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SOME COMPOUND INTEREST APPROXIMATIONS

By A. W. EVANS, F.I.A.
Actuary and Manager, The London Life Association Ltd.

INTRODUCTION

When the employment of $(a_{\overline{n}|})^{-\frac{1}{2}}$ as a medium for first-difference interpolation was suggested in *J.I.A.* LXXII, 453, no mention was made of a corresponding medium for the amount of an annuity. It seemed that the matter might bear further investigation from this angle, and that it would be well to bring the allied continuous functions within the scope of the inquiry. These latter are simpler to deal with because they may be expressed in terms of only one variable $(n\delta)$.

The investigation soon developed on more general lines and led to approximations of various kinds—some of them very close. The literature on this subject is already extensive and it is difficult to avoid points of contact with previous work. It is hoped, however, that there is sufficient variation in the lines of approach adopted to justify the presentation of results which compare not unfavourably with those obtained in the past.

FIRST STEPS AND RESULTING FORMULAE

It appeared desirable in the first place to establish, if possible, a general approximate relationship between $(a_{\overline{n}|})^{-p}$ and $(s_{\overline{n}|})^{-q}$, with the prime consideration in mind that the value of p (known by experiment to be fractional) should enable $(a_{\overline{n}|})^{-p}$ to be treated as a nearly linear function of i. If the relationship were close enough it would then follow that $(s_{\overline{n}|})^{-q}$ also would be nearly linear.

We may commence by writing down the expansions of $(n/a_{\overline{n}|})^p-1$ and $1-(n/s_{\overline{n}|})^q$ as far as the second power of i. [For convenience the suffix $_{\overline{n}|}$ is omitted hereafter.]

We have

$$(n/a)^p-1=\tfrac{1}{2}(n+1)p.i+\tfrac{1}{24}(n+1)\,[n(3p-1)p-(5-3p)p]\,i^2, \tag{A}$$

$$1-(n/s)^q=\tfrac{1}{2}(n-1)q.i+\tfrac{1}{24}(n-1)\,[n(1-3q)q-(5-3q)q]\,i^2. \tag{B}$$

Dividing (A) by (B), it may be shown that

$$\frac{(n/a)^p-1}{1-(n/s)^q}=\frac{n+1}{n-1}\frac{p}{q}\left\{1+\frac{n[3(p+q)-2]+3(p-q)}{12}i\right\}$$

very nearly.

If now we choose p and q so that $3(p+q)=2$, we arrive at the formula

$$\frac{(n/a)^p-1}{1-(n/s)^q}=\frac{n+1}{n-1}\frac{p}{q}\left(1+\frac{p-q}{4}i\right). \tag{1}$$

With continuous functions the corresponding expansions to (A) and (B) are

$$(n/\bar{a})^p-1=\tfrac{1}{2}p(n\delta)+\tfrac{1}{24}(3p-1)p(n\delta)^2, \tag{C}$$

$$1-(n/\bar{s})^q=\tfrac{1}{2}q(n\delta)+\tfrac{1}{24}(1-3q)q(n\delta)^2, \tag{D}$$

so that if

$$\frac{p}{q}=\frac{3p-1}{1-3q}\frac{p}{q},$$

i.e. if $3(p+q)=2$, as before, the ratio of (C) to (D) is p/q exactly, that is,

$$\frac{(n/\bar{a})^p-1}{1-(n/\bar{s})^q}=\frac{p}{q}. \qquad (2)$$

It is easy to pass from formulae (1) and (2) to formulae linking the rate or force of interest with present values or amounts of an annuity. Writing x for $(n/a)^p$, y for $(n/s)^q$ and $\bar{x}$ and $\bar{y}$ for the corresponding continuous functions, we may obtain from (1)

$$\frac{1}{s}=\frac{1}{n}\left(1-\frac{x-1}{t}\right)^{1/q}, \text{ where } t=\frac{n+1}{n-1}\frac{p}{q}\left(1+\frac{p-q}{4}i\right).$$

Hence

$$\begin{aligned} i &= 1/a-1/s \\ &= \frac{1}{a}-\frac{1}{n}\left(1-\frac{x-1}{t}\right)^{1/q}. \end{aligned} \qquad (3)$$

Also

$$1/a=[1+t(1-y)]^{1/p}/n.$$

Hence

$$\begin{aligned} (1+i)^n &= s/a \\ &= s[1+t(1-y)]^{1/p}/n. \end{aligned} \qquad (4)$$

From (2) the corresponding formulae are

$$\delta=\frac{1}{\bar{a}}-\frac{1}{n}\left[1-\frac{q}{p}(\bar{x}-1)\right]^{1/q}, \qquad (5)$$

and

$$e^{n\delta}=\frac{\bar{s}}{n}\left[1+\frac{p}{q}(1-\bar{y})\right]^{1/p}. \qquad (6)$$

The concluding step is to decide what are the best combinations of p and q. Previous experiment with present values suggested that $1/p$ should be in the neighbourhood of 2 so that the choice of convenient combinations is rather limited. The following five combinations satisfy the necessary link between p and q and would not be awkward to apply arithmetically:

$1/p$	$1/q$
1·8	9
2	6
2·4	4
2·5	3·75
3	3

From the second of these combinations we appear justified in regarding $s^{-\frac{1}{6}}$ as a natural complement to $a^{-\frac{1}{2}}$ for interpolation purposes. The convenience of the square root in the latter case outweighs any slight advantage there might be in using a value of $1/p$ rather above 2. If, however, we want the most accurate all-round results from formulae (3) and (5) it is found that by a narrow margin the best values of $1/p$ and $1/q$ to use are 2·4 and 4 respectively. The same is true of formulae (4) and (6) provided that the value of ni or $n\delta$, as the case may be, is not very high.

With this particular combination of $1/p$ and $1/q$ and a comprehensive range of examples up to ni (or $n\delta$) = 10, errors found by formulae (3) and (5) do not exceed about ·08%. The construction of these two formulae is such that they do not break down for high values of ni or $n\delta$, because the rate or force of interest is then practically identical with the reciprocal of the annuity-value. Nevertheless, the standard of accuracy is surprisingly high when the rather slender foundation for the formulae is remembered.

Formulae (4) and (6) have a more limited range (ni or $n\delta < 5$), but within that range approximations to i or δ are only slightly less accurate than as mentioned in the preceding paragraph.

For the two formulae dealing with yearly annuities the adjustment $1+\frac{1}{4}(p-q)i$ can be ignored except when a short term (say <20 years) is coupled with a high rate of interest. When the adjustment is used, a rough estimate of the value of i must of course be made.

When $p=q=\frac{1}{3}$, the last of the five combinations given above, it will be noted that for the continuous functions (expressions (C) and (D)) the coefficient of $(n\delta)^2$ vanishes. A further interesting point is that formula (2) becomes exact as far as the *third* power of $n\delta$. In these circumstances the formula is particularly accurate for low values of $n\delta$. Although the expansions of the yearly functions do not work out so neatly there is a corresponding reflected advantage in formula (1) for low values of ni. It should be added that a value of $\frac{1}{3}$ for p or q is not a very good one for purposes of first-difference interpolation.

Provided that $p=q=\frac{1}{3}$, formula (2) may be written

$$(n/\bar{a})^{\frac{1}{3}}+(n/\bar{s})^{\frac{1}{3}}=2,$$

i.e.

$$(n/\bar{a})^{\frac{1}{3}}(1+v^{\frac{1}{3}n})=2,$$

which leads to

$$\bar{a}=(1+3v^{\frac{1}{3}n}+3v^{\frac{2}{3}n}+v^n)\,n/8.$$

This is, of course, the quadrature formula which has already been mentioned as a means of determining the rate of interest in an annuity-certain (J. Spencer, *J.I.A.* L, 53. Cf. also G. F. Hardy, *J.I.A.* XXIV, 101).

In the remainder of this Note (until the final section) $1/p$ is taken as 2 and $1/q$ as 6, although, as already indicated, these are not necessarily the best values, either singly or in combination.

CONSIDERATION OF THE PARTICULAR CASE WHEN $1/p=2$

Practical evidence of the nature of the function $a^{-\frac{1}{2}}$ can be gained by studying specimen values of the expression $[(n/a)^{\frac{1}{2}}-1]/ni$, details of which are given in Table A.

Table A. Values of $[(n/a)^{\frac{1}{2}}-1]/ni$

100i	n = 5	10	20	40	60	100
0	·3000	·2750	·2625	·2563	·2542	·2525
1	·2997	·2753	·2638	·2593	·2588	·2596
2	·2995	·2756	·2649	·2615	·2615	·2616
3	·2992	·2758	·2657	·2629	·2624	·2597
4	·2989	·2759	·2664	·2635	·2619	·2550
5	·2986	·2760	·2668	·2634	·2601	·2489
6	·2983	·2761	·2671	·2627	·2574	·2422
8	·2976	·2760	·2670	·2598	·2504	·2286
10	·2969	·2757	·2664	·2556	·2423	·2162

For each value of n, the comparative lack of variation in the expression tabulated (except when ni becomes high) is very noticeable. Denoting the expression by F, and $(n/a)^{\frac{1}{2}}$ by x, we have

$$x = 1 + niF.$$

It is apparent that, over most of the area covered, interpolation by means of $a^{-\frac{1}{2}}$ must give good results; while if a rate i' close to the true rate has first been found then the latter must be capable of close representation by the formula $i = i'(x-1)/(x'-1)$. Similar conclusions are justified when dealing with $\bar{a}^{-\frac{1}{2}}$ and the force of interest.

Except when $n < 8$ the values of F rise to a maximum as i increases and then diminish steadily. For terms of 12 and over the maximum occurs at a point closely indicated by the formula $(n+10)i = 2{\cdot}2$. There is of course a corresponding expression $\bar{F} = [(n/\bar{a})^{\frac{1}{2}} - 1]/n\delta$, which can be regarded as a function of $(n\delta)$ and whose maximum occurs when $n\delta = 2{\cdot}1$.

Tables of $[1-(n/s)^{\frac{1}{2}}]/ni$ and $[1-(n/\bar{s})^{\frac{1}{2}}]/n\delta$ exhibit roughly similar features to those of F and $\bar{F}$ respectively, as would be expected from the connexions established by formulae (1) and (2).

For values of $n\delta$ up to 3, $\bar{F}$ can be represented extremely closely in the form $A + B(n\delta) - C(n\delta)^2$, but, apart from leading to an approximate quadratic solution for $n\delta$ from $\bar{a}$, this line of inquiry has no apparent use. Approximations to v^{n+1} and v^n may be obtained by assuming F and $\bar{F}$ respectively to be absolute constants, for each value of n, in relation to i and δ respectively.

Thus, in the case of F, we have

$$v^n = 1 - ni/x^2 = 1 - ni/(1+niF)^2.$$

Differentiating with regard to i, and remembering that on the given assumption $dx/di = nF = (x-1)/i$, we arrive at the formula

$$v^{n+1} = (2-x)/x^3 \quad \text{or} \quad (a/n)(2-x)/x. \tag{7}$$

This formula is fairly reliable (a) for short terms or (b) for longer terms when F is at or near its maximum, but its range is in consequence strictly limited.

FURTHER DEVELOPMENT USING THE FUNCTION θ

It seemed worth while to investigate whether $\theta\left(=\dfrac{1/a - 1/n}{i}\right)$, which is known to vary slowly with i, could be made the basis for a new line of approach. The method adopted was to express θ in powers of $(x-1)$, i.e. to assume

$$\theta = A + B(x-1) + C(x-1)^2 + \dots.$$

By expanding each side of this equation in powers of i and putting $i=0$ at successive stages it follows that

$$A = (n+1)/2n,$$

$$B = (n-1)/3n.$$

Inserting these values in the original equation it can be shown that, as far as the third power of i, C is very nearly equal to

$$-\frac{(n-1)(n+5)}{n(n+1)}\left[\frac{1}{18} + \frac{i}{120}(n-7)\right] \Big/ \left[1 + \frac{i}{12}(n-7)\right].$$

This expression suggests that the average numerical value of $-C$ may well lie between $\frac{1}{18}$ and $\frac{1}{10}$. In fact, it proves to be the case that $-C$ is a near-constant with a mean value of about $\frac{1}{15}$.

Accordingly we may write

$$\theta=\frac{n+1}{2n}+\frac{n-1}{3n}(x-1)-\tfrac{1}{15}(x-1)^2, \tag{8}$$

whence
$$i=\left[\frac{1}{a}-\frac{1}{n}\right]\Big/\left[\frac{n+1}{2n}+\frac{n-1}{3n}(x-1)-\tfrac{1}{15}(x-1)^2\right], \tag{9}$$

or, in terms of x and n,

$$i=[x+1]\Big/\left[\frac{n+1}{2(x-1)}+\frac{n-1}{3}-\frac{n(x-1)}{15}\right]. \tag{9a}$$

For such a simple formula the results are very accurate and errors do not exceed 1*d.* in the rate per cent provided that $ni \leqslant 3$. When $ni > 3$ there are negative errors. Taking two extreme cases, the 8% and 10% approximations for term 100 are 7·909 and 9·925 respectively.

A corresponding formula for δ in terms of $\bar{x}$ and n can be obtained by a similar process, but the modified 'C' in this case has a slightly greater range of variation.

DEVELOPMENT USING $\log_e s_{\overline{n}|}$

As a companion formula to (8), it was found that $(1/ni)\log_e s/n$ could be expanded in terms of $1-(n/s)^{\frac{1}{3}}$ and powers thereof with very satisfactory results. The formula that ultimately emerged is

$$i=\left[\frac{1}{n}\log_e\frac{s}{n}\right]\Big/\left[\frac{n-1}{2n}+\frac{n-5}{2n}(1-y)-\frac{n^2+10n-27}{4n(n-1)}(1-y)^2+\tfrac{1}{11}(1-y)^3\right], \tag{10}$$

where y is written for $(n/s)^{\frac{1}{3}}$.

The coefficients of the first two powers in the denominator were found by orthodox means, but the coefficient of $(1-y)^3$, although indicated as a possibility between certain limits, was in effect determined by trial. It is another near-constant over a considerable range.

Formula (10) is a powerful one and a table of results by it is perhaps worth giving (Table B).

Table B. Values of $100i$ by formula (10)

n	2%	4%	6%	8%	10%
100	2·000	4·001	6·003	8·003	9·999
80	2·000	4·000	6·002	8·002	9·999
60	2·000	4·000	6·001	8·002	10·002
40	2·000	4·000	6·001	8·002	10·003
20	2·000	4·000	6·000	8·001	10·002
10	2·000	4·000	6·000	8·001	10·001

A corresponding formula for δ from $\bar{s}$ is of course a particular case of the approximate solution for z of the general equation

$$(e^z-1)/z=X,$$

and the formula, with specimen results, is given below:

$$z = \frac{\log_e X}{\frac{1}{2} + \frac{1}{2}Y - \frac{1}{4}Y^2 + \frac{1}{10}Y^3}, \tag{11}$$

where $$Y = 1 - X^{-\frac{1}{2}}.$$

Values of z

True	Approximate
1	1·000
2	2·000
3	2·999
4	3·998
5	4·998
6	5·997
7	6·998
8	8·001
9	9·006
10	10·015
15	15·138

The figures speak for themselves and indicate close approximation when z ($=n\delta$ in our case) does not exceed 9.

It will be noted that the third-power coefficient was taken as $\frac{1}{10}$ in this case. Formula (11), which is simpler than formula (10), could be used to find δ (and thence i) from $s_{\overline{n}|}$ by making use of the approximate relationship

$$\bar{s}_{\overline{n-\frac{1}{2}}|} = s_{\overline{n}|} - \tfrac{1}{2}.$$

Although formula (11) was designed for use with positive values of z it is of interest to find that it can also be used reliably with negative values as far as $z = -3$. For $z = -4$ the approximation is $-4{\cdot}027$, and thereafter errors become substantial. It follows of course that, within the range $n\delta = 0$ to 3, the formula could be employed to find δ from $\bar{a}$, while an appropriate adaptation of formula (10) could be made with similar effect, that is, to find i from a.

To sum up the merits of the formulae so far given, it may be said that (10) and (11) stand out as the best for all-round accuracy and use, but they have their limitations when applied to present values. Almost as accurate, in connexion with present values, are formulae (3), (5) and (9a), the latter having the additional advantage of simplicity but lacking the range of the other formulae.

It has already been noted that variations of p and q, within certain limits, are possible with the early formulae, and it seems likely that the same is true of the later formulae, if suitably adjusted.

YIELDS ON REDEEMABLE SECURITIES

When the method adopted is first-difference interpolation, partial effect can be given to the linear property of $a^{-\frac{1}{2}}$ by operating on the square root of the reciprocal of the compound interest function A.

Determination of yields by analytical means, when the security is repayable in one sum, can be very simply and effectively approached by concentrating on the function θ, as was pointed out in *J.I.A.* LX, 344.

Assuming therefore that one is not working with interest tables, good approximations to θ have some utility though they necessarily involve an advance estimate of i, the rate to be determined. The simplest expression for θ in terms of the functions dealt with in this note is

$$\theta=(x^2-1)/ni=[(1+niF)^2-1]/ni$$
$$=2F+niF^2. \qquad (12)$$

If we work with the limitation in mind that $ni\leqslant 3$ (which is not a serious limitation in practice) we can see from a fuller version of Table A that it is possible to assign a close average value to F for each term. For example:

Term	F (average)
5	·299
10	·276
15	·269
20	·266
25	·265
30	·264
40	·2625
50	·262
60 and upwards	·261

Thus by taking a trial rate for i we can approximate to θ by formula (12).

A slightly better approximation results from the use of formula (8). Putting $x-1=niF$ this leads to

$$\theta=\frac{n+1}{2n}+\frac{n-1}{3n}(niF)-\frac{(niF)^2}{15}. \qquad (13)$$

This formula has the additional advantage that it provides the means of obtaining reasonable approximations to $d\theta/di$, if required.

The degree of accuracy by this second formula is illustrated in Table C.

Table C. Values of θ by formula (13)

n	2%		4%		6%	
	True	Approximate	True	Approximate	True	Approximate
10	·5663	·5663	·5823	·5823	·5978	·5978
30	·5658	·5660	·6124	·6121	·6553	·6547
50	·5912	·5910	·6638	·6629	·7241	·7256

When a value for θ has been found, the yield is of course ascertained from

$$i=\frac{g+k/n}{1-\theta k},$$

where g and k (discount) have their usual meanings.

A more elaborate use may be made of θ, in conjunction with its first differential coefficient. Thus, denoting $\frac{g+k/n}{1-\theta k}$ by f_i and introducing a trial rate i', we may write

$$f_i=f_{i'}+(i-i')\frac{df_{i'}}{di'}, \quad \text{as a close approximation.}$$

But $f_i = i$ and
$$\frac{df_{i'}}{di'} = \frac{g+k/n}{(1-\theta' k)^2} k \frac{d\theta'}{di'} = f_{i'} \frac{k}{1-\theta' k} \frac{d\theta'}{di'}.$$

Hence we arrive at the approximation
$$i - i' = (f_{i'} - i') \Big/ \left(1 - f_{i'} \frac{k}{1-\theta' k} \frac{d\theta'}{di'}\right). \tag{14}$$

Taking as an illustration the often quoted text-book example with
$$g = \cdot 025, \quad k = \cdot 045455 \quad \text{and} \quad n = 50,$$
we obtain, at a trial rate of ·02,
$$f_{i'} = \cdot 026624,$$
$$\theta' = \cdot 5910 \text{ from Table C above,}$$
and $\quad d\theta'/di' = 3\cdot 82$, by differentiation of formula (13),

and the final result for i is the correct one of ·026656.

A nearer trial rate would certainly be necessary to obtain corresponding accuracy in cases involving a high value of k.

It has been suggested in the past that θ might be tabulated, and a case could also be made for $d\theta/di$. To be thorough, however, such tabulation would take up appreciable space. It would be simpler to tabulate $\bar{\theta}\left(=\left[\frac{1}{\bar{a}} - \frac{1}{n}\right]\Big/\delta\right)$, because this may be written $\frac{1}{1-e^{-n\delta}} - \frac{1}{n\delta}$ and is therefore a function of $n\delta$.

A short table of values of $\bar{\theta}$ and $d\bar{\theta}/d(n\delta)$ is given at the end of this Note. This table may be employed for finding θ and $d\theta/di$ by making use of the close relationships
$$\theta = \bar{\theta} + \frac{1}{2n} - \frac{i}{12n},$$
$$\frac{d\theta}{di} = \frac{n}{1+i} \frac{d\bar{\theta}}{d(n\delta)} - \frac{1}{12n}.$$

YIELD SOLUTION FROM CUBIC EQUATION

In *J.I.A.* LV, 99 there was given a neat approximate solution derived from a quadratic equation with $v^{\frac{1}{2}n}$ as the variable. This has hardly received the subsequent notice that it seems to deserve. It is fairly simple to apply, avoids the use of interest-tables, and gives close approximations except in extreme cases.

Briefly, the method is based on the three-term quadrature formula known as Simpson's Rule, and the value of v^n determined by solving the resulting quadratic is used to find i by means of the identity
$$i = g + ki/(1 - v^n),$$
that is
$$i = \frac{g}{1 - k/(1-v^n)}.$$

Much of the accuracy is due to the fact that, as the above expression shows, the approximation to v^n need not be unduly close. This may be illustrated by imposing a severe test such as an example based on
$$n = 50, \quad i = \cdot 07, \quad g = \cdot 04, \quad \text{and} \quad k = \cdot 41402.$$

The approximate method gives a value of ·03022 for v^{50}, whereas the true value is ·03394, so that the percentage error is considerable. But the approximation to i is ·06980, an error of only ·3%.

This last result suggests that, on grounds of accuracy, little further improvement can be expected. But it may be of interest to carry the matter a stage further by showing that the four-term quadrature formula represented for this purpose by

$$\bar{a} = \tfrac{1}{8}n(1 + v^{\frac{1}{3}n})^3$$

can also lead to a simple solution of the problem. We have already seen that this formula is identical with formula (2) when $p = q = \frac{1}{3}$. (No other combination of p and q seems to provide a workable alternative in this particular connexion.)

Writing x for $(n/a)^{\frac{1}{3}}$ and $\bar{x}$ for $(n/\bar{a})^{\frac{1}{3}}$ we find from the above quadrature formula that

$$v^n = (2/\bar{x} - 1)^3$$
$$= (2m/x - 1)^3,$$

where

$$m = (i/\delta)^{\frac{1}{3}}.$$

The basic equation for the yield is

$$i = g + k/a,$$

that is

$$\frac{1 - v^n}{a} = g + \frac{k}{a}.$$

Substituting for v^n its approximate equivalent, and replacing $1/a$ by x^3/n we obtain the cubic equation

$$(1-k)\,x^3 - (2m - x)^3 = ng,$$

which leads to

$$(2-k)\,x^3 - 6mx^2 + 12m^2x - 8m^3 = ng.$$

Denote $2m/(2-k)$ by r. Then

$$x^3 - 3rx^2 + 6mrx = 4m^2r + rng/2m. \tag{15}$$

A solution for x is obtainable as follows:

First put $x = z + r$ with the object of eliminating the second power of the new variable.

After the substitution of $z + r$ for x in (15) the adjusted equation in z becomes

$$z^3 + 3r(2m - r)\,z = 2r^3 - 6mr^2 + 4m^2r + rng/2m$$
$$= 2r(r-m)(r-2m) + rng/2m.$$

Therefore

$$z = \frac{ng}{6m(2m-r)} - \tfrac{2}{3}(r - m) - \frac{z^3}{3r(2m-r)}.$$

The final term, which is always small, can for practical purposes be replaced by $z^3/3$ because $r(2m-r)$ when expressed in terms of m and k is seen to be close to unity. Therefore

$$z = \frac{ng}{6m(2m-r)} - \tfrac{2}{3}(r - m) - \frac{z^3}{3}.$$

This could be solved as a cubic equation in z; but this course is unnecessary because (for cases likely to arise in practice) z^3 is always very small in relation to z and its value can be estimated sufficiently closely after calculation of the first two terms in the above expression for z.

Taking the same example as that used for the quadratic solution we have first to estimate the value of m which equals $1 + i/6$ approximately. Assume that 1·011 is the estimate made.

Then $$r=\frac{2m}{2-k}=\frac{2\cdot022}{1\cdot5860}=1\cdot2749.$$

Therefore
$$\begin{aligned}z&=\frac{2}{6\cdot066\times\cdot7471}-\tfrac{2}{3}\times\cdot2639-\frac{z^3}{3}\\&=\cdot44132-\cdot17593-z^3/3\\&=\cdot26539-z^3/3\\&=\cdot26539-\cdot00583\text{ say}\\&=\cdot25956.\end{aligned}$$

Therefore $$x=z+r=1\cdot5345$$

and
$$\begin{aligned}i&=g+k(1\cdot5345)^3/n\\&=\cdot06992.\end{aligned}$$

This method is likely to be even more accurate than the quadratic method and is also thought to have an advantage as regards application.

When $z^3/3$ is small enough for its value to have hardly any sensible effect on the final result (as is often the case) we may, by leaving it entirely out of account, deduce from the foregoing work a virtually explicit formula for the yield, viz.

$$i=g+\frac{k}{27n}\left[\frac{ng(2-k)}{4m^2(1-k)}+\frac{2m(3-k)}{2-k}\right]^3. \tag{16}$$

In the above example $z^3/3$ has a significant value, but even so its omission and the calculation of i from formula (16) does not result in a very serious error, the approximation being ·07026. For the example taken on p. 245, formula (16) gives the true value of ·026656.

Table D. Values of $\bar{\theta}=\left(\frac{1}{\bar{a}}-\frac{1}{n}\right)\Big/\delta$ and of $\frac{d\bar{\theta}}{d(n\delta)}$

$n\delta$	$\bar{\theta}$	$d\bar{\theta}/d(n\delta)$	$n\delta$	$\bar{\theta}$	$d\bar{\theta}/d(n\delta)$
0	·50000	·08333	1·875	·64780	·07050
·125	·51041	·08326	2	·65652	·06898
·25	·52081	·08307	·125	·66504	·06743
·375	·53118	·08275	·25	·67337	·06583
·5	·54150	·08230	·375	·68150	·06421
·625	·55175	·08173	·5	·68943	·06258
·75	·56192	·08104	·625	·69714	·06093
·875	·57200	·08024	·75	·70466	·05927
1	·58198	·07933	·875	·71196	·05762
·125	·59183	·07831	3	·71906	·05597
·25	·60155	·07721	·125	·72596	·05433
·375	·61113	·07601	·25	·73265	·05271
·5	·62055	·07473	·375	·73913	·05111
·625	·62981	·07339	·5	·74542	·04953
·75	·63889	·07197			

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