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SOME FURTHER EXPERIMENTS IN THE USE OF THE INCOMPLETE GAMMA FUNCTION FOR THE CALCULATION OF ACTUARIAL FUNCTIONS

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INTRODUCTION

THE experiments described in this paper arose from some calculations made in connexion with the recent discussion on the mortality of life-office annuitants (*J.I.A.* LXXVIII, 27) and are a natural development of the principles described in the paper submitted to the Centenary Assembly of the Institute (*Proceedings*, II, 89). They fall into two groups. Those in the first, arising from an attempt to find an alternative method on which to base projections, have an affinity with those set out by Starke in his recent paper (*J.I.A.* LXXVIII, 171), in that a formula for expressing mortality is derived essentially dependent on year of birth. There are, however, fundamental differences between the approaches, in that Starke is concerned mainly with an *ad hoc* expression for a derived function of q_x without regard to the problem of computing functions, whereas the present experiments are concerned with (i) further developing actuarial technique based on the curve of deaths rather than on rates of mortality, and (ii) the calculation of actuarial functions without recourse to the usual methods of commutation columns. The second group of experiments is an extension of the gamma-function technique to the calculation of joint-life functions, including contingent assurances.

2. It will be convenient to recapitulate first the underlying principles of the gamma-function technique, which is, basically, to fit a Pearson Type III curve to a modified curve of deaths and to calculate values of actuarial functions from tabulated values of the Type III curve or incomplete gamma function. The advantage of this particular curve is that functions at different rates of interest can be found from tabular values by a process equivalent to a change in age. The extension to joint-life functions now developed consists of finding an appropriate Type III curve for the joint-life curve of deaths or other functions from the values for the single-life Type III curve.

3. The modified curve of deaths is defined by $(\mu l)_t - \kappa l_t$, where κ is a constant which for the A 1924–29 ult. data was taken as approximately equal to μ_{20} and is the minimum value reached by μ_x. This modified curve is fitted by a curve of the form $e^{-\gamma(\omega-t)}(\omega-t)^p$, where ω is a limiting age; by integration the following expressions for certain functions are found:

$$ {}_tp_x = \frac{e^{-\kappa t}\displaystyle\int_0^{(\gamma+\kappa)(\omega-t-x)} e^{-z} z^p\, dz}{\displaystyle\int_0^{(\gamma+\kappa)(\omega-x)} e^{-z} z^p\, dz}, \tag{1} $$

$$\mu_t = \kappa + \frac{(\gamma+\kappa)\, e^{-(\gamma+\kappa)(\omega-t)} \{(\gamma+\kappa)(\omega-t)\}^p}{\int_0^{(\gamma+\kappa)(\omega-t)} e^{-z} z^p\, dz}, \tag{2}$$

$$\bar{A}_x - \kappa \bar{a}_x = \frac{e^{-(\kappa+\delta)(\omega-x)} (\gamma+\kappa)^{p+1} \int_0^{(\gamma-\delta)(\omega-x)} e^{-z} z^p\, dz}{(\gamma-\delta)^{p+1} \int_0^{(\gamma+\kappa)(\omega-x)} e^{-z} z^p\, dz}. \tag{3}$$

4. Pearson's tables of the incomplete gamma function give values of

$$I(u,p) = \int_0^x e^{-z} z^p\, dz \Big/ \Gamma(p+1)$$

to seven decimal places for various values of p and for an argument

$$u = x(p+1)^{-\frac{1}{2}}$$

progressing by intervals of ·1 from 0 to such a value that $I(u,p) = 1{\cdot}0$. It will be noted that for a given mortality table p remains the same for all rates of interest; thus values of $I(u,p)$ for a single value of p will suffice for all single-life calculations.

APPLICATION TO ANNUITANTS' MORTALITY

5. The starting-point of the experiments was the table of female ultimate data for the period 1921–48. Study of the run of exposed to risk and deaths suggested that a natural division would be into the two periods 1921–35 and 1936–48, and after some arithmetical trials a set of constants κ, γ, p and ω was found which gave a reasonable representation of the values of q_x for the earlier period. Attention was next given to the O^{af} data and a further set of constants found, it being noted that the same values of p and a $(=p/\gamma)$ would provide a reasonable representation of the O^{af} data, provided that the value of ω was reduced. Consideration of the $a(f)$ ultimate data then suggested that the same values of p and a could be used, and that the value of ω was approximately collinear with the values for the O^{af} and the recent data. Finally, some trials were made with the data for the period 1936–48, from which it appeared that a reasonable result could be obtained with the same values of p and a but with a slightly larger value of ω than for the earlier period. The values of κ showed a decrease in geometrical progression over the whole period covered by the experiments.

6. It was clear, from the progression of the values of the constants, that it would be reasonable to extrapolate on the values of ω and κ and so obtain values of q_x for future calendar years; thus the preliminary object of the experiments was formally attained. However, actuarial functions calculated on the basis of the values of q_x for a given calendar year would have little theoretical meaning, and attention was next directed to finding a method of calculating functions for given entry ages in any calendar year. Although the values of ω progressed in arithmetical progression over the period from the O^{af} data to that for 1936–48, and a mathematical transformation of the incomplete gamma function could

be devised to cope with this variation, the geometrical variation of κ could not be similarly dealt with. Accordingly, an attempt was made to find sets of the constants κ, γ, p and ω which depended only on the year of birth; a solution was found in which ω increased by one-twentieth of a year of age for each year of birth and κ decreased in geometrical progression, being halved each 50 years.

7. The values of the constants finally found were as follows:

$$p = 11{\cdot}0 \qquad \omega = 115{\cdot}15 + {\cdot}05s$$

$$a = 31{\cdot}75 \qquad \kappa = {\cdot}0077 \times 2^{-{\cdot}02s}$$

$$\gamma + \kappa = {\cdot}346456 \qquad \text{where } s = (\text{year of birth} - 1860)$$

and values of q_x appropriate to the years 1878, 1911–12, 1936, 1950 and 1980 were found from formula (1). The first three sets of values were then applied to the exposed to risk for the O^{af}, $a(f)$ and the 1921–48 data (in quinary groups adjusted by $\frac{1}{25}$ of the second central difference), and the expected deaths so found were compared with the actual deaths, with the result given in Table 1.

8. Study of the figures in Table 1 shows that the formula produces values of q_x above the experience values for the period 1900–20 and below the experience values for 1921–48. It would of course be more appropriate, since q_x differs in each calendar year, to calculate the expected deaths for each year, but the above results are adequate for the present purpose and it has not been considered necessary to perform all the required calculations. It should be noted that the 'fitting' process does not involve an agreement between the totals of the expected and actual deaths.

9. A comparison of the values of q_x given by the constants of paragraph 7 for 1950 and for 1980 with the values calculated by the Joint Mortality Committee is given in Table 2. From a practical point of view the projected values are equally acceptable and, in addition, the new set of values has certain advantages, since mortality functions can be calculated without the need for prepared commutation columns. It is of interest to note that, although the new projection is based fundamentally on a linear extrapolation in ω, it is not so radically different from the geometric method adopted by the Committee, since a linear change in age will be approximately equivalent to a geometric change in q_x.

10. A comparison of 3% ultimate annuity-values for entrants in 1955 by the formula (3) and the constants of paragraph 7 with those given by the Committee is given in Table 3. Although this has been carried back to age 20, it is important to note that the values of q_x below age 40 by the formula would be too high, the reasonable agreement in the values of $\bar{a}_x$ being possible only because the influence of the mortality at these young ages is comparatively small. At the very advanced ages the formula values run below those found by the Committee, the reason being the cut-off of the formula at an age ω, whereas $\bar{a}_x$ by the Perks graduation has a minimum value approximating to

$${\cdot}5 + (D/B - 1) = 1{\cdot}0.$$

A small error is also introduced because $\bar{a}_x$ for the Committee figure has been taken as $a_x + {\cdot}5$. It should also be noted that the formula values are for durations

Table 1. Comparison of expected and actual deaths.
Females, durations 5 and over

O^{af}				$a(f)$				1921–48			
Central age of group	Expected deaths	Actual deaths	q_x (formula)	Central age of group	Expected deaths	Actual deaths	q_x (formula)	Central age of group	Expected deaths	Actual deaths	q_x (formula)
52	50	49	·0144	47·5	61	43	·0080	52·5	127	134	·0070
57	119	108	·0182	52·5	149	168	·0096	57·5	411	481	·0095
62	289	256	·0250	57·5	382	329	·0126	62·5	1,270	1,328	·0142
67	682	673	·0368	62·5	884	878	·0183	67·5	3,535	3,472	·0229
72	1,299	1,219	·0562	67·5	1,825	1,691	·0285	72·5	7,703	7,665	·0383
77	1,766	1,732	·0867	72·5	3,201	3,028	·0457	77·5	12,038	12,255	·0628
82	1,659	1,703	·1320	77·5	4,297	4,076	·0732	82·5	13,124	13,674	·1006
87	963	996	·1987	82·5	4,087	4,141	·1147	87·5	9,327	10,331	·1568
92	357	374	·2973	87·5	2,569	2,719	·1761	92·5	3,770	3,776	·2392
97	56	63	·4464	92·5	1,024	994	·2665	97·5	969	926	·3631
				97·5	187	138	·4029				
				102·5	23	12	·617				
Total	7,240	7,173			18,689	18,217			52,274	54,042	

5 and over, whereas the table given by the Committee is designed for a one-year select period.

11. The next set of experiments related to the male lives, and preliminary calculations were made in an endeavour to express the male mortality as equivalent to the female for a few years older with an adjustment to the value of κ. These indicated that acceptable results would be obtained only with a variable adjustment to the age. Calculations were then made of the expected deaths by the O^{am}, $a(m)$ and the 1921–48 ultimate data, using the female formula with κ increased by 60% and a rating-up in age equal to ·05 (year of birth – 1786). This formula gave satisfactory results for the O^{am}, the $a(m)$ and the 1921–48 data, but the known changes in the nature of the data suggested that the formula was not suitable for projection purposes. It seemed reasonable to adopt an upper limit to the rating-up in age—4 years was adopted, this being reached for year of birth 1866—and to reduce the increase in κ from 60 to 20% for entrants from the year 1950. The justification for these particular amendments is slight, but they do ensure that the male mortality rates run parallel to the female, and avoid the divergence which would occur if projection were to be made on the unadjusted formula. There is a serious break between years of entry 1949 and 1950, but this break could clearly be avoided by grading over a longer period if the method were to be developed.

Table 2. Values of q_x in 1950 and in 1980. Females

Age	1950		1980	
	By formula	By Committee	By formula	By Committee
50	·0052	·0056	·0035	·0040
55	·0068	·0076	·0047	·0055
60	·0098	·0108	·0071	·0081
65	·0158	·0166	·0120	·0128
70	·0266	·0265	·0210	·0210
75	·0448	·0434	·0367	·0356
80	·0737	·0723	·0621	·0611
85	·1174	·1218	·1010	·1060
90	·1820	·1850	·1589	·1660
95	·2778	·2626	·2440	·2429

Table 3. Values of $\bar{a}_x$ at 3% for entrants in 1955. Females

Age at entry	By formula	By Committee	Age at entry	By formula	By Committee
20	26·87	27·04	60	15·23	15·25
25	26·00	26·13	65	12·96	13·09
30	25·01	25·08	70	10·68	10·89
35	23·87	23·87	75	8·51	8·75
40	22·56	22·49	80	6·56	6·80
45	21·04	20·94	85	4·91	5·12
50	19·32	19·21	90	3·54	3·78
55	17·37	17·31			

Table 4. Comparison of expected and actual deaths.

Males, durations 5 and over

O^{am}				$a(m)$				1921–48			
Central age of group	Expected deaths	Actual deaths	q_x (formula)	Central age of group	Expected deaths	Actual deaths	q_x (formula)	Central age of group	Expected deaths	Actual deaths	q_x (formula)
52	25	25	·0238	47·5	42	45	·0141	52·5	62	55	·0126
57	61	67	·0287	52·5	101	71	·0168	57·5	202	175	·0170
62	124	115	·0370	57·5	245	218	·0216	62·5	639	632	·0251
67	253	231	·0504	62·5	510	505	·0299	67·5	1,670	1,616	·0367
72	453	451	·0714	67·5	888	944	·0437	72·5	3,664	3,600	·0610
77	582	612	·1027	72·5	1,406	1,322	·0653	77·5	5,010	5,192	·0935
82	510	532	·1481	77·5	1,729	1,735	·0978	82·5	4,701	4,699	·1410
87	303	318	·2131	82·5	1,591	1,593	·1450	87·5	2,894	2,900	·2093
92	84	88	·3070	87·5	998	989	·2130	92·5	993	934	·3086
97	20	14	·4456	92·5	333	272	·3115	97·5	159	154	·4570
				97·5	59	45	·4579				
Total	2,415	2,453			7,902	7,739			19,994	19,957	

Table 5. Values of $\bar{a}_x$ at 3% for entrants in 1955. Males

Age at entry	By formula	By Committee	Age at entry	By formula	By Committee
20	25·81	26·24	60	13·25	13·25
25	24·85	25·20	65	11·01	11·10
30	23·76	24·00	70	8·84	9·01
35	22·51	22·61	75	6·88	7·09
40	21·06	21·05	80	5·19	5·42
45	19·42	19·31	85	3·81	4·07
50	17·54	17·41	90	2·75	3·04
55	15·48	15·37			

The comments in paragraph 10 regarding Table 3 apply also to Table 5.

12. The constants finally adopted for the male mortality were:

$$\begin{array}{lll} p=11\cdot0 & \omega=115\cdot15+\cdot05(s-t) & \\ a=31\cdot75 & \kappa=1\cdot6\times\cdot0077\times2^{-\cdot02(s-t)} & \text{year of entry}<1950 \\ \gamma+\kappa=\cdot346456 & \quad=1\cdot2\times\cdot0077\times2^{-\cdot02(s-t)} & \text{year of entry}\geqslant1950 \\ & s=\text{year of birth}-1860 & \\ & t=\cdot05\,(\text{year of birth}-1786)\text{ or }4\text{, whichever is less} & \end{array}$$

functions being calculated at the age increased by t years. The comparison of actual and expected deaths for the three periods is given in Table 4.

13. Projected values of q_x for male lives have not been calculated; but it may be noted that they will be equal to the female values for an age 4 years older, with an addition of $\cdot2\kappa$ approximately. Annuity-values at 3% interest for entrants in 1955 have been calculated, and are given in Table 5 with the values obtained by the Committee.

14. The experiments described in the foregoing paragraphs were undertaken mainly with a view to finding an alternative approach to the problem of projection; they are not intended to be interpreted otherwise than as an arithmetical exercise, although they are sufficiently successful to suggest that the technique might provide an adequate method for practical application. Although the formulae may appear rather elaborate, calculations of isolated values are relatively straightforward and an example is given below, setting out the entire calculations.

Required: $\bar{a}_{63}$ at 3% for an entrant in 1958—female.

Year of birth 1895

$$\omega=115\cdot15+\cdot05(1895-1860)=116\cdot90$$

$$\kappa=\cdot0077\times2^{-\cdot02(35)} \qquad =\cdot00474$$

$$\begin{array}{r|ll} \gamma+\kappa=\cdot34646 & \log(\gamma+\kappa)=\bar{1}\cdot539648 & \\ \gamma=\cdot34172 & \log(\gamma-\delta)=\underline{\bar{1}\cdot494377} & \\ \delta=\cdot02956 & \text{difference}=\cdot045271\times12=\cdot543252 & \\ \gamma-\delta=\cdot31216 & \qquad\qquad \log^{-1}=3\cdot4934 & \text{(A)} \\ \kappa+\delta=\cdot03430 & (\kappa+\delta)(\omega-x)\log_{10}e \quad =\cdot802912 & \\ \omega-x=53\cdot90 & \qquad\qquad \log^{-1}=6\cdot3520 & \text{(B)} \end{array}$$

$$u_1=(\omega-x)(\gamma-\delta)(p+1)^{-\frac{1}{2}}=4\cdot8571 \qquad I(u_1,11)=\cdot908933$$

$$u_2=(\omega-x)(\gamma+\kappa)(p+1)^{-\frac{1}{2}}=5\cdot3908 \qquad I(u_2,11)=\cdot959623$$

$$[(A)\times I(u_1,11)]/[(B)\times I(u_2,11)]=\cdot52092 \qquad \text{(C)}$$

$$\bar{a}_{63}=[1-(C)]/[\kappa+\delta]=13\cdot97.$$

Interpolating in the table calculated by the Committee for female entrants in 1955, and increasing by the recommended additions, the value of $\frac{1}{2}+a_{63}$ is found to be 14·04.

If, for example, it were desired to find the value of $\bar{a}_{72}$ for a male entrant in 1962, the calculation would follow the above scheme for an age 76, substituting $1\cdot2\times\cdot0077=\cdot00924$ for the factor ·0077.

APPLICATIONS TO JOINT LIFE FUNCTIONS

15. Certain suggestions were put forward in the original paper on the extension of the incomplete gamma-function technique to the calculation of joint-life functions. These suggestions were not entirely satisfactory, and a further method is now described which is more appropriate to the underlying principles of the method. The basis of the technique is the approximation to the curve of deaths by a mathematical function with certain convenient properties. The natural extension would be to find the corresponding functions for the joint-life curve of deaths but, unfortunately, the analytical approach has not yielded any expression except in the case of two lives of equal age. Recourse has accordingly been had to arithmetical methods.

16. From formula (1) we may write, for lives subject to the same mortality table,

$$ {}_tp_{xy} = \frac{e^{-2\kappa t}\int_0^{(\gamma+\kappa)(\omega-t-x)} e^{-z} z^p\,dz \int_0^{(\gamma+\kappa)(\omega-t-y)} e^{-z} z^p\,dz}{\int_0^{(\gamma+\kappa)(\omega-x)} e^{-z} z^p\,dz \int_0^{(\gamma+\kappa)(\omega-y)} e^{-z} z^p\,dz} $$

$$ = e^{-2\kappa t}[I(u_1,p)\,I(u_2,p)]/[I(u_3,p)\,I(u_4,p)] $$

where $u_2 = u_1 + (\gamma+\kappa)(x-y)(p+1)^{-\frac{1}{2}}$.

If we now form the products $I(u_1,p)\,I(u_2,p)$ we have a table of values equivalent to $e^{2\kappa t}\,l_{x+t:y+t}$. By fitting these values with a new Type-III curve, joint-life functions for an age difference $(x-y)$ can be calculated as for single-life values.

17. The products $I(u, 10{\cdot}2)\,I(u+h, 10{\cdot}2)$, for values of h from 0 to 6·5 by intervals of ·5 and for values of u from 0 by intervals of ·5 to the value such that $I(u, 10{\cdot}2) = 1{\cdot}0$, have been calculated to seven decimal places and the moments found from the first differences. The value of 10·2 for p was selected as forming the basis of the earlier experiments with the A 1924–29 table, and also of the hypothetical table described in the appendix to the earlier paper. The values of the moment functions derived from these calculations are given in Table 6.

18. The values of $2\beta_2 - 6 - 3\beta_1$ are sufficiently close to zero to suggest that a Type-III curve will be adequate for calculating approximate values for most practical applications; but a complication arises, since the value of p will not, in general, be one for which $I(u,p)$ is tabulated. To avoid interpolations for the non-tabular values of p we can select a convenient value of p and find the other constants keeping the first two moments unchanged, a process equivalent to introducing a third moment error, which is known to have only a small effect on the resulting calculations. In fact, this process can be continued further and the value of p taken as the single life value, so that it is only necessary to calculate the first two moments of $I(u,p)$ products. Thus, once the moments of the $I(u,p)$ products have been calculated for one value of p, approximate values of joint-life actuarial functions can be determined for any combination of mortality tables.

19. To calculate the constants appropriate to the A 1924–29 table from the values of Table 6 and for a value of $p = 10{\cdot}2$ we note that $u = 0$ corresponds to $\omega = 110$ and the scale factor is given by $u = {\cdot}3023(\omega - x)/\sqrt{(11{\cdot}2)}$ so that $h = 1$ corresponds to 11·0706 years of age. The terminating age for the joint life

table is found from the formula $\omega' = 110 - 11{\cdot}0706\,(\text{mean} - \sqrt{[11{\cdot}2\mu_2]})$ and the a' given by $a' = a\sqrt{\mu_2}$. The resulting values are given in Table 6.

Table 6. Moments and constants for two-life calculations

h	Moment functions of $\Delta I(u,\ 10{\cdot}2)\ I(u+h,\ 10{\cdot}2)$					Joint-life constants A *1924–29* ult.	
	Mean	μ_2	β_1	β_2	$2\beta_2-6-3\beta_1$	ω'	a'
0	3·90457	·85543	·37823	3·66009	·18549	101·041	31·207
·5	3·69064	·86586				103·617	31·397
1·0	3·54397	·89250	·39370	3·65915	·13720	105·767	31·876
1·5	*3·45200*	*·92491*				107·415	32·450
2·0	3·39915	·95370	·39019	3·60641	·04225	108·551	32·951
2·5	3·37117	·97442				109·252	33·307
3·0	3·35744	·98715	·37068	3·55731	·00258	109·642	33·524
3·5	3·35114	·99406				*109·840*	*33·641*
4·0	3·34842	·99744	·36054	3·54002	−·00158	109·932	33·698
4·5	3·34732	·99896				109·973	33·724
5·0	3·34689	·99960	·35775	3·53633	−·00059	109·990	33·735
5·5	3·34673	·99985				109·996	33·739
6·0	3·34667	·99995	·35723	3·53576	−·00017	109·999	33·740
6·5	3·34665	·99998				110·000	33·741
∞	*3·34664*	*1·00000*	*·35714*	*3·53571*	−·00000	110·000	33·741

20. To illustrate the use of this table the calculations for $\bar{a}_{20:50}$ by the A 1924–29 table at 3 % interest are now given.

Diff. in age $(y-x) = 30$ years, $\omega' = 109{\cdot}449$, $a' = 33{\cdot}413$.

$$\gamma' + 2\kappa = p/a' = 10{\cdot}2/33{\cdot}413 = {\cdot}305270. \quad \log = \bar{1}{\cdot}484684$$

$$\gamma' - \delta \qquad = {\cdot}271111. \quad \log = \underline{\bar{1}{\cdot}433147}$$

$$\text{difference} = {\cdot}051537$$

$$\text{difference} \times (p+1) = {\cdot}577214$$

$$\log^{-1} = 3{\cdot}7776 \qquad \text{(A)}$$

$$(\omega' - y) = 59{\cdot}449$$

$$(2\kappa+\delta)(\omega'-y)\log_{10} e = {\cdot}88191 \qquad \log^{-1} = 7{\cdot}6192 \qquad \text{(B)}$$

$$u_2 = (\omega'-y)(\gamma'+2\kappa)(p+1)^{-\frac{1}{2}} = 5{\cdot}4227 \qquad I(u_2,\ 10{\cdot}2) = {\cdot}967970$$

$$u_1 = (\omega'-y)(\gamma'-\delta)(p+1)^{-\frac{1}{2}} = 4{\cdot}8160 \qquad I(u_1,\ 10{\cdot}2) = {\cdot}918243$$

$$[(\text{A}) \times I(u_1,\ 10{\cdot}2)]/[(\text{B}) \times I(u_2,\ 10{\cdot}2)] = {\cdot}470306 \qquad \text{(C)}$$

$$\bar{a}_{20:50} = [1 - (\text{C})]/(2\kappa+\delta) = 15{\cdot}51.$$

(Tabular value, $a_{20:50} + {\cdot}5 = 15{\cdot}57$.)

Values of $\bar{a}_{xy}$ for various combinations of ages calculated by this method are given in Table 7, together with values interpolated, where necessary, from the A 1924–29 tabulated values.

21. The errors in Table 7 are the combined effect of a number of factors. First, the single-life functions are themselves an approximation to the A 1924–29 values; secondly the A 1924–29 annuity-values are based on interpolation in the tabulated values; thirdly $\frac{1}{2}+a$ is used as an approximation to $\bar{a}$; fourthly the joint-life values are based on a Type-III approximation to the curve of deaths with the value of p fixed at 10·2. To test the last error, joint-life values have been calculated for the hypothetical table described in the appendix to the first paper, since the corresponding life table columns are available. Some results are given in Table 8.

Table 7. Values of $\bar{a}_{xy}$, A 1924–29 ult., 3%

x	y	Approx.	Tabular	x	y	Approx.	Tabular
20	20	22·96	22·95	20	30	21·45	21·43
30	30	20·43	20·41	30	40	18·40	18·38
40	40	17·05	17·04	40	50	14·56	14·58
50	50	12·96	13·06	50	60	10·36	10·42
60	60	8·78	8·84	60	70	6·56	6·50
70	70	5·28	5·20	70	80	3·70	3·64
80	80	2·79	2·82	80	90	1·81	2·00
90	90	1·16	1·56				
20	40	18·95	18·94	20	50	15·51	15·57
30	50	15·27	15·32	30	60	11·44	11·48
40	60	11·15	11·18	40	70	7·60	7·51
50	70	7·31	7·21	50	80	4·54	4·49
60	80	4·29	4·22	60	90	2·38	2·59
70	90	2·21	2·38				
20	60	11·52	11·57	20	70	7·71	7·64
30	70	7·69	7·62	30	80	4·65	4·62
40	80	4·63	4·58	40	90	2·47	2·71
50	90	2·45	2·68				

Table 8. Values of $\bar{a}_{xy}$ at 3%. Hypothetical Table

x	y	Approx.	True	x	y	Approx.	True
20	20	23·39	23·39	20	40	19·30	19·31
40	40	17·33	17·34	50	70	7·52	7·49
60	60	8·99	8·93	70	90	2·32	2·39
80	80	2·93	3·02	20	80	4·86	4·86
90	90	1·37	1·83	30	90	2·63	2·66

The true values were calculated by approximate integration and the approximate values by the method of paragraph 20 with appropriate values of the various constants. Apart from the very high age combinations the formula gives an acceptable method of approximation. In considering this table it will be appreciated that the entire calculations are based on the four constants κ, γ, ω and a and the values of $I(u, p)$ for $p = 10{\cdot}2$.

22. The method can obviously be extended to calculations involving more than two lives, although the subsidiary tables of ω and a become more extensive

and more troublesome to use. However, calculations have been made for combinations of 3 lives, and the means and variances are given in Table 9. In Table 10 are given values of ω and a corresponding to the A 1924–29 approximations. For example, to calculate $a_{x:x+r:x+s}$, the values of ω and a corresponding to differences h, k in u of $\frac{r}{11\cdot0706}$ and $\frac{s-r}{11\cdot0706}$ would be found by interpolation in Table 10. The calculations would be treated as a single-life annuity at age $x+s$ with a constant of 3κ and $p=10\cdot2$.

Table 9. Moments of $\Delta I(u, 10\cdot2)\, I(u+k, 10\cdot2)\, I(u+h+k, 10\cdot2)$

h	k	Mean	μ_2	h	k	Mean	μ_2	h	k	Mean	μ_2
0	0	4·21377	·78043	1	0	3·99375	·80744	2	0	3·92432	·83873
0	1	3·69044	·82425	1	1	3·57994	·86440	2	1	3·55086	·88518
0	2	3·44408	·91708	1	2	3·40853	·94323	2	2	3·40072	·95161
0	3	3·36787	·97503	1	3	3·35915	·98474	2	3	3·35777	·98666
0	4	3·35019	·99491	1	4	3·34909	·99633				
0	5	3·34716	·99913								
3	0	3·90806	·85158	4	0	3·90508	·85477	5	0	3·90463	·85533
3	1	3·54503	·89114	4	1	3·54411	·89229				
3	2	3·39936	·95338								

Table 10. Constants for three-life values, A 1924–29

h	k	w	a	h	k	w	a	h	k	w	a
0	0	96·081	29·808	1	0	99·078	30·319	2	0	100·486	30·901
0	1	102·781	30·633	1	1	104·814	31·370	2	1	105·547	31·745
0	2	107·352	32·312	1	2	108·249	32·770	2	2	108·494	32·915
0	3	109·299	33·317	1	3	109·578	33·483	2	3	109·629	33·515
0	4	109·866	33·655	1	4	109·905	33·679				
0	5	109·978	33·727								
3	0	100·925	31·137	4	0	101·022	31·195	5	0	101·038	31·205
3	1	105·729	31·852	4	1	105·762	31·872				
3	2	108·543	32·946								

Table 11. Values of $\bar{a}_{xyz}$, A 1924–29 ult., 3%

x	y	z	Approx.	True	x	y	z	Approx.	True
20	20	20	21·39	21·39	30	30	50	14·63	14·68
30	30	30	18·79	18·77	30	50	50	12·55	12·64
40	40	40	15·31	15·29	20	20	60	11·28	11·34
50	50	50	11·17	11·31	20	60	60	8·64	8·71
60	60	60	7·16	7·25	30	30	70	7·57	7·48
70	70	70	4·03	3·99	30	70	70	5·22	5·14

23. Approximate values of three-life annuity-values on A 1924–29 ult. at 3% are given in Table 11, in which 'true' values are taken from the official table or from *T.F.A.* xv, 107. As with the two-life calculations, the errors given are the combined result of a number of factors.

CONTINGENT ASSURANCES

24. Although various convenient approximations are available for the calculation of functions such as $\bar{A}^1_{xy}$, it was considered worth while to develop the present technique for such functions.

We have $$\bar{A}^{1}_{x:x-s}=\int_0^\infty v^t{}_tp_{x:x-s}\,\mu_{x+t}\,dt.$$

Using formulae (1) and (2) and making some reductions we find

$$\bar{A}^{1}_{x:x-s}=\kappa\bar{a}_{x:x-s}+\frac{e^{-(\delta+2\kappa)(\omega'-x+s)}(\gamma')^{p+1}NI_2(u_5,p)}{I_1(u_3,p)I_1(u_4,p)(\gamma'-\delta-2\kappa)^{p+1}}, \tag{4}$$

where
$$u_3=(\omega-x)(\gamma+\kappa)(p+1)^{-\frac{1}{2}},$$
$$u_4=(\omega-x+s)(\gamma+\kappa)(p+1)^{-\frac{1}{2}},$$
$$u_5=(\omega'-x)(\gamma'-\delta-2\kappa)(p+1)^{-\frac{1}{2}}$$

and I_1 refers to the single-life functions. (In place of the product $I(u_3,p)\,I(u_4,p)$ the equivalent joint-life function may be used.) I_2 is derived by fitting the product $(\gamma+\kappa)\,e^{-u\sqrt{(p+1)}}\{u\sqrt{(p+1)}\}^p\,I(u+h,p)$ with a new Type-III curve for which the terminating age is ω' and γ' is equal to p/a' for this curve.

25. Approximate moments of I_2 were found for the A 1924–29 table by forming the products $(\gamma+\kappa)\,e^{-u\sqrt{(p+1)}}\{u\sqrt{(p+1)}\}^p\,I(u+h,p)$ for various values of h at intervals of ·5 in u. These are given in Table 12. The error in calculating the moments may be gauged by comparison of the values for $h=0$ and $h=6{\cdot}0$ with Table 6.

26. The values of ω', a' and N corresponding to the A 1924–29 table are also given in Table 12.

Table 12. Moments and Constants for contingent assurances (see paragraph 25)

h	Mean	μ_2	A 1924–29 ult.		
			ω'	a'	N
0	3·90192	·85280	101·017	31·159	·49895
·5	3·71824	·88229	103·637	31·693	·64093
1·0	3·58164	·91618	105·812	32·296	·76473
1·5	3·48714	·94316	107·376	32·768	·85924
2·0	3·42449	·96292	108·445	33·110	·92224
2·5	3·38652	·97740	109·138	33·358	·96001
3·0	3·36526	·98684	109·550	33·519	·98052
3·5	3·35421	·99255	109·778	33·615	·99075
4·0	3·34892	·99560	109·893	33·667	·99547
4·5	3·34654	·99713	109·948	33·693	·99750
5·0	3·34552	·99783	109·972	33·705	·99833
5·5	3·34512	·99811	109·982	33·710	·99867
6·0	3·34497	·99823	109·986	33·712	·99879

27. Values of $\bar{A}^1_{x:y}$ at 3% calculated from formula (4) are given in Table 13 together with 'true' values calculated by the formulae

$$\bar{A}^1_{xx}=\tfrac{1}{2}\bar{A}_{xx},$$
$$\bar{A}^1_{xy}=(\bar{a}_{xy}-p_x\bar{a}_{x+1:y})+\tfrac{1}{2}(p_x\,p_{x+1}\,\bar{a}_{x+2:y}-2p_x\bar{a}_{x+1:y}+\bar{a}_{xy})+\ldots$$

and the results need no comment.

Table 13. Values of $\bar{A}^{1}_{xy}$, A 1924–29 ult., 3%

x	y	Approx.	True	x	y	Approx.	True
20	20	·1604	·1608	40	20	·365	·360
30	30	·1977	·1984	60	40	·583	·582
40	40	·2475	·2482	80	60	·751	·760
50	50	·3077	·3069				
60	60	·3690	·3693	60	20	·632	·628
70	70	·4233	·4232	80	40	·848	·848
80	80	·4652	·4583				

CONCLUSION

28. These further notes on the incomplete gamma function technique are presented in the hope that in the principles developed may be found some ideas for development. They are essentially of a research character, since it is clear that for many practical problems the classical methods based on prepared tables and commutation columns would be more appropriate.