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SOME NOTES ON GRADUATION

By R. E. BEARD, M.B.E., F.I.A., F.S.S., A.S.A.
Assistant General Manager, Pearl Assurance Co. Ltd.

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INTRODUCTION

AFTER the publication of the volume of statistics of the data derived from the Continuous Mortality Investigation 1924–29, experiments were made by the author in fitting the curves developed by Perks in 1931 (*J.I.A.* LXIII, 12) to various sections of the data. The experiments were later extended to include the data for the Light and Heavy tables and the experience for the period 1924–38. Although these experiments are now mainly of historical interest they represent useful material for research in testing and otherwise and, together with the results of some other published and unpublished graduations, they form the background of this paper.

2. Apart from the general aspects of the various graduations and some practical points arising therefrom, the main topics treated in this paper relate to the use of grouping and to the theory and application of the mean-deviation test. With regard to the former, three aspects are considered, namely, grouping data for the purpose of graduation, grouping deviations when testing a graduation, and the effect on the grouped deviations of the choice of age-groups. With regard to the mean-deviation test the experiments also throw some additional light on the r_x technique, introduced by Redington and Michaelson (*Trans. Twelfth Int. Cong. Act.*, Lucerne, 1940, I, 225), and further discussed by Daw (*J.I.A.* LXXII, 174, [1945]).

3. The emphasis throughout the paper on the mean-deviation test should not be regarded as implying that it is considered that such summary tests are adequate to provide a judgment on the results of a graduation—this subject has recently been discussed at the Institute (H.A.R. Barnett, *J.I.A.* LXXVII, 15 [1950]). Also, the use of the particular mathematical curves should not be regarded as implying a belief in any philosophical basis justifying their use in the graduation of mortality data, although the author hopes to be able to submit certain suggestions in this regard in the not too distant future.

GROUPING

4. The first point to be discussed is the error arising through combining data in quinary groups, and this will be illustrated by a graduation of the A 1924–29 ultimate data (i.e. durations 3 and over) by means of the curve

$$q_x = (A + Bc^x)/(1 + Dc^x).$$

The results of this graduation are set out on p. 9 of the volume of *Extracts and Discussions* (1935). Comparison of the actual and expected deaths shows a slight excess of expected deaths over actual deaths. While there may be good reason for adopting graduated rates of mortality higher than those disclosed by the statistics (see *J.I.A.* LXVIII, 54 et seq. [1936]), or the condition of equality of actual and expected deaths may not be imposed (see Barnett, *J.I.A.* LXXVII, 15), in the present context the disagreement is a defect in the process of fitting. It arises because the constants in the formula were found from equations of the form

$$A\Sigma^r E_x + B\Sigma^r c^x E_x = \Sigma^r \theta_x + D\Sigma^r c^x \theta_x,$$

in which the summations were based on quinary groups (corrected by deduction of $\frac{1}{25}$th of the second central differences), and the expected values were calculated at individual ages, and is thus a measure of the error introduced by the use of the central value of c^x in each group.

5. To produce a more polished graduation, values of $c^x E_x$ and $c^x \theta_x$ were calculated at individual ages and fresh values of the constants determined from quinary groupings of these values. The results of this re-graduation, together with those of the earlier graduation, are summarized in Table 1. The re-graduation shows negligible differences as compared with the earlier graduation, but the excess of expected over actual deaths is now reduced from 124 to 12. Of the excess of 12 deaths, 10 are accounted for by the first and last groups which were not used in the fitting process. This experiment is of value in illustrating the small magnitude of the error introduced by treating the data in quinary groups, an important consideration in the early stages of a graduatioı when experiments are being made for a suitable formula and, in cases such as the present, for a suitable value of c.

6. The next point arises from the practice of grouping deviations when summarizing the results of a graduation. In his paper on graduation tests (*J.I.A.* LXXI, 10 [1941]), Seal contended that the grouping of deviations is open to serious criticism, and supported his contention by an analysis of the Kenchington graduation of the O^{JF} data. Now, provided that the deviations at individual ages are random variates from a binomial or normal population, grouping them will merely exhibit the effect of sampling from such a population, assuming that the standard deviation of the values within the group are sensibly the same. Thus, ignoring the small adjustment required by the use of King's quinquennial pivotal value formula, grouping the deviations in fives should produce grouped values which are approximately normally distributed with a standard deviation $\sqrt{5}$ times the standard deviation of the individual values.

7. We can find an approximation to the effect of the use of King's formula by noting that it combines 15 separate observations in the following scheme, where e_y is the observation at age y (i.e. E_y for the exposed to risk or θ_y for the deaths):

$$\text{Group value} = \frac{1}{25}\left\{-\sum_{y=-7}^{-3} e_y + 27\sum_{y=-2}^{2} e_y - \sum_{y=3}^{7} e_y\right\}.$$

If we make the assumption that E_y is constant over the range considered and ignore the influence of the increasing nature of q (i.e. assume that the standard deviations are the same and equal to σ), then the e_y are proportional to the deviations and the group deviation will have a standard deviation of

$$\sigma \times \sqrt{\{5(2\times 1^2 + 27^2)/25^2\}} \doteqdot 1\cdot 08\sqrt{5}\,\sigma.$$

8. We can now consider the application to a particular graduation. It is assumed that the data are grouped in fives and that a formula is fitted to the quinary groups adjusted by King's method. The deviations found by applying the graduated values of q to the adjusted group totals will be termed group deviations. The deviations found at each age by applying the graduated q's will

Table 1. Graduation of A 1924–29 data—durations 3 and over

Central age of group x	Actual deaths minus expected deaths: Official		Re-graduation (a)		Previous graduation (b)		Graduated $q_x \times 10^5$: Age	Official	Re-graduation (a)	Previous graduation (b)
	+	−	+	−	+	−				
$12\frac{1}{2}$	.	7	.	8	.	8	10	156	198	199
$17\frac{1}{2}$	.	2	5	.	5	.	15	206	203	203
$22\frac{1}{2}$	12	.	56	.	56	.	20	235	211	212
$27\frac{1}{2}$	.	28	.	40	.	43	25	235	226	226
$32\frac{1}{2}$	8	.	.	110	.	113	30	241	252	252
$37\frac{1}{2}$	.	34	.	48	.	48	35	286	297	298
$42\frac{1}{2}$	142	.	282	.	282	.	40	388	378	378
$47\frac{1}{2}$	.	116	.	113	.	124	45	527	520	520
$52\frac{1}{2}$	23	.	.	74	.	82	50	764	769	769
$57\frac{1}{2}$	.	5	3	.	.	9	55	1,190	1,203	1,202
$62\frac{1}{2}$	100	.	135	.	123	.	60	1,973	1,950	1,952
$67\frac{1}{2}$	.	220	.	223	.	235	65	3,188	3,212	3,215
$72\frac{1}{2}$	37	.	190	.	175	.	70	5,327	5,273	5,278
$77\frac{1}{2}$	.	2	.	41	.	57	75	8,497	8,463	8,472
$82\frac{1}{2}$	34	.	.	39	.	53	80	12,910	13,009	13,027
$87\frac{1}{2}$	.	12	.	6	.	11	85	18,676	18,784	18,815
$92\frac{1}{2}$	.	9	28	.	27	.	90	25,611	25,132	25,183
$97\frac{1}{2}$	.	23	.	7	.	7	95	33,675	31,094	31,168
$102\frac{1}{2}$	.	6	.	2	.	2	100	42,960	35,920	36,015
Totals	356	464	699	711	668	792				
Total*	820		1410		1460					

* Without regard to sign.

Formula: $q_x = (A + Bc^x)/(1 + Dc^x)$.

	Origin	c	A	B	D
Graduation (a)	$57\frac{1}{2}$	$(1{\cdot}05875)^2$	·001923	·013811	·030683
Graduation (b)	$57\frac{1}{2}$	$(1{\cdot}05875)^2$	·001926	·013819	·030603

be termed individual deviations, and the sums of quinary groups of individual deviations will be termed grouped deviations. Then noting that the number of groups is approximately one-fifth of the number of individual values, the sum of grouped deviations without regard to sign will be approximately $1/\sqrt{5}$ times the sum of individual deviations without regard to sign. This follows by noting that if there are n independent values normally distributed about a mean of zero with a S.D. of σ, the expected value of $\Sigma\,|\text{deviations}|$ is $\cdot 8n\sigma = M$ say. Samples of 5 from this population will be normally distributed about zero with a S.D. of $\sqrt{5}\,\sigma$ and the expected value of $\Sigma\,|\text{deviations}|$ of $n/5$ values from this latter population will be $\cdot 8\sigma n/\sqrt{5} = M/\sqrt{5}$. Similarly, the sum of the group deviations will be about 1·08 times the sum of the grouped deviations without regard to sign.

Table 2. Comparison of individual, grouped and group deviations

Data (1)	Graduation (2)	Sum of individual deviations (3)	Col. (3) $\div\sqrt{5}$ (4)	Sum of grouped deviations (5)	Sum of group deviations (6)
A 1924–29 ult.	$q_x = \dfrac{A+Bc^x}{1+Dc^x}$	+1593 −1605 3198	+ 712 − 718 1430	+ 699 − 711 1410	+ 827 − 731 1558
	Spencer 21 term (official)	+1454 −1561 3015	+ 650 − 698 1348	+ 356 − 464 820	
O^M	$q_x = \dfrac{A+Bc^x}{Ec^{-x}+1+Dc^x}$	+1317 −1312 2629	+ 589 − 587 1176	+ 508 − 503 1011	+ 547 − 524 1071
	Spencer 21 term	+1192 −1150 2342	+ 533 − 514 1047	+ 314 − 272 586	
O^{JF}	$q_x = A+Bc^x$	+ 103·7 − 104·5 208·2	+ 46·4 − 46·7 93·1	+ 45·2 − 46·0 91·2	+ 47·7 − 47·8 95·5
	Kenchington	+ 92·8 − 97·9 190·7	+ 41·5 − 43·8 85·3	+ 15·2 − 20·3 35·5	

9. Table 2 has been prepared to illustrate the foregoing by application to three sets of data. The first set comprises the A 1924–29 ultimate data for which the deviations at individual ages for the official graduation were found by calculating the expected deaths from values of $q_{x+\frac{1}{2}}$ found by interpolation from the published tabular values. The details for the Perks graduation were obtained from the re-graduation described in § 5 above. The second set comprises the O^M data, for which the individual deviations for the Spencer graduation were calculated from the values of q_x given in *J.I.A.* XXXVIII, 342 and *J.I.A.* XLI, 378. The details of the Perks graduation were taken from the calculations for the graduation given in *J.I.A.* LXVII, 53. The third set consists of the O^{JF} data, no formula graduation for which was available. A rough Makeham graduation was accordingly made, details for which are given in Table 3.

10. From Table 2 we may first consider the comparison of the sum of the grouped deviations with $1/\sqrt{5}$ times the sum of the individual deviations, and it is seen that in all three cases the formula graduation produces a reasonable comparison. As regards the summation graduation the sum of the grouped deviations in all cases is well below $1/\sqrt{5}$ times the sum of the individual deviations. It is thus clear that Seal's contention regarding grouping requires closer analysis. If a formula graduation is involved, it seems unlikely that the use of grouped deviations will lead to any erroneous conclusions as regards the success of the graduation, but if a summation formula is involved then the size of the individual deviations will be greater than the grouped deviations would suggest. Cases will, of course, arise in practice where the sum of the grouped deviations will differ substantially from $1/\sqrt{5}$ times the sum of the individual deviations from purely random variation. This topic has been discussed by Barnett, and it is of interest to note that such a case arises in the graduation of the A 1924–29 select data given in Table 6 of his paper.

Table 3. Graduation of O^{JF} data

Age-group	Actual deaths minus expected deaths*				$\sqrt{(Epq)}$
	Kenchington		Makeham†		
	+	−	+	−	
20–24	·	2·2	·	5·8	3·7
25–29	1·9	·	·	1·8	7·2
30–34	·	·8	·5	·	9·5
35–39	·	1·2	16·3	·	10·8
40–44	2·3	·	11·1	·	11·7
45–49	·	·1	·	18·0	12·3
50–54	2·2	·	·	13·7	12·5
55–59	3·3	·	3·7	·	12·2
60–64	·	4·0	5·9	·	11·5
65–69	5·5	·	7·7	·	10·1
70–74	·	12·0	·	6·7	8·2
Totals	15·2	20·3	45·2	46·0	109·7
Total‡	35·5		91·2		

* Grouped individual deviations.
† $q_x = \cdot01017 + \cdot003019\,(3)^{x/10}$, origin $x=47$.
‡ Without regard to sign.

11. Now in the above calculations for the formula graduations a similar grouping has been made to that adopted in the fitting process. It is possible, however, to group the deviations in different ways, and one set of groupings, the assumptions underlying which would be materially the same, would be the five series that can be formed from consecutive groups of five values. Accordingly, Table 4 has been calculated for the three formula graduations on this basis. The table is interesting since it gives some indication of the variation introduced by the grouping process. It will be noted that for the grouping adopted in Table 2 the A 1924–29 deviations are about equal to the average of the five groupings, but in the case of the O^M and O^{JF} they happen to be the lowest of the five possible groupings. All values are, however, reasonably close to their expected values. The standard deviations shown are those calculated from the five values given in Table 4.

Table 4. Sums for various groupings of grouped deviations arising from the graduations by formulae

Grouping	A 1924–29	OM	OJF
$x-(x+4)$	1,410	1,011	91·2
$(x+1)-(x+5)$	1,176	1,071	116·2
$(x+2)-(x+6)$	1,380	1,061	114·4
$(x+3)-(x+7)$	1,530	1,315	140·8
$(x+4)-(x+8)$	1,650	1,077	119·6
Mean	1,429	1,107	116·4
Standard deviation	159	107	15·8
'Expected' value (from Table 2)	1,430	1,176	93·1

12. We may now compare the sums of the grouped deviations with the group deviations. In this comparison similar groupings must be adopted and, of course, formula graduations only are involved. The ratios of group to grouped deviations are given in Table 5. In § 7 it was shown that the effect of

Table 5. Ratio of Σ (group deviation) to Σ (grouped deviation)
Formula graduations

Data	Ratio
A 1924–29	1·105
OM	1·059
OJF	1·047

King's formula would be to increase the standard deviation by a factor of approximately 1·08. The above values are thus not unreasonable having regard to the assumptions made in deriving the factor and to the limited number of values involved.

13. Finally, we may give some consideration to the sum of the grouped deviations arising from the summation graduations. If we postulate the existence of a true underlying rate of mortality q_x^{T}, then we say that the deviations

$$\epsilon_x = \theta_x - \mathrm{E}_x q_x^{\mathrm{T}} = \mathrm{E}_x (q_x^{\mathrm{O}} - q_x^{\mathrm{T}})$$

are distributed normally about a mean of zero with a standard deviation σ. If we now operate on q_x^{O} with a summation formula defined by

$$q_x^{\mathrm{G}} = \sum_{-n}^{n} \alpha_t q_{x+t}^{\mathrm{O}}$$

we obtain a set of deviations

$$\mathrm{E}_x (q_x^{\mathrm{O}} - q_x^{\mathrm{G}}) = \mathrm{E}_x (q_x^{\mathrm{O}} - q_x^{\mathrm{T}}) - \mathrm{E}_x \sum_{-n}^{n} \alpha_t (q_{x+t}^{\mathrm{O}} - q_{x+t}^{\mathrm{T}}) + \mathrm{E}_x \left(q_x^{\mathrm{T}} - \sum_{-n}^{n} \alpha_t q_{x+t}^{\mathrm{T}} \right)$$

$$= \epsilon_x - \sum_{-n}^{n} \alpha_t \epsilon_{x+t}$$

(assuming that q_x^{T} reproduces itself by the summation formula),

$$= \sum_{-n}^{n} \beta_t \epsilon_{x+t},$$

where $\beta_t = 1 - \alpha_0$ when $t = 0$ and $-\alpha_t$ for other values of t. Assuming that σ is constant over the range of operation, the expected value is zero and the variance is

$$\left(\sum_{-n}^{n} \beta_t^2\right) \sigma^2 = r_1^2 \sigma^2, \text{ say.}$$

If the deviations are now grouped in fives, we require the standard deviation of

$$\sum_{-(n+2)}^{n+2} (\beta_{t-2} + \beta_{t-1} + \beta_t + \beta_{t+1} + \beta_{t+2}) \epsilon_{x+t}.$$

The expected value is zero and the variance is

$$\left[\sum_{-(n+2)}^{n+2} (\beta_{t-2} + \beta_{t-1} + \beta_t + \beta_{t+1} + \beta_{t+2})^2\right] \sigma^2 = r_2^2 \sigma^2, \text{ say.}$$

We should expect the ratio of the sum of the grouped deviations to the sum of the individual deviations to approximate to $(r_2\sigma/5)/r_1\sigma = r_2/5r_1$. Table 6 shows that theory agrees with fact for the A 1924–29 and O^M tables but that there is something yet to be explained for the O^{JF} table.

Table 6. Ratios for summation graduations

Data (1)	$\dfrac{\Sigma \text{ Grouped deviation}}{\Sigma \text{ Individual deviation}}$ (2)	$\dfrac{r_2}{5r_1}$ (3)
A 1924–29	·272	·240
O^M	·250	·240
O^{JF}	·186	·304

From now on the paper is restricted to graduations by formulae.

MEAN-DEVIATION AND χ^2 TESTS

14. In the preceding sections the discussion has centred mainly in the relative magnitude of the deviations in grouped and ungrouped data, and consideration is now given to the significance of the deviations. Perks's comments in the discussion on Daw's paper (*J.I.A.* LXXII, 198 and 201 [1945]) are pertinent in this respect, but a more detailed analysis may well be of some value.

15. In view of the symmetry of the normal curve, sampling from a normal population without regard to sign may be regarded as sampling from half a normal population. The moments of the 'half-normal' distribution, $e^{-x^2/2\sigma^2}$ over the range (0, ∞), can readily be found (*J.I.A.* LXXIII, 379 [1946]) and are as follows:

$$\text{Mean} = \sigma\sqrt{\frac{2}{\pi}} \doteqdot \cdot 8\sigma,$$

$$\mu_2 = \sigma^2\left(1 - \frac{2}{\pi}\right) \doteqdot \cdot 36\sigma^2,$$

$$\beta_1 = \cdot 9906, \quad \beta_2 = 3 \cdot 8692.$$

The moments of the distribution of the mean deviation for samples of n measured from the population mean can thus be written down as

$$\text{Mean} \doteqdot \cdot 8\sigma,$$
$$\mu_2 \doteqdot \cdot 36\sigma^2/n,$$
$$\beta_1' = \cdot 9906/n, \quad \beta_2' = 3 + \cdot 8692/n.$$

From these values it would be possible to calculate the probability integral for various values of n, but for the values of n likely to arise in mortality data the error introduced by assuming a normal distribution will clearly be small.

16. It may be noted, however, that tables of the probability integral of the mean deviation in normal samples of n observations have been calculated (Hartley, *Biometrika*, XXXIII, 257 [1945]), and approximations to the required distributions could be found from them. The tables give the distribution of the mean deviation measured from the sample mean in terms of the population standard deviation. In this case (*Biometrika*, XXXIII, 252) we have

Mean deviation in samples of n observations:

$$n = \frac{\sigma}{n}\sum_{i=1}^{n} |x_i - \bar{x}| \quad (\bar{x} = \text{sample mean});$$

Expectation of m: $$\bar{m} = \sigma\sqrt{\frac{2(n-1)}{n\pi}};$$

$$\text{Variance} = \frac{2(n-1)\sigma^2}{n^2\pi}\left\{\frac{\pi}{2} + \sqrt{[n(n-2)]} - n + \sin^{-1}\frac{1}{n-1}\right\}.$$

Values of the moment-ratios β_1 and β_2 calculated from values of the third and fourth semi-invariants of m calculated from expansions given by Dr Geary are also given.

The change of the variance with n is small, as the following values show:

n	2	3	5	10	20	∞
$\frac{n}{\sigma^2} \times$ variance	·3634	·3508	·3547	·3588	·3611	·3634 $\left(=1-\frac{2}{\pi}\right)$

and the approximation $\mu_2 = \cdot 36\sigma^2/n$ will suffice for most practical problems.

17. As regards the β_1, β_2 values, Table 7 sets out a comparison of the values given in *Biometrika* XXXIII, 252 with the values of $\cdot 9906/(n-\frac{1}{2})$ and

Table 7. Comparison of true with approximate values of β_1 and β_2

	Sample size n					
	4	5	6	10	15	20
β_1	·299	·230	·187	·106	·069	·051
$\cdot 9906/(n-\frac{1}{2})$	·283	·220	·180	·104	·068	·051
β_2	3·244	3·194	3·160	3·093	3·060	3·045
$3+\cdot 8692/(n-\frac{1}{2})$	3·248	3·193	3·158	3·091	3·060	3·045

$3 + \cdot 8692/(n-\frac{1}{2})$ respectively. These figures show that with a slight shift in the mean, the tabulated probability integral will suffice for the distribution in § 15 provided it is entered with $n+\frac{1}{2}$, where n is the number in the sample.

18. In deriving the distribution of the mean deviation for use in a graduation test we may need to have regard to the number of constraints used in obtaining the graduated values, and this problem does not appear to have been fully investigated. When the number of groups is large the adjustment will clearly be small, but in the present applications there are a limited number of groups only and the adjustment cannot therefore be ignored. Having regard to the form of the moments of the sampling distribution of the mean deviation, by analogy with the χ^2 test and from consideration of the limiting values, the following approximation is indicated:

Expected mean deviation

$$=\sigma\sqrt{\frac{2}{\pi}}\sqrt{\frac{n-l}{n}}\doteqdot\cdot 8\sigma\sqrt{\frac{n-l}{n}},$$

$$\text{Variance} \qquad \doteqdot\sigma^2\left(1-\frac{2}{\pi}\right)\left(\frac{n-l}{n}\right)\doteqdot\cdot 36\sigma^2\left(\frac{n-l}{n}\right),$$

$$\beta_1 \qquad \doteqdot \quad \cdot 9906/(n-l+\tfrac{1}{2}),$$

$$\beta_2 \qquad \doteqdot 3+\cdot 8692/(n-l+\tfrac{1}{2}),$$

where n is the number of groups and l the number of constraints.

19. In the foregoing discussion the method of applying the mean-deviation test has been based on the sum of standardized deviations, i.e. in the case of mortality data, the sum of values of

$$|\,\text{Actual deaths}-\text{Expected deaths}\,|\div\sqrt{(\text{E}pq)}.$$

This corresponds to the usual method of applying the χ^2 test from values of $(\theta-\text{E}q)^2/\text{E}pq$. As Perks pointed out, however, there are reasons why it may be preferable in the case of the graduations being considered to adopt weighted values of these standardized deviations, and it is desirable to consider the effect if these modified tests are adopted. Where no constraints are involved the problem of finding the expected mean deviation or expected weighted χ^2 is not difficult, but when a number of constraints are involved the determination of the sampling distribution becomes difficult.

20. First considering the χ^2 test, a problem equivalent to that of weighting has been discussed by Patnaik (*Biometrika*, XXXVI, 231 [1949]). He is concerned with ascertaining the distribution of the sum of squares of independent normal variates with different means and variances. In the present application we are considering values of $w(\theta-\text{E}q)^2/\text{E}pq$, i.e. means of zero but variances of w. If w is made equal to $\text{E}pq$, it is found that in place of χ^2 we have $\Sigma(\theta-\text{E}q)^2$, which is approximately distributed as a χ^2 distribution with a first moment of $\Sigma\text{E}pq$ and a variance $2\Sigma(\text{E}pq)^2$. Thus, $\dfrac{(\Sigma\text{E}pq)\,\Sigma(\theta-\text{E}q)^2}{\Sigma(\text{E}pq)^2}$ is distributed as χ^2 with $n'=\dfrac{(\Sigma\text{E}pq)^2}{\Sigma(\text{E}pq)^2}$ degrees of freedom. Clearly n' may be materially less than n in the distributions arising from mortality data. If we are testing the hypothesis that the particular set of observed rates of mortality is expressed by the formula used in the graduation, then it would be appropriate to make allowance for the constraints, and it would seem that the reduced number of degrees of freedom would approximate to $n'\left(\dfrac{n-l}{n}\right)$. If we are testing the hypothesis that the particular set of observed rates can be regarded as a random set from a population defined by the rates given by the graduation formula, then it could be argued

that the number of degrees of freedom should be n'. In the subsequent sections of this paper the former, and more stringent, test is applied.

21. In considering the mean-deviation test we can similarly consider weighting the standardized deviations, and in this case we consider the use of weights equal to $\sqrt{(Epq)}$. By analogy with the χ^2 test we then obtain an expected mean deviation of $\cdot 8\sqrt{\frac{n-l}{n}}\Sigma\sqrt{(Epq)}$ and a standard deviation of $\cdot 6\sqrt{\frac{n-l}{n}}\sqrt{(\Sigma Epq)}$. Higher moments for the sampling distribution could be found from the approximation given in § 17, the denominator being taken as $(n'+\frac{1}{2})\left(\frac{n-l}{n}\right)$, when $n'=\frac{[\Sigma\sqrt{(Epq)}]^2}{\Sigma Epq}$.

22. Before using the above in practical tests it is necessary to note that the r_x technique devised by Redington and Michaelson (*Trans. Twelfth Int. Cong. Act.* Lucerne, 1940) and further discussed by Daw (*J.I.A.* LXXII, 174 [1945]) and the paper by Solomon (*J.I.A.* LXXIV, 94 [1948]) show that we cannot make the assumption that the deaths are binomially distributed with variance Epq. The main reason for this is the presence of duplicates. In default of knowledge as to the incidence of duplicates (see Seal, *J.I.A.* LXXI, 41 [1941] and Daw, *J.I.A.* LXXII, 178 [1945]) it seems necessary to make some assumption so that appropriate statistical tests may be applied. Daw's σ_r measures the departure from unit variance so that we might regard $\sigma_r\sqrt{(Epq)}$ as the appropriate standard deviation, but this includes the effect of other disturbing factors which should not be eliminated by the testing process, and accordingly a factor k_r is used in place of σ_r to represent the correction for the presence of duplicates. Usually it will be necessary to take $k_r=\sigma_r$, but occasionally there may be evidence to enable σ_r to be modified. In the following sections k_r has been taken as 1·5 for the A 1924–29 ultimate data, 1·16 for the O^{M} and 1·0 for the O^{JF}. It is assumed that k_r may be regarded as constant over the whole range of the data. This point is, however, discussed in §§ 53, 54 and 55.

23. Summarizing the foregoing, we are led to consider four summary tests of graduation. In general, these will show different results because they weight the various aspects of the graduation differently. The weighted-χ^2 test gives maximum weight to the regions where the data are greatest; the weighted-mean-deviation test, which might be termed a weighted-standardized-deviation test or a wD test, similarly tests fidelity in the region where the data are most dense, although the weights used above give rather less weight than in the χ^2 test. Next, the mean-deviation test, which might be called a D test, disregards the extent of the data and measures fidelity over the whole range. Finally, the χ^2 test further emphasizes the tails of the data as compared with the D test. If one test only is adopted, regard must be had to these considerations in drawing conclusions from the result. As pointed out in the discussion on Barnett's paper these points can be conveniently summarized in the statement that the χ^2 and D tests do not take into account all the available information and thus are not 'efficient' tests in the language of theoretical statistics.

24. It is convenient now to summarize the practical application in the case of formula graduations of group mortality data. If the grouping is quinary, the factor adjusting for duplicates k_r, the number of groups n, and the number of constraints l, we have,

(*a*) *Weighted-χ^2 test*:

$$\frac{\Sigma \mathrm{E}pq\,\Sigma(\theta-\mathrm{E}q)^2}{(1{\cdot}08k_r)^2\Sigma(\mathrm{E}pq)^2} \text{ is distributed as } \chi^2 \text{ with } \frac{(\Sigma \mathrm{E}pq)^2}{\Sigma(\mathrm{E}pq)^2}\cdot\frac{n-l}{n} \text{ degrees of freedom.}$$

(*b*) *Weighted-mean-deviation* (wD) *test*:

The moments of the distribution of $\Sigma\,|\,\theta-\mathrm{E}q\,|$ are

$$\text{Mean} = {\cdot}8 \times 1{\cdot}08 \times k_r \times \sqrt{\frac{n-l}{n}}\,\Sigma\sqrt{(\mathrm{E}pq)},$$

$$\text{Standard deviation} = {\cdot}6 \times 1{\cdot}08 \times k_r \times \sqrt{\frac{n-l}{n}}\sqrt{(\Sigma \mathrm{E}pq)}.$$

(*c*) *χ^2 test*:

$$\Sigma\frac{(\theta-\mathrm{E}q)^2}{(1{\cdot}08k_r)^2\,\mathrm{E}pq} \text{ is distributed as } \chi^2 \text{ with } (n-l) \text{ degrees of freedom.}$$

(*d*) *Mean-deviation* (D) *test*:

The moments of the distribution of $\Sigma\dfrac{|\theta-\mathrm{E}q|}{\sqrt{(\mathrm{E}pq)}}$ are

$$\text{Mean} = {\cdot}8 \times 1{\cdot}08 \times k_r \times \sqrt{\frac{n-l}{n}} \times n,$$

$$\text{Standard deviation} = {\cdot}6 \times 1{\cdot}08 \times k_r \times \sqrt{\frac{n-l}{n}} \times \sqrt{n}.$$

In cases (*b*) and (*d*) Hartley's tables may be used if $(n-l)$ is small, but it has been considered sufficient in the present applications to use tables of the probability integral of the normal curve.

Table 8. Comparison of graduation tests

Formula graduations

Test	A 1924–29	$\mathrm{O^M}$	$\mathrm{O^{JF}}$
(*a*)	·26	·66	·29
(*b*)	·23	·60	·22
(*c*)	·15	·28	·27
(*d*)	·20	·23	·21

25. The foregoing tests have been applied to the three sets of data considered in § 9, and the results are given in Table 8. It has been considered desirable to set out the tests in the usual probability form, i.e. the chance that a deviation exceeds that actually found. In all cases the graduations fall comfortably within the usual significance levels. The results of the A 1924–29 tests suggest that there is a slightly poorer agreement near the ends of the data. The graduation of the $\mathrm{O^M}$ data appears to be least satisfactory at the ends of the data, the trouble in this case arising from age groups 20–24, 25–29 and 40–44. The tests of the $\mathrm{O^{JF}}$ graduation suggest a satisfactory result.

26. To conclude this section Table 9 sets out the results of applying the wD test to the individual values of the three graduations. Within the limits of sampling fluctuations the values are reasonable compared with the results of Table 8 above, and we may assume that the results shown by the group data will be adequate for practical work.

Table 9. wD test—individual deviations

Data (1)	$\cdot 8k_r\,\Sigma\,\sqrt{(\mathrm{E}pq)}\,\sqrt{\frac{n-l}{n}}$ (2)	$\Sigma\,\lvert\theta-\mathrm{E}q\rvert$ (3)	Col. (3) − col. (2) (4)	$\cdot 6k_r\,\sqrt{(\Sigma\mathrm{E}pq)}\,\sqrt{\frac{n-l}{n}}$ (5)	P (6)
1924–29	3114	3198	+84	287	·39
M	2681	2629	−52	247	·58
JF	190·3	208·2	+17·9	19·9	·18

Table 10. Graduation of A 1924–29 data—durations 3 and over

Central age of group x (1)	Actual deaths minus expected deaths (a) (2)		(b) (3)		(c) (4)		x (5)	$q_x \times 10^5$ Graduation (a) (6)	Table 1 Re-graduation (7)
	+	−	+	−	+	−			
18	1	.	6	.	2	.	15	205	203
23	59	.	118	.	66	.	20	213	211
28	.	50	61	.	.	45	25	227	226
33	.	133	.	27	.	127	30	253	252
38	.	66	.	38	.	74	35	299	297
43	306	.	207	.	297	.	40	378	378
48	.	138	.	385	.	148	45	522	520
53	.	62	.	372	.	66	50	770	769
58	5	.	.	220	7	.	55	1,204	1,203
63	183	.	164	.	190	.	60	1,950	1,950
68	.	211	.	1	.	202	65	3,214	3,212
73	216	.	620	.	225	.	70	5,278	5,273
78	.	117	229	.	.	113	75	8,474	8,463
83	.	69	.	43	.	71	80	13,031	13,009
88	.	11	.	177	.	15	85	18,814	18,784
93	49	.	.	51	47	.	90	25,149	25,132
98	.	8	.	26	.	6	95	31,062	31,094
Totals	819	865	1,405	1,340	834	867			
Total*	1,684		2,745		1,701				

* Without regard to sign.

Formula (*a*), (*b*): $\mu_x = (\mathrm{A} + \mathrm{B}c^x)/(1 + \mathrm{D}c^x)$.

Formula (*c*): $\mu_x = (\mathrm{A} + \mathrm{B}c^x)/(\mathrm{K}c^{-x} + 1 + \mathrm{D}c^x)$.

	Origin	c	A	B	D	K
Graduation (*a*)	58	$(1{\cdot}05875)^2$	·001936	·013809	·023375	—
Graduation (*b*)	58	$(1{\cdot}055)^2$	·001639	·014275	·015148	—
Graduation (*c*)	58	$(1{\cdot}05875)^2$	·001957	·013789	·023246	·000380

FITTING OF μ_x COMPARED WITH q_x

27. In a further graduation experiment the formula $(\mathrm{A}+\mathrm{B}c^x)/(1+\mathrm{D}c^x)$ was fitted to μ_x for the A 1924–29 ultimate data using quinary grouped data adjusted by deduction of $\frac{1}{24}$th of the second central differences. Two values of c were used, and the resulting graduations were almost identical with those found from fitting q_x. The results in Table 10 show the effect of variations in c and illustrate the importance of careful initial selection of this constant. In practice it should be possible to secure a reasonable fit, using not more than three values of c. For

comparative purposes the values of q_x derived from the graduation of μ_x (col. 2) are given in col. 6 of the table. The differences compared with the values given in Table 1 are negligible. It may be noted that the best fit in all these experiments has been deemed to be that in which the sum of the deviations regardless of sign is a minimum, although for the present purpose it has not been considered necessary to find the value of c at the true minimum and carry through the calculations using this value. By analogy with a minimum-χ^2 method the theoretical basis is that of a minimum weighted standardized deviation method, the appropriateness of which for mortality data was pointed by G. F. Hardy

Table 11. Frequency curve graduation of A 1924–29 data—durations 3 and over

Central age of group x	$\left(\frac{\theta_x}{E_x}\right) E'_x$	Graduated values	Deviation +	Deviation −	Central age of group	Actual deaths minus expected deaths +	Actual deaths minus expected deaths −
$15\frac{1}{2}$	2	6	.	4	$12\frac{1}{2}$	.	15
$20\frac{1}{2}$	29	26	3	.	$17\frac{1}{2}$	.	7
$25\frac{1}{2}$	112	113	.	1	$22\frac{1}{2}$	19	.
$30\frac{1}{2}$	408	421	.	13	$27\frac{1}{2}$	.	37
$35\frac{1}{2}$	1,342	1,344	.	2	$32\frac{1}{2}$	.	66
$40\frac{1}{2}$	3,770	3,641	129	.	$37\frac{1}{2}$	.	11
$45\frac{1}{2}$	8,232	8,293	.	61	$42\frac{1}{2}$	270	.
$50\frac{1}{2}$	15,665	15,687	.	22	$47\frac{1}{2}$	.	174
$55\frac{1}{2}$	24,051	24,333	.	282	$52\frac{1}{2}$	.	130
$60\frac{1}{2}$	30,835	30,509	326	.	$57\frac{1}{2}$	1	.
$65\frac{1}{2}$	30,151	30,434	.	283	$62\frac{1}{2}$	174	.
$70\frac{1}{2}$	24,213	23,733	480	.	$67\frac{1}{2}$	.	181
$75\frac{1}{2}$	13,976	14,183	.	207	$72\frac{1}{2}$	182	.
$80\frac{1}{2}$	6,232	6,353	.	121	$77\frac{1}{2}$	.	109
$85\frac{1}{2}$	2,115	2,080	35	.	$82\frac{1}{2}$	.	94
$90\frac{1}{2}$	497	483	14	.	$87\frac{1}{2}$	20	.
$95\frac{1}{2}$	87	77	10	.	$92\frac{1}{2}$	78	.
$100\frac{1}{2}$	8	9	.	1	$97\frac{1}{2}$	13	.
					$102\frac{1}{2}$	1	.
Total	161,725	161,725	997	997		758	824
Total*			1994			1582	

* Without regard to sign.

(*Construction of Tables of Mortality, etc.*, p. 36, 1909). The same technique can, of course, be used for minimum χ^2, for minimum weighted-χ^2 and for minimum standardized deviations regardless of sign.

28. Col. (4) of Table 10 sets out the result of a trial fitting of μ_x by the formula

$$\mu_x = (A + Bc^x)/(Kc^{-x} + 1 + Dc^x).$$

In this case the extra constant afforded no improvement in the fit, although a different value of c might have produced a better result.

29. The value of B/D derived from the graduation of μ_x (Table 10, col. 2) is ·5908. This is the limiting value of μ_x as $x \to \infty$, and it is of interest to note the nearness of this to the value of ·57 adopted for the force of mortality of non-

elect lives by R. D. Anderson in his paper on Select Mortality Tables (*J.I.A.* XVIII, 223 [1936]). The value of B/D from the graduation of q_x is ·4501, implying limiting value for μ_x of ·5980, which latter value shows the close relationship f the two graduations. Graduations of other data show values of this ratio narkedly different from this value, and it would not seem therefore that any pecial philosophical significance should be attached to it.

FREQUENCY CURVE GRADUATION

30. In a further experiment with the A 1924–29 data the method of graduaion by a frequency curve, fully described in Sir William Elderton's *Frequency Curves and Correlation*, was used to graduate the data, and the results may be of some interest. Most of the details of the calculations are omitted. For the basic unction a normal curve with origin at age 53 and standard deviation 10 was used, and Table 11 sets out the relevant results. The moments of the distribution of $(\theta_x/E_x)E'_x$ were found to be $\mu_2=4{\cdot}162519$, $\beta_1=\cdot0246720$, $\beta_2=3{\cdot}0376952$, the criterion indicating a Pearson type VI. However, consideration of the moments suggested that a type III would be a sufficiently close approximation, and this was accordingly used. The comparison of actual and expected deaths resembles that of the graduation by the formula $q_x=(A+Bc^x)/(1+Dc^x)$, and no further comment appears necessary except to mention that owing to the use of the type III curve the graduated values of q_x reach a maximum at about age 96 and then decrease, a feature which would not have arisen had a type VI curve been used. The graduated values of q_x also show a minimum at about age 23. In considering the results of this graduation it should be remembered that the variance of Eq is again not (Epq) but k_r^2 (Epq).

A1924–29 SECTIONAL DATA

31. The next group of experiments relates to attempts to fit various sections of the A 1924–29 ultimate data by means of the curve

$$q_x=(A+Bc^x)/(1+Dc^x).$$

For this purpose the medical and non-medical data for durations 3 and over were combined to determine exposed to risk and deaths for the four classes: whole life with and without profits and endowment assurance with and without profits. Various values of c were used in an endeavour to produce the best fits. The whole of this set of calculations was carried through on quinary grouped data, the exposed to risk and deaths being adjusted by deduction of $\frac{1}{25}$th of the second central differences and the expected deaths being found by applying the graduated values of q_x to the adjusted groups. In all cases the constants were found from equations of the form

$$A\Sigma^r E_x+B\Sigma^r c^x E_x=\Sigma^r\theta_x+D\Sigma^r c^x\theta_x.$$

32. The relative importance of the different sections of the data is shown by the following summary:

	Exposed to risk	Deaths
Whole life with profits	1,858,383	66,431
Whole life without profits	355,454	7,710
Endowment assurance with profits	5,377,054	32,623
Endowment assurance without profits	952,635	6,663
Total	8,543,526	113,427

It was expected that the best fit by formula graduation would be closer in the case of the sectional data than in the case of the combined data since the disturbance caused by the running off of the endowment assurance data would be absent.

Table 12. Graduation of A 1924–29 whole life with profits data—durations 3 and over

Central age of group x	Actual deaths minus expected deaths						Actual deaths	$\sqrt{(\mathrm{E}pq)}$ (c)	Graduated $q_x \times 10^5$ (c)
	(a)		(b)		(c)				
	+	−	+	−	+	−			
$17\frac{1}{2}$	.	2	.	1	.	1	.	1	207
$22\frac{1}{2}$	5	.	10	.	12	.	42	5	222
$27\frac{1}{2}$	.	17	.	8	.	4	62	8	246
$32\frac{1}{2}$	.	33	.	19	.	12	127	12	289
$37\frac{1}{2}$	.	58	.	40	.	32	268	17	361
$42\frac{1}{2}$	6	.	19	.	26	.	629	24	484
$47\frac{1}{2}$	24	.	20	.	17	.	1,218	35	693
$52\frac{1}{2}$	12	.	.	27	.	47	2,220	47	1,047
$57\frac{1}{2}$	187	.	109	.	71	.	4,148	63	1,640
$62\frac{1}{2}$	185	.	93	.	49	.	6,971	82	2,625
$67\frac{1}{2}$	.	148	.	188	.	204	10,697	102	4,227
$72\frac{1}{2}$	78	.	155	.	196	.	14,043	114	6,752
$77\frac{1}{2}$	.	182	.	52	13	.	13,128	108	10,539
$82\frac{1}{2}$	.	125	.	75	.	58	8,441	85	15,830
$87\frac{1}{2}$	.	28	.	55	.	72	3,517	53	22,470
$92\frac{1}{2}$	62	.	39	.	27	.	830	24	29,863
$97\frac{1}{2}$	17	.	13	.	11	.	88	7	37,017
Totals	576	593	458	465	422	430	66,429	787	
Total*	1169		923		852				

* Without regard to sign.

Formula: $q_x = (\mathrm{A}+\mathrm{B}c^x)/(1+\mathrm{D}c^x)$

	Origin	c	A	B	D
Graduation (a)	$57\frac{1}{2}$	$(1{\cdot}05875)^2$	·002520	·013903	·030462
(b)	$57\frac{1}{2}$	$(1{\cdot}05625)^2$	·002091	·014618	·028198
(c)	$57\frac{1}{2}$	$(1{\cdot}055)^2$	·001865	·014978	·026735

WHOLE LIFE WITH PROFITS

33. The results for three values of c are given in Table 12. It was found that a lower value of c was required than that used in the graduation of the combined data, but it is doubtful if further reduction below $(1{\cdot}055)^2$ would have improved the fit. In order to apply tests to the graduation it is necessary to have some knowledge of the value of k_r. Daw found a value of 1·59 for σ_r for the combined whole life data for durations 5 and over, but it would seem possible that σ_r differs as between the with and without profits data. Accordingly, σ_r was calculated for these two sets of data for durations 5 and over (see Table 27) with the result given in Table 13, the final values being taken from Daw's paper.

34. This table shows that σ_r for the without profits data is not significantly different from unity, and that the effect of combining the two sets of data is to produce a value of σ_r larger than either component. Daw's suggestion regarding σ_r was that the main effect was due to the presence of duplicates, the effect of

heterogeneity being relatively small. Whilst there would be cross-duplicates as between the with and without profits data it seems unlikely that amalgamation would result in an increase in the proportion of duplicates of the order shown. The inference from the figures seems to be that the low value of σ_r for the without profits data is probably a statistical fluctuation, as is the high value for the combined data, and therefore in deciding on a factor to use for adjusting for the effect of duplicates a lower value than 1·59 should be used for the combined data.

35. For the with profits data k_r has been taken to be 1·5. From Table 12, using the graduation with $c=(1{\cdot}055)^2$, $\Sigma|\text{deviations}|$ is ± 852. Using the formulae given in § 24 the expected mean deviation is 883. The difference of -31 between these two values may be compared with an approximate standard deviation of 208 leading to a P of ·56. The graduation may thus be regarded as satisfactory when judged by the *w*D test.

Table 13. Standard deviation of $r_x\,(\sigma_r)$

Data	Number of values of r_x	Standard deviation of $r_x\,(\sigma_r)$
Whole life with profits, durations 5 and over	65	$1{\cdot}49 \pm {\cdot}20$
Whole life without profits, durations 5 and over	65	$1{\cdot}15 \pm {\cdot}15$
Combined whole life, durations 5 and over	65	$1{\cdot}59 \pm {\cdot}21$

WHOLE LIFE WITHOUT PROFITS

36. The results of the graduation of these data are given in Table 14 and proved to be the least successful of the four sets. The best fit was again provided by the lower value of $c=(1{\cdot}055)^2$, and the figures suggest that a slight improvement might be gained by a still lower value. Using the formulae of § 24 and with $k_r = 1{\cdot}15$, the expected mean deviation becomes 249, the actual figure being ± 422. The difference of 173 compares with a standard deviation of 55, so that the graduation is defective when judged by the *w*D test. (If k_r were taken as 1·5, P would be ·08.) It will be noted that there are substantial deviations in the 65–69 and 80–84 age-groups, the former possibly being connected with the incidence of duplicates. The large deviation in the group 80–84 may possibly be linked up with the inclusion of paid-up policies in the exposed to risk without the corresponding deaths being included, an explanation supported by the lower level of the rates of mortality at the higher ages as compared with the with profits data. By way of illustrating the differences between the with and without profits graduations the figures in Table 15 have been calculated.

37. These figures show that the without profits rates are below the with profits, the smallest percentage difference being in the middle range. On the particular weighting the over-all with profits experience is about 5 % heavier than the without profits experience. The difference at the younger ages is not apparent from the data for durations 5 and over, and it is possibly due to the higher proportion of early durations included in the without profits data; for ages up to 44½ the percentages of exposures for durations 3 and 4 included in the 'durations 3 and over' data is 35 % for the without profits and 26 % for the with profits data. As mentioned earlier the deficiency at the older ages may be accounted for by the inclusion of paid-up policies in the exposed to risk. This is discussed in *J.I.A.* LXVIII, 57–58 (1936), but the inference from the present

figures is that the effect is rather less than is there indicated; in considering these figures it may also be noted that the major part of any trouble would probably rest in the without profits data. It seems, therefore, on the evidence

Table 14. Graduation of A 1924–29 whole life without profits data—durations 3 and over

Central age of group x	Actual deaths minus expected deaths						Actual deaths	$\sqrt{(Epq)}$ (c)	Graduated $q_x \times 10^5$ (c)
	(a)		(b)		(c)				
	+	−	+	−	+	−			
$22\frac{1}{2}$	7	.	8	.	8	.	15	3	184
$27\frac{1}{2}$	.	4	.	2	.	1	21	5	210
$32\frac{1}{2}$	.	11	.	9	.	7	43	7	254
$37\frac{1}{2}$	.	4	.	2	.	1	100	10	329
$42\frac{1}{2}$	14	.	14	.	13	.	203	14	456
$47\frac{1}{2}$	.	12	.	18	.	21	297	18	672
$52\frac{1}{2}$	66	.	56	.	51	.	562	22	1,034
$57\frac{1}{2}$	2	.	.	9	.	14	733	27	1,635
$62\frac{1}{2}$	28	.	24	.	22	.	1,019	31	2,617
$67\frac{1}{2}$	.	130	.	122	.	118	1,142	35	4,172
$72\frac{1}{2}$	.	52	.	35	.	27	1,284	35	6,524
$77\frac{1}{2}$	.	18	.	4	3	.	1,136	32	9,838
$82\frac{1}{2}$	70	.	72	.	73	.	754	24	14,079
$87\frac{1}{2}$	45	.	41	.	39	.	309	15	18,877
$92\frac{1}{2}$	.	1	.	5	.	7	87	8	23,607
$97\frac{1}{2}$	.	15	.	16	.	17	4	4	27,677
Totals	232	247	215	222	209	213	7,709	290	
Total*	479		437		422				

* Without regard to sign.

Formula: $q_x = (A + Bc^x)/(1 + Dc^x)$.

	Origin	c	A	B	D
Graduation (a)	$57\frac{1}{2}$	$(1{\cdot}05875)^2$	·001897	·014878	·048534
Graduation (b)	$57\frac{1}{2}$	$(1{\cdot}05625)^2$	·001623	·015332	·044724
Graduation (c)	$57\frac{1}{2}$	$(1{\cdot}055)^2$	·001479	·015566	·042516

Table 15. Comparison of 'With profits' and 'Without profits' graduations on whole-life without-profits data—durations 3 and over

Age-group	Expected deaths		Actual deaths
	On rates by 'With profits' graduation (c)	On rates by 'Without profits' graduation (c)	
Up to $44\frac{1}{2}$	401	370	382
$45\frac{1}{2}$–$74\frac{1}{2}$	5,229	5,144	5,037
$75\frac{1}{2}$ and over	2,449	2,199	2,290
	8,079	7,713	7,709

available to be doubtful whether there is any significant difference between the whole life with and without profits mortality, and it is also of interest to note that the failure of the without profits graduation is associated with doubt as to the accuracy of the data.

COMBINED WHOLE-LIFE DATA

38. The preceding graduations were made on the data as originally published. In *J.I.A.* LXVIII, 82–83 [1936] adjustments of the data were given as a result of investigation into inconsistent figures. The information there given is insufficient to enable a complete calculation to be made, but the major adjustments arose in the whole life and endowment assurance with profits classes and approximate allowance can be made. Since the data relate to a period of 25 years ago and these adjustments are not the only criticism of the data, it was not considered worth while to make a complete set of recalculations. However, the whole life data for durations 3 and over were amalgamated and re-graduated. In Table 16 figures are set out showing:

(i) Addition of the sectional results.
(ii) Graduation of combined whole life data.
(iii) Graduation of whole life data using an additional 300 deaths (details of an adjustment of 339 deaths are given in *J.I.A.* LXVIII).

39. Comparison of (i) and (ii) of Table 16 shows that the sum of the deviations of the separate data compares very closely with the result of graduating the combined data. Comparison of (ii) and (iii) shows that the adjustment to the deaths produces a slight improvement in the graduation. The expected mean deviation for the combined data, using $k_r = 1\cdot5$, is 946, with a standard deviation of 221 so that the difference of $-116 (=830-946)$ indicates a satisfactory fit ($P = \cdot70$).

40. However, it is of interest to study the graduation more closely, and it is to be noted that the very large deviation in the group centred around age $67\frac{1}{2}$ appears to cause distortion, being reflected in the deviations in the three groups on either side. To study the graduation in this region the values of q_x for the combined data (graduation (*d*)) were calculated and the individual deviations found over the range $55\frac{1}{2}$–$79\frac{1}{2}$. The results are given in Table 17. The individual figures do not suggest that any appreciable distortion is present, and the deviations are not unreasonable if allowance be made for the effect of duplicates. Col. (5) sets out the grouped deviations, and col. (6) an alternative grouping. It is interesting to note that the run of the grouped deviations in col. (6) is satisfactory. These results show how desirable it is to study the detailed figures before passing final judgment on a graduation. It is important, however, to distinguish the reasons here in relationship to the comments made earlier. For a summation graduation the individual deviations are related, and grouping reflects this relationship by a reduction in the average value. For a formula graduation no such relationship exists, and any peculiarity in the grouped deviations arises from the nature of the fit or from random fluctuations. In the present case the latter factor has given rise to the effect shown.

ENDOWMENT ASSURANCE WITH PROFITS

41. The results from three values of c are shown in Table 18, and the best fit is obtained by a value of c slightly lower than that for the combined data and slightly higher than that for the whole life data. The outstanding feature is the deviation in the group centred on age $42\frac{1}{2}$. It will also be noted that for a value of c between $(1\cdot05625)^2$ and $(1\cdot055)^2$ the D constant would vanish, suggesting that a Makeham curve would fit the data fairly well. Such a curve would not, of course, necessarily be suitable outside the range of the data. The D constant

Table 16. Graduation of A 1924–29 whole life data—durations 3 and over

Central age of group x	Actual deaths minus expected deaths										Additional deaths	Actual deaths plus additional deaths	$\sqrt{(Epq)}$ (e)
	Addition of sectional deviations (i)				Combined data (ii)				Combined data (adjusted) (iii)				
	(a)		(b)		(c)		(d)		(e)				
	+	−	+	−	+	−	+	−	+	−			
$17\frac{1}{2}$	.	1	.	1	.	2	.	1	.	1	.	.	1
$22\frac{1}{2}$	18	.	20	.	17	.	20	.	20	.	.	57	6
$27\frac{1}{2}$	.	10	.	5	.	11	.	5	.	5	.	83	9
$32\frac{1}{2}$	.	28	.	19	.	30	.	22	.	20	.	170	14
$37\frac{1}{2}$	.	42	.	33	.	44	.	34	.	32	.	368	20
$42\frac{1}{2}$	33	.	39	.	32	.	38	.	39	.	.	832	28
$47\frac{1}{2}$	2	.	.	4	5	.	.	1	.	1	.	1,515	39
$52\frac{1}{2}$	29	.	4	.	36	.	9	.	5	.	.	2,782	52
$57\frac{1}{2}$	100	.	57	.	109	.	63	.	50	.	.	4,881	69
$62\frac{1}{2}$	117	.	71	.	120	.	73	.	66	.	20	8,010	88
$67\frac{1}{2}$	.	310	.	322	.	315	.	327	.	321	55	11,894	108
$72\frac{1}{2}$	120	.	169	.	110	.	161	.	179	.	85	15,412	119
$77\frac{1}{2}$	.	56	16	.	.	60	13	.	15	.	70	14,334	113
$82\frac{1}{2}$	.	3	15	.	6	.	30	.	27	.	45	9,240	88
$87\frac{1}{2}$	.	14	.	33	.	3	.	22	.	24	20	3,846	55
$92\frac{1}{2}$	34	.	20	.	31	.	16	.	15	.	5	922	25
$97\frac{1}{2}$	.	3	.	6	.	7	.	10	.	10	.	92	8
Totals	453	467	411	423	466	472	423	422	416	414	300	74,438	842
Total*	920		834		938		845		830				

* Without regard to sign.

Formula: $q_x = (A + Bc^x)/(1 + Dc^x)$.

	Origin	c	A	B	D
Graduation (a)	$57\frac{1}{2}$	$(1{\cdot}05625)^2$	—	—	—
Graduation (b)	$57\frac{1}{2}$	$(1{\cdot}055)^2$	—	—	—
Graduation (c)	$57\frac{1}{2}$	$(1{\cdot}05625)^2$	·001995	·014706	·029681
Graduation (d)	$57\frac{1}{2}$	$(1{\cdot}055)^2$	·001786	·015050	·028127
Graduation (e)	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]

controls the shape of the curve at the older ages, and in this case the limited data in this region must give rise to a high uncertainty in the value of D.

42. To apply the *w*D test to this graduation it is necessary to find an adjustment for the effect of duplicates and Daw has shown that σ_r for the endowment data is 1·02 and may not be significantly different from unity. If k_r is taken as unity, the expected mean deviation is 394 with a standard deviation of 95, so that the actual difference of 134 (= 528 − 394) indicates a P of about ·08. This low value is rather surprising, as is the suggestion that duplicates are absent. The use of a higher value of k_r would produce a better result (e.g. if $k_r = 1{\cdot}2$, P = ·32), but the fact still remains that if σ_r be regarded as a measure of the departure from unit variance then the graduation must be regarded as borderline.

Table 17. Graduation of A 1924–29 whole life data—durations 3 and over

Age x (1)	Actual deaths (2)	Expected deaths (3)	Actual − expected (4)		Grouped deviations (5)		Re-grouped deviations (6)	
			+	−	+	−	+	−
55½	792	777	15	.	.	.	.	.
56½	859	866	.	7	.	.	.	.
57½	993	967	26	.	.	.	.	.
58½	1,070	1,070	.	.	.	.	.	.
59½	1,207	1,181	26	.	60	.	.	.
60½	1,246	1,302	.	56	.	.	.	11
61½	1,561	1,436	125	.	.	.	.	.
62½	1,533	1,584	.	51	.	.	.	.
63½	1,790	1,742	48	.	.	.	.	.
64½	1,888	1,904	.	16	50	.	.	.
65½	2,052	2,074	.	22	.	.	84	.
66½	2,194	2,249	.	55	.	.	.	.
67½	2,262	2,438	.	176	.	.	.	.
68½	2,622	2,618	4	.	.	.	.	.
69½	2,690	2,761	.	71	.	320	.	.
70½	3,087	2,876	211	.	.	.	.	87
71½	3,015	2,961	54	.	.	.	.	.
72½	3,020	3,030	.	10	.	.	.	.
73½	2,965	3,058	.	93	.	.	.	.
74½	3,065	3,079	.	14	148	.	.	.
75½	3,053	3,048	5	.	.	.	.	58
76½	2,906	2,977	.	71	.	.	.	.
77½	2,978	2,882	96	.	.	.	.	.
78½	2,705	2,695	10	.	.	.	.	.
79½	2,467	2,446	21	.	61	.	.	.

ENDOWMENT ASSURANCE WITHOUT PROFITS

43. Results for three values of *c* are given in Table 19 and show that the results for the two lower values are equally acceptable. As with the with profits data a Makeham curve could easily be fitted. In view of the difference in the values of σ_r shown between the with and without profits whole life data, values were calculated for the endowment assurance data with the results shown in Table 20 (the value for the combined data is taken from Daw's paper).

44. None of the values of σ_r is significantly different from unity, but as commented in §42 an absence of duplicates seems to be unlikely. The proportion of duplicates may be higher in the without profits data than in the with profits data, but the data are of course silent on this point. For the purpose of applying the mean-deviation test a value of $k_r = 1{\cdot}2$ for the without profits data has been used, but the limitations of this value must be kept in mind.

Table 18. Graduation of A 1924–29 endowment assurance with profits data—durations 3 and over

Central age of group x	Actual deaths minus expected deaths (a) +	(a) −	(b) +	(b) −	(c) +	(c) −	Actual deaths	$\sqrt{(Epq)}$ (b)	Graduated $q_x \times 10^5$ (b)
$17\frac{1}{2}$	4	.	5	.	5	.	39	6	208
$22\frac{1}{2}$	26	.	33	.	37	.	523	22	220
$27\frac{1}{2}$	.	21	.	11	.	6	1,264	36	240
$32\frac{1}{2}$	.	85	.	85	.	84	1,876	44	275
$37\frac{1}{2}$	.	32	.	49	.	57	2,912	54	335
$42\frac{1}{2}$	217	.	187	.	173	.	4,224	63	439
$47\frac{1}{2}$	.	68	.	90	.	100	5,210	73	618
$52\frac{1}{2}$	1	.	15	.	22	.	6,040	77	926
$57\frac{1}{2}$	.	32	11	.	33	.	5,684	75	1,454
$62\frac{1}{2}$	.	25	.	6	4	.	3,385	58	2,349
$67\frac{1}{2}$	.	11	.	24	.	31	1,162	34	3,853
$72\frac{1}{2}$	25	.	12	.	5	.	273	16	6,328
Totals	273	274	263	265	279	278	32,592	558	
Total*	547		528		557				

* Without regard to sign.

Formula: $q_x = (A + Bc^x)/(1 + Dc^x)$.

	Origin	c	A	B	D
Graduation (*a*)	$57\frac{1}{2}$	$(1{\cdot}05875)^2$	·001986	·013364	·048022
Graduation (*b*)	$57\frac{1}{2}$	$(1{\cdot}05625)^2$	·001919	·012835	·015068
Graduation (*c*)	$57\frac{1}{2}$	$(1{\cdot}055)^2$	·001883	·012569	−·001895

45. For the graduation with $c = (1{\cdot}05625)^2$ the expected mean deviation is 218, and the difference of 45 ($=263-218$) compares with a standard deviation of 52 so that on the *w*D test the graduation is satisfactory ($P = {\cdot}19$).

46. It is interesting to note that the without profits mortality is again below that of the with profits class and Table 21 illustrates this feature. On the particular weighting the without profits deaths are some $4\frac{1}{2}$% below the with profits, the deficiency being mainly at the middle and younger ages.

47. As with the whole life data a re-graduation of the endowment assurance data was made using an adjustment of deaths in accordance with the findings of *J.I.A.* LXVIII, 82–83. Table 22 sets out results of the following calculations:

(*a*) Addition of sectional results,

(*b*) Graduation of combined endowment assurance data,

(*c*) Graduation of combined endowment assurance data using an additional 210 deaths.

Table 19. Graduation of A 1924–29 endowment assurance without profits data—durations 3 and over

Central age of group x	Actual deaths minus expected deaths (a) +	(a) −	(b) +	(b) −	(c) +	(c) −	Actual deaths	$\sqrt{(Epq)}$ (b)	Graduated $q_x \times 10^5$ (b)
$17\frac{1}{2}$	.	.	.	.	.	.	3	2	182
$22\frac{1}{2}$	11	.	13	.	14	.	79	8	194
$27\frac{1}{2}$	.	26	.	23	.	22	158	13	214
$32\frac{1}{2}$	.	15	.	15	.	14	256	16	248
$37\frac{1}{2}$	10	.	8	.	7	.	420	20	308
$42\frac{1}{2}$	74	.	68	.	65	.	682	25	411
$47\frac{1}{2}$	.	27	.	34	.	37	875	30	588
$52\frac{1}{2}$	.	32	.	33	.	34	1,151	34	893
$57\frac{1}{2}$	.	14	.	5	.	2	1,282	36	1,417
$62\frac{1}{2}$	26	.	36	.	40	.	1,102	32	2,312
$67\frac{1}{2}$	.	22	.	21	.	21	484	22	3,830
$72\frac{1}{2}$	10	.	7	.	5	.	148	11	6,367
$77\frac{1}{2}$	2	.	.	.	.	1	22	4	10,512
Totals	133	136	132	131	131	131	6,662	253	
Total*	269		263		262				

* Without regard to sign.

Formula: $q_x = (A + Bc^x)/(1 + Dc^x)$.

	Origin	c	A	B	D
Graduation (*a*)	$57\frac{1}{2}$	$(1{\cdot}05875)^2$	·001747	·013002	·034470
Graduation (*b*)	$57\frac{1}{2}$	$(1{\cdot}05625)^2$	·001665	·012647	·010062
Graduation (*c*)	$57\frac{1}{2}$	$(1{\cdot}055)^2$	·001623	·012464	−·002733

Table 20. Standard deviation of $r_x\,(\sigma_r)$

Data	Number of values of r_x	Standard deviation of r_x (σ_r)
Endowment assurance with profits, durations 5 and over	55	·92 (±·13)
Endowment assurance without profits, durations 5 and over	55	1·26 (±·18)
Endowment assurance combined, durations 5 and over	55	1·02 (±·15)

Table 21. Comparison of 'With profits' and 'Without profits' graduations on endowment assurance without profits data—durations 3 and over

Age-group	Expected deaths: On rates by 'With profits' graduation (*b*)	Expected deaths: On rates by 'Without profits' graduation (*b*)	Actual deaths
Up to $44\frac{1}{2}$	1,686	1,547	1,598
$45\frac{1}{2}$–$64\frac{1}{2}$	4,588	4,446	4,410
$65\frac{1}{2}$ and over	670	668	654
Total	6,944	6,661	6,662

48. Comparison of (*a*) and (*b*) in Table 22 shows that the deviations from the graduation of the combined data present an almost identical pattern with that found by addition of the deviations from the separate sectional graduations. Comparison of (*b*) and (*c*) shows that the additional deaths make very little difference to the resulting graduation. Applying the *w*D test to the results of (*c*) we find an expected mean deviation of 445 ($k_r = 1{\cdot}0$) compared with the actual figure of ± 741. The standard deviation is approximately 107 so that the

Table 22. Graduation of A 1924–29 endowment assurance data—durations 3 and over

Central age of group x	Actual deaths minus expected deaths: Addition of sectional deviations (*a*) +	−	Combined data (*b*) +	−	Combined data (adjusted) (*c*) +	−	Additional deaths	Actual deaths plus additional deaths (*c*)	$\sqrt{(Epq)}$ (*c*)
$17\frac{1}{2}$	5	.	5	.	5	.	.	42	6
$22\frac{1}{2}$	46	.	46	.	46	.	.	602	24
$27\frac{1}{2}$	.	34	.	36	.	36	.	1,422	38
$32\frac{1}{2}$	.	100	.	99	.	98	.	2,132	47
$37\frac{1}{2}$	.	41	.	40	.	40	.	3,332	58
$42\frac{1}{2}$	255	.	257	.	259	.	5	4,911	68
$47\frac{1}{2}$	.	124	.	124	.	119	15	6,100	79
$52\frac{1}{2}$	.	18	.	19	.	25	20	7,211	85
$57\frac{1}{2}$	6	.	5	.	.	5	40	7,006	83
$62\frac{1}{2}$	30	.	28	.	44	.	75	4,562	66
$67\frac{1}{2}$	.	45	.	45	.	45	40	1,686	41
$72\frac{1}{2}$	19	.	20	.	18	.	15	436	20
$77\frac{1}{2}$	.	.	.	.	.	1	.	22	4
Totals	361	362	361	363	372	369	210	39,464	619
Total*	723		724		741				

* Without regard to sign.

	Origin	c	A	B	D
Graduation (*a*)	$57\frac{1}{2}$	$(1{\cdot}05625)^2$	—	—	—
Graduation (*b*)	$57\frac{1}{2}$	$(1{\cdot}05625)^2$	·001887	·012784	·014057
Graduation (*c*)	$57\frac{1}{2}$	$(1{\cdot}05625)^2$	·001886	·012767	·005592

fit judged by this test is not very satisfactory ($P < {\cdot}01$). In considering this result it must not be overlooked that the calculations of σ_r have been based on the data for durations 5 and over, whereas the data graduated are for durations 3 and over. During the period concerned the endowment assurance data had been increasing fairly rapidly, and it is possible that the value of σ_r would be higher for durations 3 and over, so that the above results may be inadequate to judge the graduation.

49. In order to investigate further this rather unexpected result the deviations at individual ages from $30\frac{1}{2}$ to $54\frac{1}{2}$ have been calculated and are given in Table 23. Clearly the fit is giving trouble in the range $40\frac{1}{2}$–$44\frac{1}{2}$, and there appears to be a wave at this point which the curve naturally fails to reproduce. The presence

of this wave, in data which rapidly increased over the period, suggests that the difficulty of obtaining a good fit to this, as well as to the with profits data, might well be due to neglected selection. This suggestion is supported by the fact that the graduated curve tends to produce higher values of q_x for ages below 40 and lower values thereafter as compared with the actual values, a feature consistent with a shorter average duration at the younger ages. An alternative grouping of the deviations is also given in the table and shows how fine detail may be obscured by grouping.

Table 23. Graduation of A 1924–29 endowment assurance data—durations 3 and over

Age x	Actual deaths	Expected deaths	Actual − expected		Grouped deviations		Re-grouped deviations	
			+	−	+	−	+	−
30½	393	376	17	.	.	.	.	.
31½	396	408	.	12	.	.	.	.
32½	392	447	.	55	.	.	.	.
33½	478	488	.	10	.	.	.	.
34½	492	531	.	39	.	99	.	.
35½	558	578	.	26	.	.	.	.
36½	616	625	.	9	.	.	.	.
37½	716	674	42	.	.	.	.	36
38½	684	731	.	47	.	.	.	.
39½	772	772	.	.	.	34	.	.
40½	860	819	41	.	.	.	.	.
41½	909	869	40	.	.	.	.	.
42½	989	933	56	.	.	.	90	.
43½	1,056	998	58	.	.	.	.	.
44½	1,078	1,048	30	.	225	.	.	.
45½	1,085	1,108	.	23	.	.	.	.
46½	1,165	1,180	.	15	.	.	.	.
47½	1,204	1,258	.	54	.	.	.	4
48½	1,255	1,329	.	74	.	.	.	.
49½	1,371	1,324	47	.	.	119	.	.
50½	1,432	1,336	96	.	.	.	.	.
51½	1,326	1,398	.	72	.	.	.	.
52½	1,531	1,472	59	.	.	.	56	.
53½	1,479	1,523	.	44	.	.	.	.
54½	1,370	1,449	.	79	.	40	.	.

ALL CLASSES COMBINED

50. To show the effect of aggregating the various classes Table 24 has been prepared setting out the sum of the deviations of the graduations of the sectional data and the deviations arising from the graduation of the combined data for the same two values of c. All the figures in this table are derived from the quinary group calculations. The value of c producing the lower sum of the deviations in the separate classes does not produce the best fit in the combined data, but the comparison is of value in showing the flexibility of the curves and also how the constants in the curves can adapt themselves to the effect of heterogeneity.

A1924–29 LIGHT AND HEAVY TABLES

51. The next group of experiments relates to graduations of the light and heavy data (durations 3 and over) given in *J.I.A.* LXVIII, 67 [1936]. The formula $q_x=(A+Bc^x)/(1+Dc^x)$ was used and the calculations limited to a single value of c, namely $(1\cdot05875)^2$, because the results were satisfactory for the purpose intended. The details are given in Table 25 and the application of the wD test to the results is given in Table 26.

Table 24. Graduation of A 1924–29 data—durations 3 and over

Central age of group x	Actual deaths minus expected deaths							
	Addition of sectional deviations				Combined data			
	(*a*)		(*b*)		(*c*)		(*d*)	
	+	−	+	−	+	−	+	−
17½	2	.	4	.	2	.	5	.
22½	49	.	64	.	61	.	99	.
27½	.	68	.	44	.	45	27	.
32½	.	144	.	128	.	117	.	55
37½	.	84	.	83	.	58	.	41
42½	311	.	288	.	311	.	244	.
47½	.	83	.	122	.	134	.	296
52½	47	.	11	.	.	65	.	268
57½	143	.	106	.	.	4	.	151
62½	214	.	147	.	168	.	157	.
67½	.	311	.	355	.	215	.	75
72½	61	.	139	.	204	.	466	.
77½	.	198	.	56	.	105	110	.
82½	.	55	.	3	.	51	.	41
87½	17	.	.	14	.	1	.	100
92½	61	.	34	.	44	.	.	11
97½	2	.	.	3	.	7	.	15
Totals	907	943	793	808	790	802	1,108	1,053
Total*	1,850		1,601		1,592		2,161	

* Without regard to sign.

Formula: $q_x=(A+Bc^x)/(1+Dc^x)$.

	Origin	c	A	B	D
Graduation (*a*)	57½	$(1\cdot05875)^2$	—	—	—
Graduation (*b*)	57½	$(1\cdot05625)^2$	—	—	—
Graduation (*c*)	57½	$(1\cdot05875)^2$	·001926	·013819	·030603
Graduation (*d*)	57½	$(1\cdot05625)^2$	·001736	·014124	·025414

52. In applying the wD test in the preceding paragraph, k_r was assumed to be unity. This was decided upon after calculating values of r_x and σ_r for these data. These calculations were made as part of a further analysis of the data with the aid of the r_x technique. The records of the exposed to risk and deaths at individual ages were obtained for this purpose from the C.M.I. Committee, and the values of r_x were calculated from the observed values of q_x. The individual values are given in Table 27 and the values of σ_r in Table 28. Although none of these values of σ_r is statistically different from unity it would not be unreasonable to infer that a higher value than 1 should be used for k_r; the figures also imply a proportion of duplicates rather lower than in the A 1924–29 data.

Table 25. Graduation of A 1924–29 light and heavy data—durations 3 and over

Central age of group x	Light: Actual deaths − expected deaths +	Light: Actual deaths − expected deaths −	Light: Actual deaths	Light: $\sqrt{(Epq)}$	Light: Graduated $q_x \times 10^5$	Heavy: Actual deaths − expected deaths +	Heavy: Actual deaths − expected deaths −	Heavy: Actual deaths	Heavy: $\sqrt{(Epq)}$	Heavy: Graduated $q_x \times 10^5$
17½	2	.	3	1	163	1	.	15	4	242
22½	.	2	10	3	172	18	.	57	6	255
27½	3	.	41	6	188	.	9	87	10	278
32½	.	3	72	9	217	.	13	141	12	318
37½	1	.	149	12	267	.	12	220	15	390
42½	5	.	251	16	357	1	.	325	18	516
47½	.	3	362	19	514	8	.	445	21	737
52½	4	.	467	21	791	3	.	549	23	1,122
57½	.	18	519	23	1,274	.	5	659	26	1,786
62½	.	18	567	24	2,108	11	.	752	27	2,909
67½	.	2	684	26	3,523	23	.	903	29	4,743
72½	42	.	866	28	5,853	.	21	1,019	31	7,587
77½	17	.	810	27	9,504	.	29	988	30	11,650
82½	.	9	586	23	14,807	19	.	685	24	16,826
87½	.	23	245	14	21,714	11	.	269	14	22,538
92½	.	1	73	7	29,539	.	4	79	8	27,922
97½	5	.	12	2	37,120	.	2	10	3	32,294
Totals	79	79	5,717	261		95	95	7,203	301	
Total*	158					190				

* Without regard to sign.

Formula: $q_x = (A + Bc^x)/(1 + Dc^x)$.

	Origin	c	A	B	D
Light	57½	$(1{\cdot}05875)^2$	·001508	·011492	·020615
Heavy	57½	$(1{\cdot}05875)^2$	·002248	·016336	·040268

53. Although in the preceding experiments the large standard deviation of σ_r made it difficult to obtain any very high precision in the application of the wD test, it was considered of value to analyse σ_r in further detail and to examine the values of this statistic for limited age-ranges in the various sections of the data. The results are given in Table 29, which includes, where necessary, figures taken from Daw's paper. Although very few of the values of σ_r in the table are statistically different from unity, it is very unlikely that the values when considered as a group would arise as a set of values from a population of unit mean. It is therefore appropriate to make a closer study of the figures.

54. It would be expected that the proportion of duplicates would vary with age, starting at a relatively low figure at the younger ages, increasing to a maximum at the late middle ages and then possibly falling off at the old ages as policies mature or are discontinued at ages where they cannot be replaced. The figures of Table 29 are consistent with these suggestions and also confirm Daw's suggestion that the proportion of duplicates might be expected to be lower among endowment than among whole life assurances. The figures for the various combinations of the data are also generally consistent when due allowance is made for the proportion of the data entering into the combined figures.

Table 26. wD test applied to graduations of A 1924–29 light and heavy data—durations 3 and over

	Light	Heavy
Actual mean deviation	158	190
Expected mean deviation	197	227
Difference	−39	−37
Approximate standard deviation	41	46
P	·83	·79

55. Consideration of the results of Table 29 shows that the assumption made earlier regarding the constancy of σ_r over the age range is not justified. Having regard to the differences shown, any adjustment to allow for the age variation in σ_r would have the effect of reducing the expected mean deviation and standard deviation, and this would have the effect of making the results of the tests less favourable than shown. Clearly the results show that, for the present, σ_r should be regarded only as an indication that some adjustment is called for in interpreting the results of statistical tests of graduations.

A 1924–29 SELECT DATA

56. The remaining experiments with the A 1924–29 data relate to the fitting of a formula to the select data. Perks put forward certain suggestions (*J.I.A.* LXIII, 33 [1931]) for fitting a formula to $\mu_{[x]+t}$, and for the purpose of the present experiments his formula as follows was used:

$$q_{[x]+t} = (A + Bc^{x+t})/(K_{[x]}\,\phi(t) + 1 + Dc^{x+t}).$$

Perks suggested the use of $\phi(t) = \omega^t$, but pointed out that trouble might arise in effecting a smooth junction with the ultimate table. As it was proposed to use a 3-year select period it was decided to try the form $\phi(t) = \omega^t - \omega^3$. This can be made to fit durations 0 and 1 reasonably well but means forcing duration 2.

Table 27. Values of r_x

Assured lives 1924–29

Age $x-\frac{1}{2}$	Whole life with profits, durations 5 and over	Whole life without profits, durations 5 and over	Endowment assurance with profits, durations 5 and over	Endowment assurance without profits, durations 5 and over	Light, durations 3 and over	Heavy, durations 3 and over	Light and heavy, durations 3 and over
15	.	.	.	.	− ·84	·91	·51
16	.	.	.	.	·54	−1·14	− ·87
17	.	.	.	.	− ·08	·31	·33
18	.	.	.	.	− ·07	− ·06	·00
19	.	.	.	.	− ·10	·95	·80
20	1·01	− ·44	− ·83	− ·02	·11	−1·14	−1·03
21	− ·17	1·06	·25	− ·67	·09	− ·12	− ·07
22	− ·16	− ·76	·21	1·21	− ·56	·64	·36
23	− ·26	− ·41	·25	−1·08	·07	− ·45	− ·35
24	·56	1·50	−1·14	·41	·60	·91	1·09
25	− ·44	−1·72	·92	− ·44	−1·01	−1·33	−1·69
26	·53	·16	− ·57	·82	1·29	1·27	1·78
27	− ·64	·89	1·12	− ·53	− ·88	− ·79	−1·14
28	·31	−1·53	−1·30	− ·12	·14	− ·01	·07
29	·32	1·28	·13	·12	·02	− ·01	− ·01
30	−1·04	− ·39	1·21	·73	·39	·71	·82
31	1·06	− ·27	−1·35	−1·42	− ·79	−1·72	−1·85
32	− ·54	·76	·76	1·00	·38	2·65	2·27
33	·08	−1·42	− ·09	− ·44	·01	−2·08	−1·59
34	1·03	1·97	·09	·25	1·20	·91	1·48
35	−1·95	−1·47	−1·34	·29	−2·54	− ·49	−2·04
36	1·45	·12	2·22	− ·44	1·73	− ·23	·96
37	− ·66	·00	− ·99	− ·76	·20	·57	·58
38	·70	1·33	− ·87	2·66	− ·26	·40	·15
39	−1·48	−2·59	1·23	−3·15	−1·20	−1·38	−1·82

Table 27 (*cont.*)

Age $x-\frac{1}{2}$	Whole life with profits, durations 5 and over	Whole life without profits, durations 5 and over	Endowment assurance with profits, durations 5 and over	Endowment assurance without profits, durations 5 and over	Light, durations 3 and over	Heavy, durations 3 and over	Light and heavy, durations 3 and over
40	1·66	2·43	−·88	2·07	1·40	1·09	1·72
41	−·36	−·89	·53	−1·52	−·02	·00	·01
42	−1·17	−·23	−·38	·95	−·83	−·51	−·93
43	1·73	−·11	·69	·40	·00	·19	·13
44	−1·60	·99	−·78	−·73	·96	−·42	·32
45	1·00	−1·79	·67	−·40	−1·07	1·19	·16
46	·75	1·99	·06	2·19	1·21	−·91	·15
47	−2·51	−1·14	−·18	−2·53	−1·15	·35	−·51
48	2·42	·10	−1·32	·31	·17	−·59	−·32
49	−1·27	·94	2·30	2·04	·93	·46	·97
50	·55	−1·80	−1·77	−2·61	−1·36	·12	−·84
51	−·25	1·65	·51	2·12	1·28	−·32	·65
52	·44	−1·25	·33	−1·40	−1·22	·48	−·48
53	−·74	·82	·67	1·61	1·44	·08	1·03
54	·80	·15	−1·47	−1·49	−·97	−·95	−1·35
55	−·81	−·87	·78	−·55	−·47	·16	−·20
56	·74	·67	−·52	1·66	1·04	·90	1·38
57	−1·31	·64	·51	−·19	·14	−·66	−·38
58	3·12	−1·30	·27	−·31	−·93	1·65	·55
59	−4·11	−·30	−·17	−·48	−·14	−2·87	−2·22
60	3·25	2·04	−1·04	·13	1·97	2·14	2·89
61	−2·13	−1·11	1·06	·17	−2·03	−·75	−1·86
62	1·37	−·76	·04	·45	·89	−·14	·48
63	−·77	1·44	−·31	−·74	−·11	·85	·54
64	·22	−2·00	·27	·80	−·65	−1·80	−1·77

Table 27 (*cont.*)

Age $x-\frac{1}{2}$	Whole life with profits, durations 5 and over	Whole life without profits, durations 5 and over	Endowment assurance with profits, durations 5 and over	Endowment assurance without profits, durations 5 and over	Light, durations 3 and over	Heavy, durations 3 and over	Light and heavy, durations 3 and over
65	1·40	2·16	− ·02	−1·43	·70	2·52	2·34
66	−2·59	− ·64	− ·34	2·32	− ·52	−1·74	−1·64
67	2·93	·06	·00	−2·10	1·80	·38	1·42
68	−3·21	− ·89	·71	1·21	−2·10	·14	−1·25
69	1·88	·57	− ·48	− ·34	·89	− ·29	·39
70	− ·49	·26	− ·84	− ·23	− ·87	− ·25	− ·78
71	1·01	− ·38	1·90	− ·08	·40	·65	·76
72	−1·02	·17	−1·17	·46	1·34	·54	1·29
73	− ·20	− ·04	·13	·09	− ·63	−1·33	−1·42
74	1·37	·88	·63	− ·35	−1·57	1·22	− ·14
75	−1·75	−1·70	.	.	2·54	− ·85	1·02
76	1·16	1·02	.	.	−2·16	− ·80	−1·99
77	− ·92	− ·22	.	.	·45	3·00	2·46
78	1·35	·56	.	.	1·05	−3·87	−2·15
79	−1·11	−1·32	.	.	−1·03	3·02	1·56
80	·67	1·78	.	.	1·20	−1·25	− ·09
81	− ·35	− ·99	.	.	− ·84	·81	− ·01
82	− ·76	− ·34	.	.	− ·91	−1·59	−1·77
83	1·82	·53	.	.	1·40	1·57	2·09
84	−1·88	− ·77	.	.	−1·03	− ·95	−1·39
85	1·22	1·07	.	.	1·57	·39	1·35
86	− ·56	− ·50	.	.	−1·34	− ·28	−1·10
87	·59	·34	.	.	·70	·48	·80
88	− ·30	− ·50	.	.	− ·77	− ·05	− ·56
89	−1·42	− ·40	.	.	− ·30	− ·77	− ·74

Since a similar forcing was used in the official graduation the comparison with this should be reasonable.

57. The graduation used for the ultimate table was that published in the volume of *Extracts and Discussions* (see §4 of this paper). The select data were collected in quinary groups of ages centred in the entry ages $7\frac{1}{2}$, $12\frac{1}{2}$, etc., and the figures were adjusted for grouping by the deduction of $\frac{1}{25}$th of the second central differences. The function

$$F_{[x]+t}=E_{[x]+t}(A+Bc^{x+t})-\theta_{[x]+t}(1+Dc^{x+t})$$

was then calculated for each group of entry ages and each duration 0, 1 and 2.

Table 28. Standard deviation of r_x

Data	Number of values of r_x	Standard deviation of r_x (σ_r)
Light	75	1·07 (±·13)
Heavy	75	1·23 (±·15)
Light and Heavy	75	1·24 (±·15)

Table 29. Values of σ_r

Age-group ...	$24\frac{1}{2}$–$49\frac{1}{2}$	$50\frac{1}{2}$–$74\frac{1}{2}$	$75\frac{1}{2}$–$89\frac{1}{2}$	$24\frac{1}{2}$–$89\frac{1}{2}$
Whole life with profits, 5 and over	1·25 (±·27)	1·83 (±·39)	1·16 (±·32)	1·49 (±·20)
Whole life without profits, 5 and over	1·30 (±·28)	1·11 (±·24)	·93 (±·25)	1·15 (±·15)
Whole life, 5 and over	1·45 (±·31)	1·84 (±·39)	1·34 (±·37)	1·59 (±·21)
Endowment assurance with profits, 5 and over	1·05 (±·22)	·82 (±·17)	—	·94 (±·14)
Endowment assurance without profits, 5 and over	1·36 (±·29)	1·21 (±·26)	—	1·28 (±·19)
Endowment assurance, 5 and over	1·09 (±·23)	1·00 (±·21)	—	1·05 (±·16)
All classes, 3 and over	1·27 (±·27)	1·63 (±·35)	1·35 (±·37)	1·44 (±·19)
Light, 3 and over	1·00 (±·21)	1·17 (±·25)	1·29 (±·35)	1·14 (±·15)
Heavy, 3 and over	1·04 (±·22)	1·20 (±·25)	1·71 (±·47)	1·28 (±·17)
Light and heavy, 3 and over	1·19 (±·25)	1·32 (±·28)	1·46 (±·40)	1·31 (±·17)

Note. In comparing these figures with others in the paper it will be noticed that differences arise from the use of slightly different ranges of ages.

To find a value for ω the values $F_{[x]+t}\div\theta_{[x]+t}$ were calculated for each group and duration and in the light of these values it was decided to use $\omega=\cdot 5$ as a first trial. Values of $\sum_{t=0}^{2}\theta_{[x]+t}(\omega^t-\omega^3)$ were then found for each group of entry ages. Study of the values of the ratio $\sum_{t=0}^{2}F_{[x]+t}\div\sum_{t=0}^{2}\theta_{[x]+t}(\omega-\omega^3)$ showed that $K_{[x]}$ would be sufficiently well represented by the form $(a+bx)$. The parameters a and b were then found from the whole of the data. From a comparison of the results of actual and expected deaths a further set of calculations was made using $\omega=\cdot 44$, but using only the data for durations 0 and 1 for finding a and b.

58. Table 30 sets out a summary comparison of the results of these two graduations and the official graduation. From a practical point of view the differences are of no significance. The lower value of ω produces perhaps a result closer to the official graduation. The detailed comparison of actual and expected deaths for this graduation is given in Table 31, from which it is clear that there is no serious departure from the data except for duration 2 where the graduation was forced. A comparison of the various values of $q_{[x]+t}$ is given in Table 32.

Table 30. Graduation of A 1924–29 select data

Duration	Actual deaths	Expected deaths	Actual deaths − expected deaths (+)	Actual deaths − expected deaths (−)
	Formula graduation (i), $\omega = \cdot 50$			
0	1,355	1,354	33	32
1	1,664	1,619	94	49
2	1,717	1,768	25	76
	Formula graduation (ii), $\omega = \cdot 44$			
0	1,355	1,362	29	36
1	1,664	1,656	76	68
2	1,717	1,792	19	94
	Official graduation			
0	1,354	1,359	29	34
1	1,667	1,664	61	58
2	1,717	1,780	35	98

Note. The slight difference between the actual deaths in the formula and official graduations arises from the use of King's formula.

59. For completeness the wD test may be applied to the data and the three durations may be tested together. After eliminating those groups for which the expected deaths are below 10 we are left with 33 groups. Using $k_r = 1$ and three constraints the expected mean deviation is found to be 314 as compared with an actual figure of 298, the difference of −16 comparing with a standard deviation of 42. On this test the graduation is satisfactory, but this serves to illustrate the danger of such summary tests, because the graduation of duration 2 has been deliberately forced away from the data.

ASSURED LIVES DATA 1924–38

60. The final experiment in connexion with data from the Continuous Mortality Investigation consists of two graduations of the combined data for durations 3 and over for the 15 years 1924–38 by means of the formula

$$q_x = (\mathrm{A} + \mathrm{B}c^x)/(1 + \mathrm{D}c^x).$$

The two values of c used were $(1\cdot 0575)^2$ and $(1\cdot 05875)^2$, and the results are given in Table 33. Of the two fittings, that with the smaller value of c provides the better result, but consideration of the deviations suggests that a slight improvement might be obtained with an intermediate value.

61. In applying the wD test to these results it is necessary to fix a value of k_r. The evidence from Solomon's paper (*J.I.A.* LXXIV, 101 [1948]) is that k_r for the

Table 31. Graduation of A 1924–29 select data

Central age of group x	Duration 0 Actual − expected +	Duration 0 Actual − expected −	Duration 0 Actual deaths	Duration 1 Actual − expected +	Duration 1 Actual − expected −	Duration 1 Actual deaths	Duration 2 Actual − expected +	Duration 2 Actual − expected −	Duration 2 Actual deaths
$12\frac{1}{2}$	.	4	2	.	.	6	.	.	6
$17\frac{1}{2}$	7	.	60	14	.	71	.	6	51
$22\frac{1}{2}$	.	16	122	19	.	166	.	1	147
$27\frac{1}{2}$	4	.	147	.	25	129	.	17	136
$32\frac{1}{2}$	3	.	129	12	.	158	.	13	141
$37\frac{1}{2}$	.	10	120	.	8	150	8	.	179
$42\frac{1}{2}$	6	.	145	.	18	158	.	27	172
$47\frac{1}{2}$	.	2	144	.	4	187	.	6	215
$52\frac{1}{2}$	4	.	192	14	.	258	.	14	256
$57\frac{1}{2}$	5	.	164	9	.	213	7	.	231
$62\frac{1}{2}$	.	.	93	.	9	111	.	3	130
$67\frac{1}{2}$	.	3	33	8	.	52	.	7	39
$72\frac{1}{2}$	.	1	3	.	3	4	4	.	12
$77\frac{1}{2}$	.	.	1	.	1	1	.	.	2
Totals	29	36	1,355	76	68	1,664	19	94	1717
Total*	65			144			113		

* Without regard to sign.

Formula: $q_{[x]+t}=(\mathrm{A}+\mathrm{B}c^{x+t})/\{(a+bx)(\omega^t-\omega^3)+1+\mathrm{D}c^{x+t}\}$.

Origin for c	57·5	$\omega=\cdot 44$
c	$(1\cdot 05875)^2$	$a=\cdot 28753$ } origin $x=7\cdot 5$
A	·001926	$b=\cdot 01104$ } origin $x=7\cdot 5$
B	·013819	
D	·030603	

Table 32. Graduation of A 1924–29 select data

Age x	Duration 0 $q_{[x]}\times 10^5$ Formula (i)	Duration 0 Formula (ii)	Duration 0 Official	Duration 1 $q_{[x]+1}\times 10^5$ Formula (i)	Duration 1 Formula (ii)	Duration 1 Official	Duration 2 $q_{[x]+2}\times 10^5$ Formula (i)	Duration 2 Formula (ii)	Duration 2 Official
$12\frac{1}{2}$	148	153	124	175	180	161	193	195	188
$17\frac{1}{2}$	148	152	156	178	183	196	199	202	220
$22\frac{1}{2}$	151	154	160	186	190	198	211	214	220
$27\frac{1}{2}$	160	162	160	201	206	200	233	236	224
$32\frac{1}{2}$	177	179	169	229	234	220	271	275	260
$37\frac{1}{2}$	211	212	211	280	287	288	340	345	348
$42\frac{1}{2}$	271	272	280	371	379	388	462	468	472
$47\frac{1}{2}$	378	378	378	530	541	542	674	684	675
$52\frac{1}{2}$	563	561	558	806	823	821	1,045	1,060	1,040
$57\frac{1}{2}$	880	875	876	1,281	1,307	1,331	1,682	1,708	1,713
$62\frac{1}{2}$	1,421	1,409	1,400	2,089	2,129	2,145	2,761	2,803	2,765
$67\frac{1}{2}$	2,326	2,301	2,230	3,430	3,492	3,516	4,536	4,603	4,593
$72\frac{1}{2}$	3,807	3,760	3,579	5,582	5,676	5,668	7,323	7,426	7,354
$77\frac{1}{2}$	6,144	6,057	5,395	8,853	8,986	8,631	11,400	11,546	11,180

Graduation (i): $\omega=\cdot 50$. (ii): $\omega=\cdot 44$.

5 years will be higher than for the A 1924–29 data, and accordingly a value of ·6 has been adopted. The entire test is set out below:

Number of groups n	17
$\Sigma \mathrm{E}pq$	287,838
$\Sigma \sqrt{(\mathrm{E}pq)}$	1,955
$\sqrt{(\Sigma \mathrm{E}pq)}$	537
Number of constants	4

Expected mean deviation	$\cdot 8 \times 1\cdot 08 \times 1\cdot 6 \times \sqrt{\tfrac{13}{17}} \times 1955 =$	2365
Actual deviation	$=$	2771
Actual – Expected	$=$	+406
Standard deviation	$\cdot 6 \times 1\cdot 08 \times 1\cdot 6 \times \sqrt{\tfrac{13}{17}} \times 537 =$	487
P	$=$	·20

62. Within the range considered none of the differences between actual and xpected deaths is significant, and it thus seems that the mean-deviation test ndicates a satisfactory graduation. It should be noted, however, that the leviations in the age-groups centred on 22½ and 32½ are substantial, even llowing for a substantial adjustment for the incidence of duplicates, and show hat there is a wave in the values of q_x which the formula fails to reproduce. Apart from this feature the graduation would be satisfactory for the preparation f a practical table, should such an instrument be required, based on these data.

Table 33. Graduation of assured lives data 1924–38 durations 3 and over

Central age of group x	Actual deaths – expected deaths (a) +	(a) −	(b) +	(b) −	Actual deaths	$\sqrt{(\mathrm{E}pq)}$ (a)	Graduated $q_x \times 10^5$ (b)
17½	.	3	.	9	141	12	174
22½	268	.	216	.	1,748	38	186
27½	110	.	.	2	4,532	66	206
32½	.	421	.	523	6,847	85	241
37½	.	190	.	213	10,022	101	304
42½	237	.	336	.	14,719	120	412
47½	.	187	44	.	20,845	145	600
52½	.	45	259	.	28,452	168	926
57½	11	.	234	.	34,231	184	1,486
62½	252	.	274	.	34,408	183	2,436
67½	.	100	.	281	34,414	182	4,012
72½	277	.	.	65	38,436	189	6,535
77½	149	.	.	163	36,997	182	10,351
82½	.	163	.	215	24,872	149	15,652
87½	.	252	.	135	10,974	93	22,209
92½	96	.	169	.	2,804	44	29,246
97½	10	.	22	.	294	14	35,738
Totals	1,410	1,361	1,554	1,606	304,736	1,955	
Total*	2,771		3,160				

* Without regard to sign.

Formula: $q_x = (\mathrm{A} + \mathrm{B}c^x)/(1 + \mathrm{D}c^x)$.

	Origin	c	A	B	D
Graduation (*a*)	57½	$(1\cdot 0575)^2$	·001583	·013676	·026901
Graduation (*b*)	57½	$(1\cdot 05875)^2$	·001673	·013520	·029225

E.L. No. 10 (Males)

63. The final experiment to be described consists of attempts to fit the E.L. No. 10 (Males) data with a curve of the form $\mu_x = (A + Bc^x)/(Kc^{-x} + 1 + Dc^x)$. Perks (*J.I.A.* LXIII, 18–22 [1931]) has described experiments in connexion with the E.C.R.D. (Males) and the E.L. No. 9 (Males) data. In the present experiments the populations and deaths were grouped in quinary age-groups, the 'actual' deaths being taken as one-third of the total deaths for the three years 1930–32. The central ordinate of each group was found by deducting $\frac{1}{24}$th of the second central difference. The constants of the formula were then found from the populations and deaths for three different values of c. The least unsatisfactory result ($c = 1{\cdot}0525^2$) is set out in Table 34, and the graduated values of μ_x for all three fittings are given in Table 35.

Table 34. Graduation of E.L. No. 10 (Males) $c = (1{\cdot}0525)^2$

Central age of group x	Actual deaths	Expected deaths	Actual − expected +	Actual − expected −	$\sqrt{\text{Expected}}/\sqrt{3}$
$17\frac{1}{2}$	4,383	4,541	.	158	39
$22\frac{1}{2}$	5,640	5,088	552	.	41
$27\frac{1}{2}$	5,373	5,556	.	183	43
$32\frac{1}{2}$	5,181	5,697	.	516	44
$37\frac{1}{2}$	6,077	6,254	.	177	46
$42\frac{1}{2}$	7,871	7,791	80	.	51
$47\frac{1}{2}$	11,035	10,369	666	.	59
$52\frac{1}{2}$	14,574	14,186	388	.	69
$57\frac{1}{2}$	18,899	19,023	.	124	80
$62\frac{1}{2}$	22,685	23,260	.	575	88
$67\frac{1}{2}$	27,025	27,347	.	322	95
$72\frac{1}{2}$	28,225	28,205	20	.	97
$77\frac{1}{2}$	24,271	23,954	317	.	89
$82\frac{1}{2}$	15,004	14,879	125	.	70
$87\frac{1}{2}$	6,493	6,519	.	26	47
$92\frac{1}{2}$	1,571	1,598	.	27	23
$97\frac{1}{2}$	205	221	.	16	9
Totals	204,512	204,488	2,148	2,124	990
Total*			4,272		

* Without regard to sign.

64. The wave of the crude values of μ_x about age 30 is not reproduced, and in this respect the graduation may be criticized as an instrument for reflecting population mortality, but apart from this the comparison of graduated and ungraduated values of μ_x is not unsatisfactory and monetary values would be unlikely to be significantly affected.

65. In considering the application of the wD test to this graduation it may be noted that Daw calculated a value of $\sigma_r = 1{\cdot}39$. This value cannot, of course arise from duplicates. Daw remarks on the effect of the assumptions made

egarding the denominator in the calculation of q_x and on the influence of age rrors. Noting that one-third of the actual deaths have been utilized in the raduation, an approximate figure for the expected mean deviation will be $3 \times 1{\cdot}08 \times 1{\cdot}4 \times \sqrt{(12/17)} \times 990 = 1006$, so that by this test the graduation is far rom satisfactory.

Table 35. Graduation of E.L. No. 10 (Males)

$$\mu_x = (A + Bc^x)/(Kc^{-x} + 1 + Dc^x)$$

Age x	Values of μ_x by graduation (i)	(ii)	(iii)	$\text{Colog}_e\, p_{x-\frac{1}{2}}$ official graduation
17½	·00272	·00265	·00259	·00259
22½	·00298	·00299	·00300	·00330
27½	·00334	·00340	·00346	·00328
32½	·00391	·00398	·00405	·00362
37½	·00483	·00489	·00494	·00475
42½	·00632	·00634	·00635	·00641
47½	·00877	·00873	·00869	·00929
52½	·01279	·01268	·01258	·03104
57½	·01936	·01920	·01905	·01908
62½	·03004	·02991	·02977	·02917
67½	·04730	·04731	·04732	·04676
72½	·07483	·07516	·07549	·07522
77½	·11784	·11857	·11930	·12019
82½	·18297	·18358	·18421	·18545
87½	·27705	·27533	·27374	·27546
92½	·40407	·39451	·38579	·38563
97½	·56077	·53369	·51013	·53991
Origin	57·5	57·5	57·5	
c	$(1{\cdot}05125)^2$	$(1{\cdot}0525)^2$	$(1{\cdot}05375)^2$	
A	·002567	·002851	·003117	
B	·017045	·016693	·016342	
D	·012133	·014685	·016956	
K	·001072	·003004	·004515	

CONCLUSION

66. This paper consists of a collection of experiments together with a limited amount of development of graduation tests. It leaves much unsaid and, indeed, raises more questions than it answers. It forms part of the background of the author's training and is written as a piece of pure research. Nevertheless, there are lessons to be learnt from such work, and this forms a justification for its presentation. From a purely practical angle the experiments show clearly that in graduating mortality data by mathematical formulae very little need be lost by working throughout on grouped data, and thus the arithmetic of graduation can be reduced to relatively small dimensions. Some attempt has been made to minimize any loss by investigation of the effect of grouping on some of the usual graduation tests. From another practical angle the paper endeavours to put the χ^2 test into a wider perspective. In many instances this test has tended to become the central test, but it is important to recognize that it is only one form of summary test, based on conditions which frequently may not be appropriate.

67. From another aspect the experiments have further shown the capabilities of the Perks (or logistic) curves in representing assured lives' data. Mortality tables are required for various practical purposes, and it is clear that the logistic curve is a very powerful instrument for devising hypothetical tables to cover widely varying forms of mortality with a very considerable economy in description—Perks's comments of twenty years ago are still very pertinent in this regard.

68. Finally, I should like to express my thanks to Mr A. C. Edwards, F.I.A., who, whilst in no way responsible for the arithmetic or algebra of this paper, has read through the manuscript and put forward a number of valuable criticisms.

ABSTRACT OF THE DISCUSSION

Mr M. T. L. Bizley, in opening the discussion, said that the paper represented the esult of many years of deep analytical thought and sheer hard work—a combination haracteristic of the best labours of an actuary.

In dealing with the grouping of deviations, the author led his readers gently through n ingenious but straightforward argument, and then, with a fine sense of the dramatic, onfronted them with Table 2. On a first reading Table 2 came as a distinct shock; or whereas the figures for the logistic curve graduations, in columns (4) and (5), ccorded reasonably well with what might be expected, those for the summation raduations were utterly confounding. The author showed that a summation graduation ught not to be judged against a formula graduation by a simple comparison of the rouped deviations; such a comparison would probably lead to the conclusion that the ummation graduation was the better—a judgment that might well be wrong. Possibly, hat feature had misled actuaries in the past and, if so, there had been every excuse for hem; but it was a chastening thought that hereafter there would be no excuse at all.

Recently, M. D. W. Elphinstone (*T.F.A.* xx, pt. 1) had shown that summation formulae night have the effect of producing sinusoidal waves of determinable period in the raduated values, and in the discussion on that paper Mr Beard had pointed out that the pencer formula might produce a wave of length 10. The importance of that in the resent context was obvious. There was a huge field for research, as yet only partly xplored; in fact, it was not too much to say that the whole subject of the testing of ummation graduations was in its infancy.

Although, as the author pointed out, the figures in columns (4) and (5) of Table 2 or the logistic curve graduations were in reasonable agreement with what might be xpected, yet each of the figures in column (4) was in excess of the corresponding figure n column (5). From a rough calculation of the theoretical standard deviation, the ifference between the two columns did not appear to be significant; but the difference vas one to which further inquiries might well be directed, particularly because it was n the same direction for both types of graduation.

There was a curious feature in Table 4. If the groupings were numbered downwards s 1, 2, 3, 4, 5 and the O^{M} values were rearranged in ascending order of magnitude, the ermutation 1, 3, 2, 5, 4 was obtained. The same process applied to the O^{JF} values gave he identical permutation. Why? Moreover, for the A 1924–29 values the same rocedure resulted in the permutation 2, 3, 1, 4, 5, the reverse of a cyclic permutation of , 3, 2, 5, 4. Obviously, it would require only very small alterations in the figures of Table 4 to upset those results, which might therefore be fortuitous; but he could not elp wondering whether the author had chosen the *mot juste* in his comment in aragraph 11, where he said that, for the O^{M} and O^{JF} graduations, the deviations '*happen* o be the lowest of the five possible groupings'.

There had long been argument whether allowance for constraints ought or ought ot to be made when testing a graduation. The problem had been ably expounded on everal occasions, but the critical question remained: if a graduation satisfied a particular est at a predetermined significance level, ignoring constraints, but failed to pass the ame test when allowing for constraints, was the graduation acceptable or unacceptable n that evidence? The paper under discussion threw the problem into sharp relief ecause, with the 5-year age-groups suggested by the author, the number of constraints ore a much higher proportion to the number of groups, and their effect on the sampling istribution of the statistic on which the test was based became correspondingly more mportant.

By way of a rough measure of the effect, he had recalculated without allowance for onstraints the expected value and the variance of the weighted mean deviation for a few f the graduations in the paper, and had compared the results with the corresponding nes calculated from the author's figures, which allowed for constraints. The following able shows the expected weighted mean deviation, expressed as a proportion of the ppropriate standard deviation.

Data	Author's figure	Revised figure
Whole life with profits	− ·1	− ·7
Whole life without profits	3·1	2·1
Endowment assurances with profits	1·4	·4
Endowment assurances without profits	·9	·0

He thought the effect was clearly one which could not be ignored.

If it were decided that allowance for constraints had to be made, the D and wD test depended upon the sufficiency of the approximate formulae given in paragraphs 18 an 24. In that connexion, he was a little sorry to see that the author followed the usu practice, customary in statistical as well as in actuarial literature, of referring to distribution and then introducing the notion of constraints as an adjustment, becaus the practice tended to obscure the fact that the constraints were fundamental. In th case of unweighted χ^2, it was known that the sum of squares in a sample of size subject to l linear constraints, from a Normal population had precisely the sam distribution as had the sum of squares in an unconstrained sample of size $n-l$. He woul emphasize the word 'precisely'; there was no question of any approximation. Th coefficients in the constraints did not affect the result; thus, it did not matter wheth a constraint was $x+y+\ldots$ etc. $=0$ or $x+2y+\ldots$ etc. $=0$; the distribution was the sam

Statisticians were so used to the idea of expressing the χ^2 distribution in terms the number of linear constraints that he thought that they were apt to argue a littl hastily by analogy and to infer that the sampling distribution of other statistics such the mean deviation might similarly be expressed in terms of that number. In h opinion, that was not so. The distribution of the mean deviation, weighted or unweighte and in particular both its expected value and its variance, depended upon the coefficien which defined the constraints imposed, or, more strictly, upon the various ratios betwee those coefficients.

He had obtained exact expressions for the expected value and standard deviation the weighted and unweighted mean deviations for the very special case where the numb of constraints was one less than the number of ages or groups. In neither instance di the results accord exactly with the formulae in the paper, but those formulae we contained as special cases of the exact results, and in general they would, he though give good approximations. For the unweighted mean deviation, Cauchy's inequali showed that the author's formulae, where not exact, would always overstate somewh both expected value and variance. In certain special cases the author's formulae woul be exact; for example, if a polynomial of degree $(n-2)$ was fitted by the method moments, the approximate formulae would give the true result if, but only if, the valu of $\sqrt{(Epq)}$ in the successive groups were proportional to the n binomial coefficients bas on $(n-1)$. For the weighted mean deviation, the binomial result which he had ju quoted still held, but it was no longer unique, and the author's formulae would n necessarily overstate the true values. In general, while the author's formulae would usual give good approximations in the cases likely to be met in practice, they would not gi theoretically exact results except for certain special sets of data.

The χ^2 test had been described by Mr Redington in a very happy phrase as a 'mon syllabic oracle'. When the mean-deviation test, or any other single summary test, w used for an uncertain theoretical distribution it was necessary to remember that the te implied the asking of questions of an 'oracle' which was not only monosyllabic, like χ but which on occasion would be unwittingly mendacious, because it would sometim answer Yes when it meant No, and vice versa.

If he was not in error in his contention that the coefficients in the constraints impos underlay the whole sampling distribution problem for statistics other than t unweighted sum of squares based on samples from a normal population, that mig explain why the search for simple, exact formulae for the first few moments had alwa proved unavailing.

As a general comment on the paper, it might be remarked that the author had worked hroughout with 5-year age-groups. The number 5 had long been a close friend of the ctuary, but it had no magic in it, and for some purposes 3-year groups or 9-year groups night be preferable. The paper demonstrated forcibly the enormous amount of work vhich had been done, was being done and doubtless would be done by actuaries to free heir minds from any misgivings about fidelity to the original data. When, however, it vas remembered that mortality was changing, that the rates at best were used as estinates of the future and were combined with other unknown factors such as the rate of nterest, actuaries might well pause and ask themselves whether their energies were)eing dissipated in descending a difficult path round a hillside when a quick jump from he top would bring them more roughly, but just as effectively, to their goal. Of course, he metaphor sounded its own warning: if the jump was too rough, there would be lisaster. Yet they were perhaps tending to emulate the legendary engineer who calculated he safety load of his bridge to ·01 oz. and then, in accordance with the usual practice, ook 5 tons as a safety margin. He hastened to add that nothing that he had said, or vould wish to say, was intended in any way to belittle the importance of the author's vork; indeed, it was his firm belief that all research of the kind in question would be horoughly well worth doing, even if it had no practical application whatever.

In his Conclusion the author stated, almost apologetically, that his paper raised more juestions than it answered; yet it was a truism, albeit a paradoxical one, that in scientific)rogress the asking of questions was often more important than the answering of them. Newton might have achieved fame when he explained what made the apple fall off the ree; but the essential advance in human thought occurred earlier, when it was first isked why an apple should fall down to the ground instead of up to the sky.

He would like to end his remarks with two lines from Addison:

> 'Tis not in mortals to command success,
> But we'll do more, Sempronius; we'll deserve it.

They might or might not feel that the author had achieved success in everything that ie had set out to do in the paper, but it would be agreed that he had deserved it.

Mr F. M. Redington expressed appreciation of the author's immense industry and his searching but always practical thought. In many ways, he said, the paper consolidated the work of the last twenty years. For example, when he, the speaker, and Michaelson first introduced the r_x technique in 1938, there had been considerable doubt about what had actually been discovered. Work had since been done by Daw and others, the mists had gradually dispersed, and, in paragraph 22 of the paper, there was a compact and, he thought, generally acceptable statement of the function of the r_x test.

He had, however, one criticism: he believed that the r_x test should be applied before graduation and not as an examination of the graduation. The r_x test was strictly part of the data: what it did was to tell the investigator what was a reasonable 'bogey' for the particular course, as was recognized by the author in his perspicacious remark in paragraph 42:

> The use of a higher value of k_r would produce a better result..., but the fact still remains that if σ_r be regarded as a measure of the departure from unit variance then the graduation must be regarded as borderline.

The thought behind that paragraph was that the value of σ_r was a fact in itself, irrespective of the reasons giving rise to that value. For example, it was possible for σ_r to be considerably greater or less than 1, purely by chance. If σ_r was high, what it said to the graduator was that the hazards of that particular course were severe and that a low score was impossible. The thought might have been pursued more thoroughly throughout the paper; indeed, there was something to be said for putting k_r in the basic formulae in paragraph 24 as σ_r without more ado.

It might be that Daw, in calculating the standard deviations of σ_r in his paper in 1945, had led his readers up an interesting but immediately irrelevant avenue. But, as a separate subject, the analysis of σ_r might be illuminating for the information it could furnish on errors in construction, heterogeneity, age mis-statements, number of duplicates and so on.

According to estimates he had made, σ_r was an approximation to the average number of policies per life. That was only a chance coincidence, but it worked for most values. Table 29 was a very useful summary of the position, but unfortunately all the values of σ_r in the Table were suspect because of the size of the standard deviations. Thus the standard deviation of σ_r became an important subject for investigation.

One way out of the problem created by those large standard deviations was to break down the data. For example, the 1924–29 experience could be broken down into separate years, and each year gave a set of r_x's thus multiplying the data, and reducing the standard deviation. If the r_x test were carried out on each year's mortality experience, there would soon be a useful body of information throwing much light on the vexed question of duplicates, on age mis-statements and on errors in the mortality experience.

Summary tests—or 'monosyllabic oracles'—were useful, and he would endorse the author's practical instinct in choosing wD as the most serviceable of them. He would however echo the opener's remarks about constraints. Tests which made allowance for constraints did answer a particular question, but he was not sure whether that was, in fact, the most fruitful question. The most fruitful question was, he thought, answered by a test without constraints, though it would take too long to elaborate his argument. He agreed also with the opener that the technique of dealing with constraints had been all too easily imported from the χ^2 test.

The author's industry had thrown a great deal of light on the value of Perks's formula $q_x=(A+Bc^x)/(1+Dc^x)$. The volume of separate experiments which the author had made showed that the formula was not as faithful to the data as it should be in theory. With a perfect graduating instrument, the resulting P (viz. probability of larger deviations) would average somewhere about ·5. It was clear that for Perks's formula, although one or two values were higher than ·5, the majority were less, and some were very considerably less. Some of the reasons for the lack of faithfulness, however, were not disadvantageous to the formula. The data contained waves that were spurious—for example, those resulting from the amalgamation of whole life and endowment business. The data also contained waves which it was inexpedient to retain—for example, the waves between ages 20 and 35. Lack of faithfulness to those waves was not discreditable. It was probable that summation formulae of graduation would prove to be too faithful to the data, and to be too faithful was as bad as being unfaithful. Moreover, the use of a mathematical formula as against a summation method had great practical advantages.

Mr H. A. R. Barnett, in a communication which was read to the meeting, said he was interested in the endeavour to put the mean-deviation test on to a sound basis, and also in the method of fitting curves to mortality data by a type of minimum-deviation method. He did not agree with the description of the method in paragraph 27 as a 'minimum weighted standardized deviation method' by analogy with a minimum-χ^2 method; the analogy should rather be with Cramér and Wold's method, which produced a fit giving the minimum χ^2 consistent with the first and second moments being zero. In the same way, the author's method, for a curve with four parameters, approximately produced the minimum deviation consistent with the first three moments being zero.

The author had not shown the considerations which led him to the choice of particular values of c; his reasons would have been both interesting and instructive. Having chosen c the author had not thought it appropriate to find those values of the other constants which would minimize the total deviation irrespective of sign. Some experiments on a minimum-deviation Makeham-fit showed that, for any value of c, the corresponding values of A and B giving the minimum produced a curve causing considerable distortion of the data at the youngest ages ($22\frac{1}{2}$–$26\frac{1}{2}$) where the number of exposed to risk, though still considerable, was lower than at the later ages. Thus standardizing had its uses, even though it might be carried too far with a minimum-χ^2 fit. The ideal fit of a particular curve to a given set of mortality data had yet to be determined, but a further study of Hardy's lectures, of the discussion at the Institute on his own paper, and of the paper under discussion, led him to the conclusion that it might be the fit giving the minimum deviation consistent with the first and second (and, in the case of a Perks curve, third) *standardized* moments being equal to zero. The author's lesson on the subject of grouping would facilitate his own experiments with that type of fit.

The sections on the graduation of the endowment assurance data were particularly interesting. There was a similarity to his own graduation of the whole A 1924–29 data, both in the values of the constants and in the special features of the graduation. He agreed strongly with paragraph 49, which suggested that the trouble caused by certain of the data was due to neglected selection; part of the trouble might also have been the effects of the 1914–18 war, which were still being felt in certain age-groups in 1924–29.

He welcomed the extensive use of the r_x technique, particularly after the critical remarks which had been made about that technique by Seal during the discussion of Daw's paper. The conclusions of paragraph 55, however, confirmed that r_x did not give the complete answer to the presence of duplicates in the data; there was no complete answer, and the paper showed once more what a pity it was that duplicates were not excluded from the continuous mortality investigation.

A further valuable lesson of the paper was the illustration in paragraph 59 of the danger of judging a graduation by a summary test alone.

The range of application of Perks's formulae was instructive. It seemed likely, from first principles, that the mortality rate was made up of two different types of decrement, one arising from those causes of death which hit the whole population with equal impact irrespective of age, and one from those which had an increasing effect as age increased. That did not mean that every basic mortality curve was a Makeham, but it did suggest that if a more complicated formula was needed there would usually be come similarity to Makeham. He wished to see an analysis of national data excluding certain causes of death, for example deaths from occupational diseases which struck at the young and middle ages, and also childbirth deaths which would be expected to cause some irregularity in the curve of mortality of females. It might well be that by excluding those, or by treating them as a different type of decrement, the remaining deaths would more easily fall into the two categories, one approximately constant and one regularly increasing with age; if curves could be fitted to national mortality rates on those lines, some interesting lessons might be learned from watching the trends in those curves.

There might be those who would speak disparagingly of the paper, those who preferred the simplicity of graphs, but he questioned whether they would be really satisfied with a standard table prepared on graphical lines. If it were agreed that a more refined technique was desirable for standard tables, then the number of the profession employing such tables was evidence that the research contained in the paper was well worth while.

Mr M. D. W. Elphinstone did not think that the comments which the author had made on the paper which he, the speaker, had presented to the Faculty were quite correct, and he had dealt with them in the appropriate place; the waves which turned up with summation formulae could sometimes be traced directly to the formula but were sometimes inherent in the data. He suspected that the wave of length 10, which the author and the opener had both mentioned, was in fact inherent in the A 1924–29 data, and good graduation tests should disclose the existence of that wave by whatever means the table was graduated. He thought that there was an unresolved problem which had been disclosed by the author's work, and by the opener's remarks on it.

One of the opener's remarks had reminded him of two successive entries in the visitors' book at the Wasdale Head Hotel. The entry 'Went up the Pillar Rock in three hours; found the rocks very easy', was immediately followed by 'Fell down the Pillar Rock in three seconds; found the rocks very hard'. If actuaries were too pushing in their graduations, the rocks would be hard.

The author was probably wise to leave summation methods at a fairly early stage in the paper. The more he thought about summation and formula graduations, the more he was impressed by the fundamental difference in logic between them. They gave answers to two distinctly different problems. The author, after merely pointing out that those differences existed, and illustrating them by his figures of the grouped data in different types of graduation, had been wise to confine his paper to formula graduations, because otherwise the scope would have been too wide.

For various reasons he had been interested recently in the logic underlying tests of graduations such as those discussed in paragraphs 14–16. The more he thought about it, the more he was convinced that it was essential to begin with a clear idea of the purpose of graduation. In the discussion on Barnett's paper, Tetley had commented on the necessity of considering possible alternatives when testing a graduation. That was sound statistical theory and fairly well accepted, but neither the present author nor any other actuary had as yet touched on it in the recent discussions of the tests of graduations. It was sound statistical theory provided that the statistical hypothesis to be tested, whether compound or simple, was straightforward and clear. When a mass of mortality data was to be graduated by a formula, it was necessary to start with a statistical hypothesis which was known from first principles to be wrong. For that reason, there was a logical difficulty in the testing of mortality tables, because it was not possible to set out any clear alternative hypothesis to the one which was being tested. For example, he did not think that useful results would be obtained, in considering a Makeham graduation, by postulating Perks's curves as an alternative hypothesis to Makeham. The truth was that in testing a Makeham graduation the question asked was not whether there was an underlying law of mortality which could be expressed by Makeham's formula, but whether by some lucky chance a Makeham curve could be found which was sufficiently close to the experience for practical life office work. For that purpose—and there he agreed with Mr Redington—he thought that it was probably advisable not to deduct constraints but to take the number of observations as being the number of degrees of freedom.

Coming back to the question of constraints and distributions of the test criteria, he was glad that the opener had mentioned the distribution of the mean deviations, since he himself was not altogether happy about it. In fact, in reading the paper he had been in some doubt about the numerical value to be given to the number of the author's constraints; it did not seem to be stated anywhere in the paper. If a mortality experience were to be used for a detailed and careful scientific investigation, with perhaps a biological rather than an actuarial purpose in view, a different set of tests would be appropriate and full allowance for constraints should be made. The difficulties would not then be all over. The opener had referred to the difficulties of computing the distribution of the mean deviation. For a minimum-χ^2 fit, the constraints could be treated as being linear, but nobody had gone into the question of what would happen for other methods of fitting, nor would the answers hold with a weighted-χ^2 test. The data could never be quite in the desired form, nor was there always the time or the ability to carry out the extensive mathematics, including arithmetic, which might be required.

The author had shown the way to squeeze the utmost possible out of what at the best were very unsatisfactory data. It was to be hoped that the data would be improved in the future, and it was by work such as the author's that progress was made. It might not lead to much more satisfactory mortality tables for use in life office work, but there would be a clearer understanding of mortality and selection and, in practical work, therefore, of underwriting.

Mr R. H. Daw described some calculations which he had made of the effect of grouping on the χ^2 test. He had used for his investigation the graduation, which had been tested by Seal, of the O^{JF} data by Kenchington's 27-term summation formula (*J.I.A.* LXXI, 5 [1941]). As the author explained, there were five different sets of groupings of the individual ages into groups of 5 and for each of those five sets he had calculated χ^2 from the grouped figures. In each set there were ten groups but he had made a deduction from that figure to allow for the constraints imposed by the graduating process. Seal had arrived at a deduction of 6 degrees of freedom for Kenchington's 27-term formula but that applied to a graduation of individual ages and was obviously inappropriate when testing grouped data. In conformity with the author's suggestion in paragraph 20 of the paper, a deduction of approximately one-fifth of Seal's figure had been made and each value of χ^2 taken as corresponding with 9 degrees of freedom. In that way he obtained five values of P, one for each set of groupings, where P was the probability of chance factors alone producing a larger value of χ^2 than that observed. Two of the values of P were ·34 and ·43, indicating a very satisfactory graduation; the other three values were ·97, ·97 and ·99, pointing to under-graduation. Seal had calculated χ^2 using individual ages, and the resulting value of P was about ·5, indicating a satisfactory graduation. Thus, three out of the five tests of the grouped figures supported Seal's criticism of the particular graduation, that grouping gave an impression of under-graduating which was not borne out by examination of the individual ages.

The figures which he had quoted and the different conclusions arrived at with different groupings supported his opinion that it was undesirable to apply over-all graduation tests, such as the χ^2 test, to grouped figures when the graduation had been made by operating on individual ages. It seemed to him that tests should not be applied to fewer groups than were used in making the graduation.

In Table 4 the author showed the effect of different age-groupings on the total of the deviations without regard to sign. The author had made only one graduation of each mortality table, basing it on one particular set of quinary age-groupings, so that all but the first line of Table 4 represented the results of the rather artificial procedure of graduating with one grouping and testing with another grouping of the individual ages. It would have been more to the point, if separate graduations had been made for each grouping, so that the five lines of Table 4 would each represent the results of both graduating and testing on the same grouping.

The r_x test had been mentioned already in the discussion and the suggestion had been made that the author's k_r should always be taken as equal to σ_r. In that connexion it should be remembered that one of the assumptions underlying the r_x test was that by taking third differences of the ungraduated rates of mortality all systematic variations with age would be eliminated and the resulting figures would be composed solely of random variations. If that was not so, the numerator of r_x contained both systematic and random variations and σ_r would be greater than k_r, the true allowance for duplicates. There did not appear to be any evidence that the effect was appreciable, but the possibility should be borne in mind and was probably one of the reasons which led the author to distinguish between k_r and σ_r in paragraph 22.

Mr A. C. Edwards drew attention to two subjects, so far not discussed. One of them was of a negative character and arose out of Tables 28 and 29. Some of the values of σ_r for combined data were greater than those for the separate parts, for example, the Light and Heavy data, where only a small increase in duplicates would be expected on combining the data. Thus it seemed worth while to follow up a remark by Perks in the discussion on Daw's paper, where he referred to the importance of the question of whether there were marked differences from age to age in the composition of the data. In the age-range 24½–74½, where the combined values of σ were greater, the speaker did not find any serious disturbance to the run of the proportions, and it seemed that the explanation of the increase in the values of σ_r, unless it was by chance, had to be sought elsewhere.

The other subject was the remarkable cyclical effect disclosed by the deviations for the graduation of E.L. No. 10 (Males) set out in Table 34. The author had referred to a wave in the crude μ_x's round age 30, but the effect might arise from a hump in the crude rates round age 47. If there were such a hump, then because the graduation was by a mathematical curve the graduated rates would be lifted up on each side, so producing the two lots of negative deviations which appeared in Table 34 on each side of ages 42½–52½. It might be that the distortion extended further up the table and caused the further crossing over of graduated and crude rates at the older ages.

The lives aged 35–55 in 1931 were 20–40 in 1916, an age-group which lost a substantial number of its fittest members during 1914–18, and the survivors who had been on active service might well, by 1931, have been suffering from delayed and cumulative after-effects. That group of lives was aged 30–50 in 1926, and it was to be noted that in several of his graduations the author found the age-group 40–44 causing outstanding positive deviations (see Tables 18, 19, 22, 23, 24).

Hocking, in his series of notes in the *Journal* on Mortality in England and Wales, gave ratios of mortality in the current year to that in 1930–32. There was a peak in the male ratios which was absent from the female ratios, and the top of the peak moved along in a most persistent manner. It was in the age-group 45–49 in 1933, and in the group 50–54 for the next 5 years, and 55–59 for the 5 years after that, and was in the age-group 60–64 for 1944–48. Presumably it had since moved into the age-group 65–69.

It appeared that from that one factor alone some marked changes in male mortality should be expected. In (say) ten years' time the rates of mortality in the 60's might be found to have improved, or might appear to have done so, and the rates in the 70's might even have got worse. It would be possible to pursue that question into the assured lives data and into the annuitants' data, and it might be that for annuitants in particular there were some useful inferences to be drawn.

A comparison of the deviations in Table 34 with the actual deaths, and of Hocking's male ratios with his female ratios, suggested that at the top of the hump the excess mortality might be about 10–15 %.

Mr W. Perks remarked that perhaps he might be allowed to take a grandfatherly interest in the paper. The author had produced a great deal of evidence concerning the usefulness of a formula which he, the speaker, had put before the Institute in 1931. The author had also followed up the idea of the mean-deviation test in the form in which the speaker had put it forward in the discussion on Daw's paper. The author had also made extensive use of the r_x technique of Redington and Michaelson. None of those three techniques received much encouragement when they were first put forward, and it was pleasant to think that there were men like the author who were not to be discouraged.

It seemed to him that the main message of the paper was 'Back to Hardy'. Hardy believed in fitting mathematical curves. He fitted Makeham whenever he could, and he stretched a few points in favour of Makeham and was probably quite right to do so; but, when Makeham would not fit, he found a curve that would. The vogue of the summation method and other makeshift methods in the past twenty-five years had been a retrograde movement. The author showed that the results of the summation method were not all that they had been cracked up to be. Incidentally, H. Ammeter had submitted a paper to the XIIIth International Congress, Scheveningen, 1951 which analysed the χ^2 test by a summation graduation and which was very relevant to the present discussion. The constraints-effects of a summation graduation had also been discussed on Seal's and Daw's papers. The effects were so paradoxical that those who favoured summation graduations might well wonder whether their mathematical model was really suitable for testing goodness of fit. It should be remembered that a test of a graduation was a test of goodness of fit, not a test of a hypothesis in quite the modern statistical sense. It was desirable to distinguish between real constraints on the data (such as $q+p=1$ or Σ frequencies = constant) and constraints introduced by the fitting process.

Hardy realized the importance of the incidence of the weight of the data and deliberately used the method of moments for fitting. In the past twenty-five years the movement away from the method of moments for fitting had come from the development of the idea of 'efficiency' in statistical estimation. There would be general agreement that where there were 'sufficient statistics' the method of moments was probably not the best to use. Fisher had developed the method of maximum likelihood, and there had also been developments on the lines of minimum χ^2 for statistical estimation. He doubted, and he thought that the present author had given good reasons for doubting, whether those principles were entirely suitable in the different field of graduation of mortality data. At any rate it was known that for large quantities of data those methods did not produce very different results. The method of moments was reasonably satisfactory in theory and it had very great advantages in practical application.

In a third field, in testing, Hardy would never have been content with a single-figure measure. He used elementary methods of testing and detailed testing. Barnett's paper harked back, as the paper under discussion had done, to the sound practice of G. F. Hardy. The author had shown reasons for and against no fewer than 8 different single-figure tests, the mean-deviation and χ^2 tests, weighted and unweighted, with and without allowance for constraints. It was interesting that those tests usually resulted in the same judgment of the acceptability of the graduation. He referred there to the same practical judgment, and by 'practical judgment' he meant whether the graduation was acceptable, not whether P was equal to $\cdot 5$ or $\cdot 75$ or $\cdot 2$ or $\cdot 1$. All those figures would make the graduation acceptable. To choose, therefore, between those various tests was of little practical significance, but from a theoretical point of view it had to be borne in mind that the choice might be between an approximate answer to the right question and an exact answer to the wrong question.

The Tchebychef and Gauss inequalities showed that the formulation of an exact probability distribution was often quite unnecessary for practical work; in fact, the use of a mildly incorrect distribution would rarely lead the actuary to a wrong judgment in practice. It was exemplified in the paper by Seal, who had calculated the effect on χ^2 of 2 constraints in 9 different sets of data; he then took the mean of those 9 effects and tested that mean with the t-distribution. In fact, on his hypothesis, the effect of 2 constraints on χ^2 was a χ^2 with 2 degrees of freedom, and the sum of 9 of them was a χ^2 with 18 degrees of freedom, so that use had been made of a t-distribution to test a χ^2 variate; yet the judgment was sound and came out correctly whichever distribution was used.

That seemed to illustrate how unimportant, in a great deal of practical work, the much-vaunted exact distributions of modern statistics really were; but he believed that at the theoretical level such researches as, for example, those of the opener into the effects of constraints in the mean deviation test were important for the light which they threw on practical work and the way in which they gave the actuary a better understanding of his figures.

The paper did not discuss the wider question whether tables based on graduated data should continue to be used or whether hypothetical tables should be adopted. His own view was quite definitely that the time had come when hypothetical tables should be used, but the tables should be mathematically smooth. However, if actuaries persisted in being conservative and wished to construct tables from graduated data he would say that they ought to do the job properly and graduate the tables by mathematical curves. But whether standard tables were based on graduated data or on hypothetical curves, graduation would still remain as an essential instrument for research.

In the paper there was once again strong evidence of the way in which the presence of an unknown amount of duplication in the data of the Continuous Mortality Investigation seriously impeded research work. He assumed that the methods of collecting data for the Continuous Mortality Investigation would have to go on, despite the arguments in favour of a change, but he had a simple suggestion to make to deal with the duplicates problem. He suggested that the offices should be asked to write a card not for each policy in force but for each policy included in the experience on becoming a claim by death. The duplicates distribution could then be obtained from the claims. If the name of the life assured, the date of birth, the date of death and class of assurance, etc., were put on the card, it would be possible to bring together all claims in all offices on the same life (the opportunity could be taken to include on the cards the cause of death as well). If the mortality did not depend on the number of policies a life had, it did not matter whether the duplicates distribution was obtained from the exposed to risk or from the claims; but if the mortality did depend on the number of policies on the life it was theoretically better to get the duplicates distribution from the claims rather than from the exposed to risk. In that way it would be possible in the various sub-groups of the data to get a measure of the disturbance to the variance by the presence of duplicates.

He was satisfied that the work involved in asking the offices to write a card for each claim would be relatively small, but it could be made a good deal smaller by asking them to write a card only for claims on deaths arising on certain dates in the year selected at random. Each office could be asked to write cards for, say, 10 % of its claims, and all the information necessary for a close estimate of the disturbance due to duplicates would then be available.

He would mention various suggestions which had been made recently in America and in Europe, and which were coming up in the papers for the Congress, with regard to the use of sampling methods to estimate the age-distribution and the duration-distribution of the exposed to risk. Provided the sampling was confined to the exposed to risk, and full information was obtained about the deaths, it seemed to him that there were great possibilities for important progress in mortality investigations on those lines.

Mr Redington had referred to σ_r as being a measure of the kind of result to be aimed at in making a graduation, a sort of 'bogey' for graduation. It had to be borne in mind that σ_r^2 was the sum of squares of standardized deviations, whereas if wD was to be accepted as the testing measure—and Mr Redington appeared to favour wD amongst the eight tests considered by the author—an appropriately weighted σ_r^2 should be taken to indicate the sort of result to be aimed at.

Mr A. W. Joseph, in closing the discussion, said that when members came to a wood as extensive as that to which the author had brought them, they could not overlook some of the trees. As the opener had pointed out, at first sight the results of Table 2 were surprising. The ratio of the sum of the grouped deviations to the sum of the individual deviations was much what was to be expected for the formula graduations but less than would be expected for the summation graduations. The explanation was given in paragraph 13, but Table 6 showed that there was something peculiar about the O^{JF} table. Mr Daw had also discovered some peculiarities in that table by means of χ^2 tests of different groupings. He, the speaker, had prepared a table for the Kenchington graduation similar to Table 4, where the author had considered five different groupings for the deviations by formula graduations. The figures were set out in the following table:

Grouping	Σ grouped deviation	$\dfrac{\Sigma \text{ grouped deviation}}{\Sigma \text{ individual deviation}}$
$x-(x+4)$	35·5	·186
$(x+1)-(x+5)$	45·9	·240
$(x+2)-(x+6)$	48·8	·256
$(x+3)-(x+7)$	93·3	·489
$(x+4)-(x+8)$	80·4	·422
Average	60·8	·319

and the average (Σ grouped deviation/Σ individual deviation) of ·319 agreed well with the calculated value of ·304.

Was there any advantage in using the mean-deviation test instead of the χ^2 test, or the weighted-mean-deviation test instead of the weighted-χ^2 test? As Mr Perks had pointed out, Table 8 showed that there was little numerical difference between any of the tests. The choice seemed to be largely one of individual preference and hinged on a leaning either to simple arithmetic or to simple algebra. The deviations had to be summed positively. The algebraist squared them so that they automatically became positive, and their sum was a function which could be dealt with quite easily. With plenty of help from text-books, the speaker had been able to follow the relatively simple theory which the author outlined in paragraph 20, and had even been able to see why the author had performed the apparently aimless operation of multiplying the sum of the deviations by the constant factor $\Sigma\, \mathrm{E}pq/\Sigma\, (\mathrm{E}pq)^2$. It was in order that the variance should be exactly twice the mean, a characteristic of all χ^2 distributions.

The arithmetician asked why it was necessary to square the values merely to make them positive; why not add them regardless of sign? Of course, theory was more difficult, but was not that what the algebraist was there for? The author had in fact found the theory difficult, and had been reduced to proceeding by analogy, but, even if the opener had not in part checked the author's results, most actuaries would trust the author's intuitions.

Several speakers—and quite rightly, because it was most important—had referred to the question of constraints. A table had been graduated by a curve which was subject to, say, 4 constraints. Should the number of degrees of freedom in the use of the χ^2 test be reduced by 4? The answer, he felt sure, was No! The effect of constraints was to reduce deviations; that was admitted. A curve with 4 parameters would be closer to the data than one with 3, but was that any justification for altering the test whether the data could reasonably have occurred from the graduated probabilities? Were not methods and tests of graduation unrelated subjects? Was it not a circular argument to insist that because a particular method of graduation was likely to reduce deviations, the test of the graduation should be made more severe?

The explanation in paragraph 36 of the large deviation in the age-group 80–84 was not convincing. At the high ages paid-up policies were said to have been included in the exposed to risk without the corresponding deaths having been included. A deficiency of deaths rather than an excess would be expected, therefore, at those old ages. The graduated curve ran at a lower level at the old ages because of a deficiency of deaths in the age-group 65–79 rather than in the age-group 80–84.

Having barked his shins against some of the trees, he came to survey the wood itself. The author had shown, by an industry which was astounding—for he had graduated no fewer than thirty-four tables and tested more than forty—that mortality tables based on assurance data might be graduated by the Perks family of curves, and that the process of fitting might be applied to data combined in quinquennial groups. It was not so clear that it was safe to test the graduation by reference to grouped deviations rather than to individual deviations, as several speakers had indicated. But the technique of grouping was a great convenience and the results of the graduations by formulae justified Perks's criticism in the discussion on the A 1924–29 table of the official (Spencer) summation graduation. Useful though a summation graduation might be to graduate a set of data, the general shape of which was unknown, actuaries had sufficient experience of assurance mortality to know the underlying trend. A summation graduation followed the ungraduated rates too closely, and a formula graduation was to be preferred. Although that might perhaps be a criticism of the particular summation formula used, and some other type, e.g. the Whittaker form of summation graduation might give better results, the author had made out his case, and the Institute was indebted to him for a most interesting paper.

The President (Mr F. A. A. Menzler, C.B.E.), in proposing a vote of thanks to the author, said that there had been a very high level of discussion, which the author would certainly regard as some compensation and reward for all his work. In referring to the discussion, the President was sure that the meeting would wish him to congratulate the opener on a very able and effective speech.

The paper illustrated what Sir William Elderton used to say so often, namely the need in all research work of the kind in question for much hard numerical work. In no other way was real mastery of the figures to be achieved. Members should not be so vain, however, as to think that the author had done all this work for *them*; he would have done it even had he no prospect of getting the paper accepted, because he was a research-minded person.

If the present meeting was not so large as meetings on less recondite aspects of the work of the actuary, the author could solace himself with the thought that all the really great actuaries, beginning with Sprague and going on through Hardy and Lidstone, had been attracted to his subject. If when he got home he would refer to the definition in the *Concise Oxford Dictionary* of 'caviare to the general', he would derive much satisfaction.

Mr R. E. Beard, in reply, expressed his thanks for the very pleasant remarks that had been made on the paper.

He thought that probably one of the important lessons which had come out of the discussion arose from the comments by Mr Edwards on Table 34. Mr Edwards pointed out that part at least of the trouble with the E.L. No. 10 (Males) graduation was due to waves in the data. The comparison of actual with expected deaths drew attention to the waves, and the waves seemed to have a definite connexion with the effect of the 1914–18 war. Some actuaries were more concerned than others with the future trends in annuitants' mortality, and the feature now emphasized in the population data had probably not been given the importance which it deserved. The wave had reached the 'annuitant' ages, and it would soon pass through them with a resultant subsequent fall in annuitants' mortality. Mr Edwards suggested that the order of magnitude of the wave was 10–15 % arising merely from a selective process acting on the particular group of lives. The attention drawn to that feature alone would have been well worth some of the work put into the paper.

There had been a great deal of discussion on the question of constraints. He was not unaware of the dangers. In paragraph 20 he pointed out that there were two points of view which could be taken and he did not there say which he accepted but adopted a particular technique for the purpose in view. There would be discussion on that question for a long time to come.

Mr Barnett had asked how the value of c was obtained. The first value of c was always the hardest to discover, but selection of an appropriate value became increasingly easy as more graduations were made. The starting point was nearly always a consideration of the ratios of successive values of q_x obtained from the adjusted group-values of deaths and exposed to risk.

Mr Joseph's calculations for the O^{JF} table showed that the value of 35·5 for the sum of the grouped deviations was an abnormally low value, and it was clear that the apparent breakdown of the theoretical formulae in paragraph 13 was due to that fluctuation, the average ratio of ·319 being in excellent agreement with the 'expected' ratio of ·304.

He was glad to see that Mr Perks had taken up the Hardy banner again, because he must confess that he did a lot of work on the paper without thinking about Hardy; subsequently he was led to go back and read a lot of what Hardy had said, and he felt that Hardy knew a great deal more about graduation than he himself would ever know, in spite of the advantage of thirty years of technique.

With regard to Mr Joseph's remarks on the advantages of using the χ^2 test, the point was that the mean-deviation test came out quite easily from the arithmetical operations already performed; to work out χ^2 it was necessary to do more arithmetic. Personally he was lazy at heart and stopped the arithmetic as soon as he could.

Mr R. E. Beard has written by way of supplementing his remarks at the meeting:

The feature of Table 2 mentioned by Mr Bizley, that the grouped deviations for the formula graduations were in all three cases less than their expectation, might arise from the method of compiling mortality data. There was a considerable overlap because any particular policy could contribute to a succession of groups of exposed to risk and thus there would be correlation between successive groups. In terms of mortality rates it would seem that the correlations would not be very strong but they might be sufficient to produce the effect shown. It would be interesting to graduate the experience for a single calendar year because this feature would then be absent, but none of the graduations he had made were suitable and he could not undertake the work at the moment. He also regretted that he could offer no suggestions on the cyclic permutation arising in Table 4, but would be surprised if the feature were not a chance one.

He was glad that Mr Bizley had attempted a more adequate investigation of the theory underlying the mean-deviation test even though the results obtained were only on the fringe of the problem. It seemed to him that as the number of degrees of freedom was increased beyond the single value used by Mr Bizley the empiric formulae of the paper would become closer to the theoretical distributions, but the problem was clearly exceedingly difficult. He had experimented with groupings other than quinary but he had not made enough experiments from which to form a reliable judgment. He felt that quinary groupings formed a very reasonable compromise and their use had the advantage that the arithmetic could be systematized, a valuable feature in experimental work.

He did not think that there was any fundamental difference between Mr Redington nd himself on the r_x test. He agreed with him that the r_x test could be regarded as setting the bogey', but in the paper he had taken the next step and devised tests which night be regarded as giving the net score for the course. A value of σ_r significantly lifferent from unity indicated that the data were unsatisfactory and that the correct ourse was to analyse the data for the source of the disturbance with a view to eliminating t. If the disturbance were not eliminated, then any statistical tool applied to the data vould be seriously blunted, as the paper had shown. If the data were satisfactory, then r_r would be taken as unity. If the view were taken that the value of σ_r after elimination f duplicates was not significantly different from unity, then the use of k_r would be egarded as equivalent to testing the graduations with a σ_r of unity after eliminating luplicates.

The lesson to be learned from this was suitably summarized by Mr Perks in his emarks and also by Mr Barnett in his written contribution; what was wanted was an nalysis of the incidence of duplicates. However much the data were subdivided and owever many values of r_x were calculated, the information so obtained was always ndirect and any sharp statistical tests would be impossible.

He thanked Mr Barnett for his valuable comments on minimum methods and ccepted the criticism of the rather loose description in the paper.

Mr Elphinstone's comments on the question whether the waves which turned up with ummation formulae were features of the data or arose from the use of the formulae drew ttention to an interesting problem that probably merited more investigation. He did ot regard the problem as having been solved one way or the other, but on the evidence vailable he was still inclined to the view that the waves arose from the summation ormulae. It seemed that the difference of opinion between himself and Mr Elphinstone vas largely one of degree and one that could only be resolved by further investigation.

Regarding Mr Daw's comment that the procedure underlying Table 4 was rather rtificial, it would appear unlikely that a recalculation of the constants of the formula for ach of the five groupings could make any material difference to the results obtained nless the fitting process was inefficient. In the present context the results of the egraduation described in paragraph 5 supported the view that the fitting process used vas efficient and therefore little would be gained by the extensive additional calculation equired.

Finally, he would like to thank all those who contributed to the discussion for the kind way in which the paper had been received and for their useful criticism and comments. He had found the investigations enjoyable and profitable and hoped that others would enefit from them.