JIA 116 (1989) 611-624

SOME ASPECTS OF THE MODELLING OF PERMANENT HEALTH INSURANCE

BY H. R. WATERS, M.A., D.PHIL., F.I.A.

[Presented at the Seminar, 'Applications of Mathematics in Insurance, Finance and Actuarial Work', sponsored by the Institute of Mathematics and Its Applications, the Institute of Actuaries, and the Faculty of Actuaries, held at the Institute of Actuaries, 6–7 July 1989.]

1. INTRODUCTION

1.1 The traditional approach in the United Kingdom to the analysis of permanent health insurance (PHI) data and to the rating of PHI business has been the Manchester Unity approach. This approach, which has its origins in Friendly Society business, is in some ways unsuitable for modern PHI business. (An interesting discussion on this subject can be found in Report No. 7 of the Continuous Mortality Investigation Bureau (*C.M.I.R.* 7, §4 (1984))[2]. For this reason the PHI Sub-Committee of the CMIB has recently been investigating the possibility of using a different approach, involving the use of a multiple state model, for analysing PHI data. A full report on the Sub-Committee's investigation will be published soon as *C.M.I.R.* 10.

1.2 This paper is concerned with the modelling of PHI in general, and with the CMIB's multiple state model in particular. We shall not attempt to duplicate the very full discussion of the model which will appear in *C.M.I.R.* 10. Instead, we shall concentrate on the reasons underlying the choice of a model for PHI.

1.3 In Sections 2, 3 and 4 we describe the multiple state model being investigated by the CMIB, outline the uses and requirements of a model for PHI and discuss briefly the extent to which the CMIB's model, and some other models for PHI, fulfil these requirements.

1.4 In Sections 5 and 6 we illustrate the power and flexibility of the CMIB's model by discussing two technical problems. The first problem is the calculation of an Incurred But Not Reported (IBNR) reserve for one-year group PHI business. Although possibly not of greater practical interest, this problem does have some interesting features and was mentioned in a recent paper by Turner (1988).[8] The second problem is, in my opinion, of much greater interest. This problem is the calculation of the statistical properties, in particular the mean, variance and covariance, of estimators for sickness rates. This is a problem which has been discussed in the British actuarial literature several times, the most recent contribution being by Daw (1986).[3]

1.5 The numerical illustrations in Sections 5 and 6 make use of graduations of PHI data which are to be published by the CMIB in *C.M.I.R.* 10. I am grateful to the CMIB for permission to use these graduations in advance of their publication.

2. THE CMIB'S MULTIPLE STATE MODEL

2.1 The multiple state model being investigated by the CMIB can conveniently be represented by the diagram in Figure 1. An individual is in one of the three states, Healthy, Sick or Dead. The possible transitions between these states are indicated by the arrows in Figure 1. It is important to note that 'Sick' is not equivalent to 'Claiming under the terms of a PHI policy'. In other words, for the CMIB's model, 'Sick' means 'Sick, no matter how short the duration'.

2.2 The transition intensities between the states of the model are indicated in Figure 1. These transition intensities are just generalizations of the force of mortality for a life table and should be interpreted as follows:

(i) for an individual who is healthy and currently aged x, the probability of
  (a) becoming sick before age $x+h$ is $h \cdot \sigma_x$;
  (b) dying before age $x+h$ is $h \cdot \mu_x$;

(ii) for an individual who is sick and is currently aged x with duration of current sickness z, the probability of
  (a) recovering before age $x+h$ is $h \cdot \rho_{x,z}$;
  (b) dying before age $x+h$ is $h \cdot \nu_{x,z}$.

where h is so small that the probability of two or more transitions before age $x+h$ can be ignored. Implicit in these statements are the assumptions that, in probabilistic terms, the future movements between the states of a healthy individual depend only on the current age of the individual and the future movements of a sick individual depend only on the current age and duration of current sickness of the individual. To be more specific, the model will regard as identical two healthy individuals of the same age, one of whom is healthy because he/she has just been accepted at normal rates for a PHI policy and the other is healthy because he/she has just recovered from a long and serious illness.

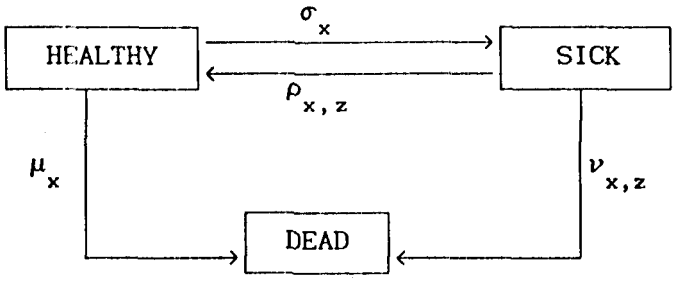

Figure 1.

3. THE RÔLE OF A MODEL

3.1 The uses to which a model for PHI would be put can be summarized as follows:

(i) the monitoring of the experience of a PHI portfolio and comparisons between different portfolios;
(ii) the derivation of a set of tables or rates to be used to calculate premiums for PHI policies;
(iii) the derivation of a set of tables or rates to be used to calculate reserves for PHI policies.

3.2 To be able to perform these functions, the model has to meet the following requirements:

(i) it has to be sufficiently realistic, in particular it should not be such a simple model that it ignores important features of PHI business;
(ii) it should be sufficiently simple to be mathematically tractable, i.e. to be possible to derive mathematically, or at least numerically, a set of tables or rates for setting premiums or reserves;
(iii) it must be possible from the data available to estimate key quantities which determine numerical values for the model, and, equally important, it should be possible to do this in such a way that the statistical properties of the estimators are known.

3.3 For the CMIB's multiple state model, the key quantities which determine numerical values for the model, and hence which have to be estimated from data, are the transition intensities. A full discussion of parameter estimation and graduation for this model will appear in *C.M.I.R.* 10 and will not be repeated here. However, two points should be made:

(i) it is possible to calculate from the data available to the CMIB point estimates of, say, $\rho_{x,z}$ for different values of x and z, which are (asymptotically) independent of other estimators and normally distributed with a variance which can be calculated; this feature of the CMIB's model is extremely useful in relation to §3.1(i);
(ii) as emphasized in §2.2, the CMIB's model is a model of sickness and not just of claims (under a PHI policy). Since the CMIB collects data only about claims and not about sicknesses there is an apparent failure of the model to meet requirement §3.2 (iii) since, for example, no data are available to the CMIB concerning sicknesses which do not last beyond the deferred period of a PHI policy. This point will be discussed further in the following two sections.

3.4 Before ending this (very brief and very general) discussion of modelling PHI, it should be pointed out that in practice different models may be appropriate for different purposes. For example, a model for reserving may well

be less complex than (and would probably have different parameter values from) a model for pricing PHI contracts. However, the advantages of having available a single relatively complex model are:

(i) simpler models can be derived from a more complex model, whereas the reverse is not true;
(ii) a more complex model can be used to check the approximations and simplifications inherent in a simpler model. This last point was made by Turner (1988),[8] § 3.4, in connection with group PHI.

4. ALTERNATIVE MODELS

4.1 In this section we shall discuss some alternative models which have been proposed and, in most cases, implemented for PHI, in particular very briefly the extent to which these alternative models meet, or fail to meet, the requirements for a model set out in the previous section. It is not practicable to discuss all alternative models, or even to describe fully those models we do discuss. The reader interested in a fuller discussion should consult Hamilton-Jones (1972).[5]

The Manchester-Unity Approach

4.2 It is difficult to regard the Manchester-Unity approach as a model at all. With this approach, premium rates and reserves are calculated using sickness rates. The central rate of sickness at age x for duration of sickness $a \rightarrow (a+b)$ weeks is denoted $z_x^{a|b}$ and defined to be the ratio of the (expected) number of weeks spent sick between ages x and $(x+1)$, counting only the time when the duration of sickness is between a and $(a+b)$ weeks, to the (expected) time spent alive between ages x and $(x+1)$. While this approach leads to very simple formulae for premiums and reserves (see, for example, Neill (1977)[7] Ch. 11) (and this advantage is not to be dismissed lightly) it does have some disadvantages. In particular:

(i) the sickness rate for an individual PHI policyholder depends on the policyholder's age but not on how long the policy has been in force, so that, for example, a policyholder who has just effected a policy should have, but will not have, a zero sickness rate for durations of sickness in excess of 1 year. This point is discussed in *C.M.I.R.* 7 (1984) § 4. It could be argued that this feature of the Manchester-Unity approach shows that it is too simple an approach for (some) practical purposes.
(ii) Although it is not difficult to estimate sickness rates from the data available to the CMIB, the statistical properties of these estimates are not so easily determined. These properties need to be known so that the raw estimates can be smoothed, i.e. graduated. The difficulties this problem has caused to the CMIB are discussed in *C.M.I.R.* 4 (1979) § 3.[1]

Both the above points will be discussed further in Section 6.

The Continuance Table Model

4.3 The Continuance Table model is commonly used in the United States of America and is described in many papers. A particularly clear description is given by Westwood (1972).[10] For the purposes of this paper, we shall regard the Continuance Table model as including the Swedish Sickness Annuities model (see Dillner (1969)[4]). The Continuance Table model can be described very briefly as follows:

each year a policy, or portfolio of policies, produces a number of claims;

once a claim has been made, i.e. a sickness has lasted beyond the deferred period, the probability of the claim terminating at any given future time is given by a 'continuance table';

claim inception rates are age-dependent, and termination rates are age- and duration of claim-dependent.

Whilst this model may be suitable for some, possibly very many, practical purposes, it does have some undesirable features:

(i) the claim inception rates are designed to be applied to all policies in force at that age, irrespective of whether the policyholder is healthy or sick, or even currently claiming. This means that observed claim inception rates will depend to some extent on the maturity of the portfolio (a feature similar to §4.2(i) concerning the Manchester-Unity approach). This point has been emphasized by Dillner (1969)[4] §2 and numerical examples relating to it will be given in *C.M.I.R.* 10.

(ii) In its published form, a Continuance Table gives rates of claim termination but does not distinguish between recoveries and deaths. This is somewhat inconvenient when monitoring the experience of a PHI portfolio.

(iii) In its application, the Continuance Table model involves some approximations. A good example of a particular approximation is given in Westwood (1972)[10], Appendix A. It is difficult to judge whether such approximations are acceptable in practice without having a more detailed model available to quantify them. (This is an example of §3.4 (ii).)

A Variation on the CMIB's Model

4.4 The CMIB's model would be somewhat simpler, at least algebraically and numerically, if the three states 'Healthy', 'Sick' and 'Dead' were replaced by the three states:

(a) alive but not claiming, i.e. healthy or sick with duration of sickness less than the deferred period of the policy;

(b) alive and claiming, i.e. sick with duration of sickness greater than the deferred period of the policy;

(c) dead.

In other words, this simpler model makes no attempt to model all sicknesses, but models only the claims under a policy. Such a model was discussed in a very interesting paper by Medin (1952),[6] although it is only in § 4 of Medin's paper that it becomes clear that he is talking about this model rather than the CMIB's. (Some of the algebra relating to this simpler model is given by Hamilton-Jones (1972)[5] Appendix 2.) A further advantage of this simpler model is that it ties in much more closely with the data available to the CMIB than does the CMIB's model itself. (See § 3.3 (ii).) However, the advantages of the CMIB's model, and hence the reasons why the CMIB decided to investigate the model described in Section 2 rather than the simplification discussed in this paragraph, are as follows:

(i) the CMIB's model is more realistic than the simpler model. In particular, we are asked to accept for the simpler model that a policyholder who has just recovered from a sickness may immediately make another claim. This may be a reasonable approximation for short deferred periods but is less acceptable for longer deferred periods.
(ii) The simpler model is, to some extent, dependent on the deferred period of the policy, which the CMIB's model is not. This makes it somewhat more difficult, with the simpler model, to compare the experiences of PHI portfolios with different deferred periods.
(iii) The simpler model makes no attempt to model the behaviour of a PHI policyholder while sick, but with duration of sickness less than the deferred period of the policy. The circumstances where this behaviour is important may be very specialized but they do exist and an example is discussed in the next section.

5. IBNR RESERVES FOR GROUP PHI

5.1 In this section we discuss a technical problem which can be dealt with using the CMIB's model but not using any of the alternative models discussed in the previous section. The problem relates to group PHI and is as follows. We suppose a group of lives are covered by a one-year group PHI contract with a deferred period of 26 weeks. All the lives are healthy at the start of the year. If any of the lives falls sick within the one-year period of cover, and if this sickness lasts beyond 26 weeks then the insurer pays an annuity at the rate of 1 p.a. continuously until the life recovers, dies or reaches age 65, whichever is sooner. At the end of the one-year period of cover, the insurer requires reserves in respect of:

(i) lives who are currently claiming, i.e. who have, at the end of the year, already been sick for more than 26 weeks, and;
(ii) lives who are currently sick, but whose sickness has not yet lasted, and may not last, beyond the 26 week deferred period and hence become a claim.

(Throughout this section we shall refer to these two types of reserve as a 'current claim reserve' and an 'IBNR reserve' respectively.)

In principle the first of these reserves is not a problem in terms of calculation and can be calculated by, for example, a Continuance Table method. The second reserve is more of a problem and is what interests us in this section. It is not our intention to suggest that the problem outlined above is one of the more important problems relating to group PHI, but it is of some interest, has been mentioned in the literature (Turner (1988)[8] § 6.3) and does allow us to illustrate the power and flexibility of the CMIB's model.

5.2 For the study of this problem we need some notation for probabilities and annuities. We denote by

$_{z,t}p_x^{hs}$ the probability that at age $(x+t)$ a life, who was healthy at age x, is sick with duration of sickness less than or equal to z. (All times and ages are measured in years.)

$_tp_{x,z}^{\overline{ss}}$ the probability that a life who is aged x and has been sick for duration z, will remain sick for a further duration t without recovering or dying,

$\bar{a}_{x,z:\overline{65-x}|}^{\overline{ss}}$ the value of an annuity payable continuously at the rate of 1 p.a. to a life currently aged x and who has been sick for duration z; the annuity ceases when the life recovers, dies or reaches age 65, whichever is sooner,

$_t|\bar{a}_{x,z:\overline{65-x}|}^{\overline{ss}}$ an annuity value defined as above except that payments do not start until the life reaches age $(x+t)$, and will not start if the life recovers, dies or reaches age 65 before age $(x+t)$.

We shall not discuss here how these probabilities and annuities can be calculated from graduations of the transition intensities σ_x, μ_x, $\rho_{x,z}$ and $\nu_{x,z}$; this will be discussed fully in *C.M.I.R.* 10. Note that the numerical illustrations in this and in the following section use graduations of the transition intensities based on data for male individual PHI policyholders, 1975–78. In particular, it should be noted that in this section we are using data on individual PHI policyholders to illustrate a problem relating to group PHI. This is because graduations relating to group PHI are not currently available.

5.3 With the above notation we can write down formulae for $_1V_x^{sc}$ and $_1V_x^{sn}$, where

$_1V_x^{sc}$ is the expected present value at age $(x+1)$ of a claim in course of payment at age $(x+1)$ to a life who was healthy at age x,

$_1V_x^{sn}$ is the expected present value at age $(x+1)$ of a claim resulting from a sickness at age $(x+1)$ whose duration is less than 26 weeks, to a life who was healthy at age x.

Note that these reserves correspond to the reserves outlined in § 5.1(i) and (ii). Note also that, although present values are calculated at age $(x+1)$, these

reserves are per policy in force (and healthy) at age x. The formulae for these reserves are as follows:

$$ {}_1V_x^{sc} = \int_{z=\frac{1}{2}}^{1} \frac{d}{dz}({}_{z,1}p_x^{hs})\, \bar{a}^{\overline{ss}}_{x+1,z:\overline{64-x}|}\, dz \tag{5.1} $$

$$ {}_1V_x^{sn} = \int_{z=0}^{\frac{1}{2}} \frac{d}{dz}({}_{z,1}p_x^{hs}) \cdot {}_{\frac{1}{2}-z|}\, \bar{a}^{ss}_{x+1,z:\overline{64-x}|}\, dz. \tag{5.2} $$

When interpreting the above expressions it may be helpful to regard the term

$$ \frac{d}{dz}({}_{z,1}p_x^{hs}) \cdot dz $$

as representing the probability that a healthy individual aged x will be sick at age $(x+1)$ with duration of sickness between z and $(z+dz)$.

5.4 Table 1 shows values of these reserves, together with some relevant probabilities, for various initial ages. All the reserves in Table 1 are calculated at a rate of interest of 9% p.a. The figures in Table 1 indicate that the IBNR reserve, ${}_1V_x^{sn}$, is numerically approximately equal to the current claim reserve, ${}_1V_x^{sc}$. This conclusion has some intuitive appeal since ${}_1V_x^{sn}$ represents a reserve for claims resulting from sicknesses starting in the second half of the year and ${}_1V_x^{sc}$ represents a reserve for claims resulting from sicknesses starting in the first half of the year. That this line of argument may be a little too simplistic is indicated by the calculations described in the next paragraph.

5.5 Suppose now that instead of calculating a current claim and an IBNR reserve immediately the year of cover ends, we calculate these reserves 3 months after the end of the year. In particular, we calculate

$$ {}_{1\frac{1}{4}}V_x^{sc} = \int_{z=\frac{1}{4}}^{1} \frac{d}{dz}({}_{z,1}p_x^{hs}) \cdot {}_{\frac{1}{4}}p_{x+1,z}^{\overline{ss}} \cdot a^{\overline{ss}}_{x+1\frac{1}{4},z+\frac{1}{4}:\overline{63\frac{3}{4}-x}|}\, dz $$

$$ {}_{1\frac{1}{4}}V_x^{sn} = \int_{z=0}^{\frac{1}{4}} \frac{d}{dz}({}_{z,1}p_x^{hs}) \cdot {}_{\frac{1}{4}}p_{x+1\frac{1}{4},z}^{\overline{ss}} \cdot {}_{\frac{1}{4}-z|}a^{\overline{ss}}_{x+1\frac{1}{4},z+\frac{1}{4}:\overline{63\frac{3}{4}-x}|}\, dz $$

so that ${}_{1\frac{1}{4}}V_x^{sn}$ represents the reserve to be held 3 months after the end of the year for claims then in course of payment, resulting from sickness starting within the year, and ${}_{1\frac{1}{4}}V_x^{sn}$ is the corresponding IBNR reserve. Note that both reserves are (expected) present values as at 3 months after the end of the year and per policy in force (and healthy) at the start of the year. Table 2 shows values of these reserves for different initial ages. The reserves in Table 2 are all calculated at a rate of interest of 9% p.a. The interesting feature of Table 2 is that the ratio of ${}_{1\frac{1}{4}}V_x^{sc}$ to ${}_{1\frac{1}{4}}V_x^{sn}$ is, very roughly, about two to one, whereas the simplistic argument in the previous paragraph would have led us to expect it to be about three to one.

5.6 A final, but not unimportant, point to note in relation to the calculations

in this section is that the 'current claim' reserves, ${}_1V_x^{sc}$ and ${}_{1\frac{1}{4}}V_x^{sc}$, are expected amounts per policy in force at the start of the year. At the time these reserves apply, the insurer should, in principle although maybe not in practice, know what proportion of the original lives are in fact claiming. (This, after all, is the distinction between our current claim and IBNR reserves.) If, for example, the insurer knows that a single life who was healthy at the start of the year is sick at the end of the year with duration of sickness, say, 9 months, the correct reserve is $\bar{a}_{x+1,\frac{3}{4}:\overline{64-x}|}$ and not, ${}_1V_x^{sc}$.

Table 1. *Reserves and probabilities at age (x+1) for a life who was healthy at the initial age* x

Initial age	Reserves at age $(x+1)$		Probabilities at age $(x+1)$		
x	${}_1V_x^{sc}\times 10^5$	${}_1V_x^{sm}\times 10^5$	Alive	Sick and claiming	Sick but not claiming
20	11	13	·99914	·00006	·00343
30	31	35	·99941	·00013	·00320
40	103	115	·99872	·00033	·00386
50	378	411	·99672	·00100	·00611
60	906	944	·99261	·00369	·01260

Table 2. *Reserves at age (x+1¼) for a life who was healthy at the initial age* x

Initial age	Reserves at age $(x+1\frac{1}{4})$	
x	${}_{1\frac{1}{4}}V_x^{sc}\times 10^5$	${}_{1\frac{1}{4}}V_x^{sm}\times 10^5$
20	11	6
30	34	18
40	120	58
50	446	207
60	838	464

6. STATISTICAL PROPERTIES OF ESTIMATORS FOR SICKNESS RATES

6.1 In two of its reports, *C.M.I.R.* 4 (1979)[1] and *C.M.I.R.* 7 (1984)[2], the CMIB has attempted to graduate estimates of sickness rates. The difficulties encountered by the CMIB are well documented in these reports and are attributed in part to the

"... lack of any wholly satisfactory ... statistical models to represent sickness data...."
(*C.M.I.R.* 4 (1979) §(3.1).

These problems can be summarized briefly as the lack of knowledge of the second and higher moments of the estimators for sickness rates, and also the lack of knowledge of the correlation between the estimators at successive ages for a given individual. These problems have been considered by Daw (1986),[3] who gives references to earlier studies.

6.2 In this section we shall show that the CMIB's multiple state model can be

used to provide some answers to these problems. (We regard this as an example of a more complex model being used to provide useful information relating to a simpler model, which itself may be adequate for some practical purposes. See the comments in §3.4.)

6.3 We define $X_x^{a/b}$ to be a random variable denoting the time spent sick between ages x and $(x+1)$ by an individual, counting only the time when the duration of sickness is between a and $(a+b)$ weeks. We shall measure this time in years even though sickness rates are typically measured in weeks. We might be tempted to regard the following expression

$$E[X_x^{a/b} | \text{the individual is alive at age } x]$$

as an appropriate sickness rate. However, this expression is not well defined in the context of the CMIB's model because the conditioning is not sufficiently precise. We can overcome this difficulty by considering the following, well defined, conditional expectation

$E[X_x^{a/b}|$the individual is alive at age x and was healthy at (some earlier) age $y]$.

For brevity we shall denote this expression

$$E[X_x^{a/b} | A \text{ at } x, H \text{ at } y]$$

This conditional expectation is almost, but not quite, a sickness rate in the usual terminology. The differences are that it includes the extra conditioning at age y and also that it is not divided by the expected time lived between ages x and $(x+1)$. However, we shall regard it as sufficiently close to a sickness rate to be able to interpret the problem outlined in §6.1 as the determination of the moments of $X_x^{a/b}$ given the conditions, 'Alive at x and Healthy at y'.

6.4 In Tables 3 and 4 we give numerical values for

$$E[X_x^{a/b} | A \text{ at } x, H \text{ at } y]$$
$$V[X_x^{a/b} | A \text{ at } x, H \text{ at } y]$$
$$r_{x,x+1}^{a/b}$$

for $x = 35$ and 55, $a/b = 1/\text{all}$ and $26/\text{all}$, and various values of y. Note that the correlation coefficient between $X_x^{a/b}$ and $X_{x+1}^{a/b}$, conditional on 'Alive at x and Healthy at y' is denoted by $r_{x,x+1}^{a/b}$. The method of calculation for these functions is an extension of a method used by the present author, Waters (1989),[9] to calculate the moments of the present value of the profit on a PHI policy. Such adjustments to, and extensions of, the formulae in Waters (1989)[9] as are necessary for our present purposes are given in the Appendix to this paper.

6.5 It should be noted that to calculate the figures in Table 3 (resp. Table 4) we have used graduations of the transition intensities based on data relating to policies with deferred period 1 week (resp. 26 weeks), which will be published in *C.M.I.R.* 10. Note also that it would have been possible, and interesting, to calculate values of these moments for other values of a/b, say $a/b = \frac{1}{3}$, and also to

calculate the correlation between $X_x^{a/b}$ and $X_x^{c/d}$ (conditional on being alive at x and healthy at y). We have not done so purely for the sake of brevity.

6.6 Particular points to note concerning the figures in Tables 3 and 4 are:

(i) the graduated sickness rate given in *C.M.I.R.* 7 for deferred period 1 week, age 35 and durations 1/all, is, after dividing by 52·18 to convert to years, $6{\cdot}55 \times 10^{-3}$. The figures in Table 3 show that this corresponds to $E[X_{35}^{1/\text{all}} | A \text{ at } 35, H \text{ at } y]$ for a value of y between 30 and 35. In this context it may be helpful to interpret the age y as the age at which the policyholder effected the policy since we may reasonably assume the policyholder was healthy at that time. The other relevant graduated sickness rates taken from *C.M.I.R.* 7 are

$$z_{55}^{1/\text{all}} \div 52.18 = 32{\cdot}9 \times 10^{-3}$$
$$z_{35}^{26/\text{all}} \div 52{\cdot}18 = {\cdot}065 \times 10^{-3}$$
$$z_{55}^{26/\text{all}} \div 52{\cdot}18 = 11{\cdot}5 \times 10^{-3};$$

(ii) the correlation coefficient is always positive, as we would expect, and can be very high (almost 90% in some cases);

(iii) it is noticeable that the values of the means and variances depend very much on y, which again we can interpret as the entry age for the policyholder.

Points (ii) and (iii) would almost certainly have been less pronounced if, instead of the unlimited sickness durations 1/all and 26/all, we had chosen to calculate numerical values for sickness durations of finite length.

Table 3. *Moment functions of estimators for sickness rates. Deferred period 1 week*

x	y	Expected value $\times 10^3$	Variance $\times 10^3$	$r_{x,x+1}^{1/\text{all}}$
35	35	5·37	1·00	·304
35	30	7·88	2·92	·588
35	25	8·43	3·41	·630
35	20	8·66	3·56	·640
55	55	13·7	4·28	·477
55	50	34·6	22·8	·779
55	45	41·4	28·9	·819
55	40	44·2	31·5	·830
55	35	45·5	32·6	·834

Expected Value $= E[X_x^{1/\text{all}} | A \text{ at } x, H \text{ at } y]$.

Variance $= V[X_x^{1/\text{all}} | A \text{ at } x, H \text{ at } y]$.

Table 4. *Moment functions of estimators for sickness rates. Deferred period 26 week*

x	y	Expected value $\times 10^3$	Variance $\times 10^3$	$r_{x,x+1}^{26/\text{all}}$
35	35	·060	·017	·526
35	30	·753	·630	·782
35	25	·946	·819	·824
35	20	1·02	·893	·835
55	55	·512	·158	·578
55	50	8·99	7·99	·849
55	45	11·8	10·7	·879
55	40	12·8	11·7	·880
55	35	13·3	12·1	·891

Expected Value $= E[X_x^{26/\text{all}} | A \text{ at } x, H \text{ at } y]$.

Variance $= V[X_x^{26/\text{all}} | A \text{ at } x, H \text{ at } y]$.

REFERENCES

(1) *C.M.I.R.* 4 (1979) *Report No. 4 of the Continuous Mortality Investigation Bureau.* Published by the Faculty of Actuaries and the Institute of Actuaries.

(2) *C.M.I.R.* 7 (1984) *Report No. 7 of the Continuous Mortality Investigation Bureau.* Published by the Faculty of Actuaries and the Institute of Actuaries.

(3) DAW, R. H. (1986) A Statistical Study of the Variability of Sickness Data. *J.I.A.* **113,** 103–150.

(4) DILLNER, C.-G. (1969) New Bases for Non-Cancellable Sickness Insurance in Sweden. *Skandinavisk Aktuarietidskrift,* 113–124.

(5) HAMILTON-JONES, J. (1972) Actuarial Aspects of Long-Term Sickness Insurance. *J.I.A.* **98,** 17–68.

(6) MEDIN, K. (1952) Some Theoretical Aspects of Permanent Sickness Insurance. *J.S.S.* **11,** 76–103.

(7) NEILL, A. (1977) *Life Contingencies.* Heinemann, London.

(8) TURNER, M. (1988) PHI Reserving. A paper presented to the Staple Inn Actuarial Society.

(9) WATERS, H. R. (1989) The Recursive Calculation of the Moments of the Profit on a Sickness Insurance Policy. (To be submitted.)

(10) WESTWOOD, W. G. (1972) The Actuarial Aspects of Disability Income Insurance. *Transactions of the Institute of Actuaries of Australia and New Zealand,* 65–111.

APPENDIX

A.1 In this Appendix we outline the procedure necessary to produce the numerical values in Tables 3 and 4 for the moments of estimators of sickness rates. The procedure is based on algorithms derived by the present author, Waters (1989),[9] to calculate the moments of the present value of the profit on a PHI policy.

A.2 In its barest outline, the method used by Waters (1989)[9] to calculate the moments of the present value of the profit on a PHI policy is as follows. Let, as in Waters (1989),[9] $E[y]$ denote the expected value of the future profit (for a given premium rate, benefit rate, interest rate and deferred period) from a policy issued to a healthy life aged y. It is then possible to derive an integral equation for $E[y]$ which, when analysed numerically, reduces to expressing $E[y]$ in terms of $E[y+h]$, $E[y+2h]$, $E[y+3h]$, . . . , where h is some small positive increment in age. By assuming the policy ceases at age 65 we can take E[65] to be zero. Hence we can calculate recursively $E[65\text{-}h]$, $E[65\text{-}2h]$, . . ., $E[y+h]$ and finally $E[y]$. The same method works for higher moments. The appropriate algorithms in Waters (1989)[9] are [4.2], [4.3], [4.4], [4.5], [4.6] and [4.7].

A.3 Now suppose we wish to calculate, for example, $E[X_x^{1/\text{all}}|A \text{ at } x, H \text{ at } y]$, in the notation of Section 6. First we note that, with an obvious extension of our notation,

$$E[X_x^{1/\text{all}}|A \text{ at } x,\ H \text{ at } y] = E[X_x^{1/\text{all}}|H \text{ at } y]/\text{P[Alive at } x|\text{Healthy at } y]$$

since $X_x^{1/\text{all}}$ will be zero if the individual is not alive at age x. The probability on the right hand side of the above expression can be calculated using algorithms in *C.M.I.R.* 10. Now let $E[y]$ denote $E[X_x^{1/\text{all}}|H \text{ at } y]$ for all $y \leqslant x+1$ with the convention that if $x<y$, then $E[y]$ denotes the time spent sick, with duration greater than 1 week, between ages y and $(x+1)$, given that the individual was healthy at age y. The algorithms in Waters (1989)[9] now apply to $E[y]$ as defined above, in particular [4.1] and [4.4], provided we assume:

(i) the interest rate is zero,
(ii) the premium rate is zero,
(iii) the benefit rate is 1 p.a. for claims being paid between ages x and $(x+1)$ and zero for claims being paid at ages less than x.

In the notation of Waters (1989),[9] the technical changes to be made to the algorithms are:

(i) $\delta = 0$
(ii) $P = 0$
(iii) the function denoted $f_2(t,u)$ in Waters (1989)[9] now becomes $f_2(t,u,y,x)$, where

$$f_2(t,u,y,x) = \min(u-w,\ y+t+u-w_1)$$
$$w = \min(d,u)$$

d = deferred period for the policy
$w_1 = \min(y+t+u, x)$.

(Note that in Waters (1989)[9] $f_2(t,u)$ represented the present value of the premiums minus the benefits payable between times 0 and $(t+u)$ by an individual aged y who pays premiums from time 0 to time t, falls sick at time t and remains sick at least until time $(t+u)$. For our present purposes, $f_2(t,u,y,x)$ represents (minus) the time spent sick by an individual now aged y, counting only the time between ages x and $x+1$, when the duration of sickness is greater than the deferred period, and given that the individual is healthy at time 0, falls sick at time $t(<d)$ and remains sick at least until time $(t+u)$.)

(iv) The parameter T appearing in the algorithms now becomes

$$T = \min(x+1-y,\ t+h).$$

Using the fact that $E[x+1]$ is zero we can use the (amended) algorithms to calculate recursively $E[x+1-h]$, $E[x+1-2h]$, . . ., $E[y+h]$, $E[y]$ for any y, and hence, using the relationship at the start of this paragraph, $E[X_x^{1/\text{all}}|A \text{ at } x, H \text{ at } y]$.

A.4 The procedure outlined in § A.3 can be used to calculate higher, in particular the second moment of $X_x^{1/\text{all}}$, with the usual conditioning. This then gives values for the means and variances in Tables 3 and 4.

A.5 To calculate the correlation coefficients in Tables 3 and 4, we calculate separately

$$E[(X_x^{1/\text{all}})^2|A \text{ at } x, H \text{ at } y]$$
$$E[(X_x^{1/\text{all}})^2|A \text{ at } x, H \text{ at } y]$$
$$E[(X_x^{1/\text{all}}) + X_{x+1}^{1/\text{all}})^2|A \text{ at } x, H \text{ at } y].$$

(The last of these can be calculated in the normal way but by starting the recursion from age $x+2$ and working back in steps of size h to age y.) These three values can then be used to calculate

$$E[X_x^{1/\text{all}} \cdot X_{x+1}^{1/\text{all}}|A \text{ at } x, H \text{ at } y]$$

which in turn can be used to calculate

$$\text{Cov}[X_x^{1/\text{all}} \cdot X_{x+1}^{1/\text{all}}|A \text{ at } x, H \text{ at } y]$$

which finally leads to the calculation of the correlation coefficient.