 1, 2, \ldots, r$, denote the central exposed to risk at age x last birthday for lives with sum assured S_i by $n_i^{(x)}$. Let ${}^{\mathrm{A}}\theta_x^\alpha$ denote the cost of benefits for those at age x last birthday. It is clear that

$$ {}^{\mathrm{A}}\theta_x^\alpha = \sum_{i=1}^{r} \mathrm{S}_i \zeta_i^{(x)} \tag{3.1} $$

where

$$ \zeta_i^{(x)} \sim \mathrm{Poisson}\left(n_i^{(x)} . \int_x^{x+1} {}^i\mu_y^\alpha \, dy\right) \tag{3.2} $$

The $\zeta_i^{(x)}$ are independent, and ${}^i\mu_y^\alpha$ denotes the force of mortality at age y among those in group i, so the m^{th} cumulant (see [4; 15.10 and 16.5]) of ${}^{\mathrm{A}}\theta_x^\alpha$ is approximately

$$\chi_m = \sum_{i=1}^{r} n_i^{(x)}(S_i)^m . \int_x^{x+1} {}^i\mu_y^\alpha \, dy \tag{3.3}$$

The central exposed to risk at age x last birthday, measured by " amounts ", is defined to be

$$^{A}E_x^c = \sum_{i=1}^{r} S_i \, n_i^{(x)} \tag{3.4}$$

and the *crude central rate of decrement by mode* α at age x last birthday, measured by "amounts", is defined to be

$$^{A}\hat{m}_x^\alpha = \frac{^{A}\theta_x^\alpha}{^{A}E_x^c} \tag{3.5}$$

The distribution of $^{A}\theta_x^\alpha$ approximates to the normal if the exposed to risk $n_i^{(x)}$ for each group (especially those in which S_i is large) is large. To help determine how close the distribution of $^{A}\theta_x^\alpha$ is to normality, we may calculate its skewness $\gamma_1 = \chi_3/\chi_2^{3/2}$ and excess $\gamma_2 = \chi_4/\chi_2^2$, which will be small if the distribution of $^{A}\theta_x^\alpha$ is close to the normal.

The corresponding values for " lives " are obtained by substituting $S_i = 1$ for $i = 1, 2, \ldots, r$ in formulae (3.1) to (3.5).

In life office and pension fund studies it is often found that mortality rates measured by amounts are lower than those measured by lives; this feature is due to the fact that lives with large sums assured or pensions tend to be healthier than those whose sums assured or pensions are small. Financial calculations should normally be based on an " amounts " table (unless the effect of using a " lives " table is to give errors on the safe side.) If, however, no such variation exists, the mean of $^{A}\hat{m}_x^\alpha$ is the same as for $\hat{m}_x^\alpha$ (the corresponding central rate on the basis of lives), although their variances will in general differ.

Example 3.1

In a certain life office investigation the central exposed to risk in respect of policyholders aged 50 last birthday with the various sums assured is as follows:

sum assured S_i (£)	central exposed to risk $n_i^{(50)}$
1,000	76
5,000	143
10,000	103
20,000	44
50,000	21
all	387

Estimate the mean, standard deviation, skewness and excess of (*a*) the number of deaths, and (*b*) the cost of claims, given that, irrespective of sum assured, mortality follows the A1967-70 ultimate table.

Solution

We first note that, according to A1967-70 ultimate mortality, $\mu^{a}_{50.5} = 0{\cdot}004805$. By formula (3.3), the m^{th} cumulant of ${}^{A}\theta^{x}_{\alpha}$ is approximately

$$\chi_m = \{(1{,}000)^m \times 76 + (5{,}000)^m \times 143 + (10{,}000)^m \times 103 + (20{,}000)^m \times 44 + (50{,}000)^m \times 21\} \times 0{\cdot}004805$$

so

$$\chi_1 = 18{,}024,\ \chi_2 = 403{\cdot}865 \times 10^6,\ \chi_3 = 14{,}886 \times 10^9,\ \chi_4 = 669{,}862 \times 10^{12}$$

The corresponding formulae for " lives " are

$$\chi_1 = \chi_2 = \chi_3 = \chi_4 = (76+143+103+44+21) \times 0{\cdot}004805 = 1{\cdot}860.$$

We therefore obtain the following results:

	lives	amounts
mean	1·860	18,024
standard deviation	1·364	20,096
skewness	0·733	1·834
excess (or kurtosis)	0·538	4·107

As might be expected, the departure from normality is greater on the " amounts " basis.

4. *Duplicate policies*

Consider a life office investigation in which the central exposed to risk, measured by lives, at age x last birthday among those with i policies is $n^{(x)}_i$ ($i = 1,\ 2,\ \ldots,\ r$). When measured by policies the central exposed to risk is

$$ {}^{P}E^{c}_{x} = \sum_{i=1}^{r} i\, n^{(x)}_i \tag{4.1}$$

and the distribution of the number of policies becoming claims, ${}^{P}\theta^{\alpha}_{x}$, is the same as for " mortality by amounts " with $S_i = i$ for $i = 1, 2, \ldots, r$. Assuming that the force of mortality μ^{α}_{y} at age y does not depend on the number of policies held, the m^{th} cumulant of ${}^{P}\theta^{\alpha}_{x}$ is, by (3.3), approximately equal to

$$\sum_{i=1}^{r} i^m\, n^{(x)}_i\, \mu^{\alpha}_{x+\frac{1}{2}} \tag{4.2}$$

Letting

$$p^{(x)}_i = \frac{n^{(x)}_i}{n^{(x)}_1 + n^{(x)}_2 + \ldots + n^{(x)}_r}\ (1 \leqslant i \leqslant r) \tag{4.3}$$

we obtain the result that ${}^{P}\theta_x^a$ has approximate mean

$$ {}^{P}E_x^c\, \mu_{x+\frac{1}{2}}^a \tag{4.4}$$

and approximate variance

$$ R_x {}^{P}E_x^c\, \mu_{x+\frac{1}{2}}^a \tag{4.5}$$

where

$$ R_x = \frac{\sum_{i=1}^{r} i^2\, p_i^{(x)}}{\sum_{i=1}^{r} i\, p_i^{(x)}} \tag{4.6}$$

Estimates of these factors R_x may be used in the graduation of the crude central death rate by policies, which is

$$ {}^{P}\hat{m}_x^a = \frac{{}^{P}\theta_x^a}{{}^{P}E_x^c} \tag{4.7}$$

This type of adjustment was made in the graduation of the A1967-70 tables for assured lives, although in these tables the function being graduated was $\hat{q}_{[x-\frac{1}{2}]+t}$ or $\hat{q}_{x-\frac{1}{2}}$ (the factors R_x being derived from an earlier study of duplicates).

5. *The analysis of surplus*

In the analysis of surplus of a life office, pension fund or friendly society one is interested in the excess (or shortfall) of the cash S payable on death over the reserve V held for the policy. The " death strain " is therefore defined by weighting the deaths by the " death strain at risk " $D = S - V$.

Suppose that for $i = 1, 2, \ldots, r$, the central exposed to risk at age x last birthday for lives with death strain at risk D_i is $n_i^{(x)}$. By arguments similar to those of section 3, the death strain at age x last birthday has mean $\left(\sum_{i=1}^{r} n_i^{(x)} D_i\right)\mu_{x+\frac{1}{2}}^a$ and variance $\left(\sum_{i=1}^{r} n_i^{(x)} D_i^2\right)\mu_{x+\frac{1}{2}}^a$ approximately. An " amounts " table of mortality should be used to compute $\mu_{x+\frac{1}{2}}^a$.

In many investigations one uses the formula

$$ n_i^{(x)} = \frac{T}{2}\left({}^{0}n_i^{(x)} + {}^{T}n_i^{(x)}\right) \tag{5.1}$$

where ${}^{0}n_i^{(x)}$ and ${}^{T}n_i^{(x)}$ are the numbers of policies in force at age x last birthday with death strain at risk D_i at times 0 and T respectively. In this case the expected death strain is approximately equal to

$$ \frac{T}{2}\left\{\sum_{i=1}^{r} {}^{0}n_i^{(x)} D_i + \sum_{i=1}^{r} {}^{T}n_i^{(x)} D_i\right\}\mu_{x+\frac{1}{2}}^a $$

$$ = \frac{T}{2}\left\{(S_0^{(x)} - V_0^{(x)}) + (S_T^{(x)} - V_T^{(x)})\right\}\mu_{x+\frac{1}{2}}^a \tag{5.2}$$

where $S_0^{(x)}$ and $V_0^{(x)}$ denote the total sums assured and reserves on lives aged x last birthday in force at time 0, and $S_T^{(x)}$ and $V_T^{(x)}$ denote these quantities at time T. Formula (5.2) is known as *Ryan's formula* (cf. [7; p. 283]) and is used in several investigations, e.g. [7; formula (14.12)], in which T = 1; [8; section 19.5] in which T = 5 and there is a change of sign to convert " strain " into " release "; and [9; appendix III], in which $q_{x-\frac{1}{2}}$ is used in place of μ_x.

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