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SOME RELATIONSHIPS BETWEEN EXTRA PREMIUMS

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It sometimes happens that a whole-life or endowment assurance is effected by means of a single or limited premium on the life of a person subject to an extra risk. The annual extra premium for the more usual policy, with premiums payable throughout, is probably tabulated or more or less readily assessable. It may even be fixed according to a tariff agreed between offices. But, in the case of the single- or limited-premium policy, an extra premium has to be specially calculated to allow for the different method of premium payment.

As a first requirement, the special extra should be related to the annual extra for the more usual type of policy, for otherwise inconsistent quotations will emerge.

Secondly, the calculation of the special extra should be as simple as possible and should involve tabulated functions only.

I. WHOLE-LIFE AND ENDOWMENT ASSURANCES BY SINGLE PREMIUM

On p. 56 of his text-book *The Treatment of Extra Risks*, C. F. Wood gives a formula for the calculation of the extra premium required for a whole-life or endowment assurance policy effected by single premium, corresponding to the annual extra required when premiums are payable throughout. He concludes that if loading is ignored the single extra premium corresponding to f per annum can be ascertained by calculating an annual extra premium at f per unit on the amount at risk $(1-\text{A})$ and commuting by the special (mortality) annuity a′.

In the usual notation his formula is

$$\text{F}=f(1-\text{A})\,\text{a}',$$

where F is the extra premium for a single-premium policy, A is calculated on the normal mortality basis, and a′ is calculated on the special mortality basis to which the life assured is subject.

The use of this formula in practice requires the evaluation of a′ which is not tabulated, so that assumptions or approximations have to be made.

A simpler formula than this can, however, be devised. Ignoring loadings, and using accented symbols to represent functions calculated on the special mortality basis, we have

$$f=\text{P}'-\text{P},$$

i.e.

$$f=\frac{d\text{A}'}{1-\text{A}'}-\frac{d\text{A}}{1-\text{A}},$$

or

$$f=\frac{d\text{A}'-d\text{A}}{(1-\text{A}')(1-\text{A})},$$

whence

$$\text{F}=\text{A}'-\text{A}=\frac{f}{d}(1-\text{A}')(1-\text{A}).$$

F may therefore be expressed in the following ways:

$$F = f(1 - A)a', \tag{1}$$

$$F = f(1 - A')a. \tag{2}$$

(1) is Wood's formula. (2) does not appear, at first sight, to be very helpful, but by substituting $(A + F)$ for A' and rearranging we obtain

$$F = f(1 - A)\frac{a}{1 + fa}, \tag{3}$$

and it is to be observed that a' in Wood's formula has been replaced by $\frac{a}{1+fa}$, an identity that can be obtained directly from the relation

$$f = P' - P.$$

Formula (3) is an explicit expression for F in terms of f and functions on the normal mortality basis. The calculation is simple and no approximations, as suggested by Wood, are necessary.

For practical purposes, loadings can be introduced by giving f the value of the office extra premium instead of the net extra premium. But it should be borne in mind that this imparts a percentage loading to the single extra which is a little less than that included in the annual extra. The denominator $(1 + fa)$ in (3) is slightly increased when the office extra is used for f.

As a corollary to (3) we may notice that F and f are in the same numerical relationship whatever assumption may be made concerning the incidence of the risk. We may compute F from f on any convenient assumption even though that assumption is unrelated to the actual extra risk. For example, if we enter a table of P inversely with the net function $(P + f)$ to find the 'rated-up age', then the same rated-up age* gives A' and hence F. Where A and P are tabulated adjacently according to age, A' can be extracted (being opposite P') without actually calculating the rated-up age. It must, however, be admitted that an inconvenient interpolation is required if $(P + f)$ does not correspond to an exact age, and this frequently happens in practice.

The method is accurate whatever the nature of the 'special' mortality; the rating in age is simply a mathematical artifice and does not restrict the solution to a particular form of extra mortality. The method seems so obvious that it must surely have been used before, but I have not seen it in print.

A percentage loading must finally be applied to the net single extra, F, but if office premium tables are used in place of net premium tables, the office single premium emerges at the last step and the loading problem is automatically solved.

II. EXTRA PREMIUM PAYABLE FOR A LIMITED NUMBER OF YEARS

Some risks, such as aviation, are covered by an annual extra premium payable for a maximum of n years, although the normal annual premium is payable throughout the duration of the policy. If, however, a policy is effected by single premium, we require the equivalent single extra.

* This result follows also from the relation $A = P/(P + d)$, which is true whatever the mortality basis.

Let $f_x^{(n)}$ = the prescribed annual extra premium limited to n years under a whole-life policy, with premiums payable throughout, effected at age x,

$f_{x\overline{n}|}$ = the annual extra premium which would be payable throughout, under an endowment assurance of term n years, to cover the same risk.

Then
$$f_x = f_x^{(n)} \frac{a'_{x\overline{n}|}}{a'_x},$$

and making this substitution in (1) we find

$$F_x = \left(f_x^{(n)} \frac{a'_{x\overline{n}|}}{a'_x}\right)(1 - A_x)\, a'_x$$

$$= f_x^{(n)} (1 - A_x) \frac{a_{x\overline{n}|}}{1 + f_{x\overline{n}|}\, a_{x\overline{n}|}}. \tag{4}$$

This formula is not very convenient as it stands because $f_{x\overline{n}|}$ would probably have to be specially calculated for the purpose. We may have available the corresponding extra premium restricted to t years' payments ($f_x^{(t)}{}_{\overline{n}|}$), where $t<n$.

Now
$$f_{x\overline{n}|}\, a'_{x\overline{n}|} = f_x^{(t)}{}_{\overline{n}|}\, a'_{x\overline{t}|},$$

and it is not difficult to show* that

$$f_{x\overline{n}|}\, a_{x\overline{n}|} > f_x^{(t)}{}_{\overline{n}|}\, a_{x\overline{t}|},$$

so that the alternative divisor $(1 + f_x^{(t)}{}_{\overline{n}|}\, a_{x\overline{t}|})$ is slightly too small, and the resulting value of F_x slightly too large.

Furthermore, $f_x^{(n)} > f_{x\overline{n}|}$ and therefore $(1 + f_x^{(n)} a_{x\overline{n}|})$ is rather too large.

It follows that the correct divisor lies between the limits

$$(1 + f_x^{(t)}{}_{\overline{n}|}\, a_{x\overline{t}|}) \quad \text{and} \quad (1 + f_x^{(n)} a_{x\overline{n}|}),$$

and if $f_x^{(t)}{}_{\overline{n}|}$ is not readily available the limits must be taken as

$$1 \quad \text{and} \quad (1 + f_x^{(n)} a_{x\overline{n}|}).$$

An example will illustrate the method:

A certain aviation risk is covered for a life aged 30 by the following extra premiums (assumed net):

$$f_{30}^{(10)} = \cdot 01 \quad \text{and} \quad f_{30}^{(7)}{}_{:\overline{10}|} = \cdot 01.$$

On the basis of A 1924–29 (select) $2\frac{1}{2}$ %,

$$f_{30}^{(10)} (1 - A_{[30]})\, a_{[30]:\overline{10}|} = \cdot 0550,$$
$$1 + f_{30}^{(7)}{}_{:\overline{10}|}\, a_{[30]:\overline{7}|} = 1 \cdot 0647,$$
$$1 + f_{30}^{(10)}\, a_{[30]:\overline{10}|} = 1 \cdot 0888,$$

so that F_{30} lies between ·0517 and ·0505 and might reasonably be taken as £5. 2*s*. %. If $f_{30}^{(7)}{}_{:\overline{10}|}$ had not been available, the limits would have been ·0550 and ·0505, and a consideration of the extra risk in question is probably the best guide to a suitable intermediate value.

* For example, this may be deduced from the inequalities given by C. F. Wood on p. 14, *ibid.*

Finally, a loading for expenses must be added to F_x, and a practical solution would be to use the above formulae with the office extras in place of the net extras, $f_x^{(m)}$, etc. The precise implications here have been mentioned earlier.

III. WHOLE-LIFE LIMITED-PAYMENT POLICIES

A number of useful expressions can be devised for the extra annual premium payable under a limited-payment policy. We shall use the following notation for extra premiums (net functions):

Symbol	Policy	Period of normal premium payments under policy	Period of extra premium payments under policy
f_x	Whole-life assurance	Throughout	Throughout
$f_x^{(n)}$	Whole-life assurance	Throughout	n years
${}_nf_x$	Whole-life assurance	n years	n years
F_x	Whole-life assurance	Single premium	Single premium
$f_{x\overline{n}\rceil}$	Endowment assurance (term n years)	n years	n years
$f_{x\,\overline{n}\rceil}^{(t)}$	Endowment assurance (term n years)	n years	t years ($t<n$)

Now $${}_nP_x = P_x\, a_x / a_{x\overline{n}\rceil},$$
and $${}_nP'_x = P'_x\, a'_x / a'_{x\overline{n}\rceil}.$$

On taking the difference between these equations we obtain a formula which ultimately reduces to
$${}_nf_x = f_x^{(n)} - {}_nP_x h(x, n), \tag{5}$$
where $$h(x, n) = 1 - (1 + f_{x\overline{n}\rceil}\, a_{x\overline{n}\rceil})/(1 + f_x\, a_x).$$

This formula shows the deduction to be made from $f_x^{(n)}$ to allow for the earlier payment of premiums under a limited-payment policy. A more useful expression which can be derived from (5) is
$${}_nf_x = [f_x(1 - h(x, n)) - P_x h(x, n)]\, a_x / a_{x\overline{n}\rceil}. \tag{6}$$

This is in a convenient form because $f_{x\overline{n}\rceil}$ enters into the calculation of $h(x, n)$ alone. We see that the required extra premium ${}_nf_x$ is an explicit function of f_x and $f_{x\overline{n}\rceil}$ and annuities on the normal mortality basis. Hence if f_x and $f_{x\overline{n}\rceil}$ are known, we have the required extra directly.

Formulae (5) and (6) reduce of course to formula (3) when $n = 1$.

In order to give some idea of the size and nature of $h(x, n)$ some specimen values have been calculated (see p. 90) for two simple types of extra risk:

Type I. The special mortality is represented by a constant addition to the age, and therefore assumes an increasing extra risk.

Type C. The special rates of mortality are calculated on the basis of a constant addition to the force of mortality at all ages.

The normal premium is assumed to be calculated on the basis of A 1924–29 (select) mortality at a rate of interest of $2\frac{1}{2}\%$.

It will be noticed from formula (6) that the limited-payment extra is least when $h(x, n)$ is greatest. For a given whole-life extra f_x, the limited-payment extras are, therefore, less for Type I mortality than for Type C (cf. C. F. Wood, *ibid.* p. 13).

Table of $h(x,n) = 1 - (1 + f_{x\overline{n}|}a_{x\overline{n}|})/(1 + f_x a_x)$

Age at entry x	Duration n	$h(x, n)$			
		Type I		Type C	
		Addition to age		Addition to q	
		5 years	10 years	005	·010
		(f_x = ·00285)	(f_x = ·00650)	(f_x = ·00338)	(f_x = ·00687)
30	10	·065	·135	·060	·113
	20	·059	·122	·040	·076
	30	·049	·095	·023	·044
	40	·030	·052	·010	·019
		(f_x = ·00630)	(f_x = ·01480)	(f_x = ·00322)	(f_x = ·00652)
45	10	·099	·199	·040	·077
	20	·073	·138	·021	·041
	30	·035	·057	·007	·015

An alternative expression to (6) may be obtained after a certain amount of algebraic manipulation. We find

$$_nf_x = \mathrm{F}_x/a_{x\overline{n}|} + f_{x\overline{n}|}(\mathrm{A}_x + \mathrm{F}_x). \tag{7}$$

This formula is useful where F_x has already been calculated. It is therefore most convenient where the tabulated extra for the ordinary whole-life policy is restricted to t years. First we find F_x by the method suggested in section II, and then proceed by formula (7) to obtain $_nf_x$. Unfortunately, $f_{x\overline{n}|}$ is required for the complete calculation and an estimate of its value will be necessary. This difficulty has been discussed in section II; if $f^{(t)}_{x\overline{n}|}$ is known, the estimate is much easier to make.

Finally, a suitable loading must be added to the net function $_nf_x$ in order to arrive at the office extra premium which is payable for the same period as the normal premiums under the policy.