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ACTUARIAL NOTE

SOME REMARKS ON THE BASIC MORTALITY FUNCTIONS

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Synopsis. In the construction of the last two series of English Life Tables a mathematical formula has been used at adult ages to obtain a graduated set of values of m_x, the central death rate at age x. An approximate method has then been used to estimate q_x, the annual death rate at age x.

We show how to avoid this approximation. More precisely, if m_x is known for all values of x (not only when x is an integer), we derive an exact formula for the associated survival function. In particular, our result is applicable if m_x is given by a mathematical formula.

The paper includes some general remarks on the mortality functions related to the survival function. Some examples are given which may perhaps be of interest.

1. *Introduction.* The mortality function l may be considered the starting-point of actuarial mathematics. If l_x is defined for $x \geqslant 0$, we let $s(x) = l_x/l_0$ and call s the survival function associated with l. A survival function may attain the value 0 at some finite age or may decrease monotonically to the limit 0. For theoretical reasons it is convenient to consider this latter case. Accordingly, in this paper, a given function s will be called a survival function if and only if the following conditions are satisfied :

$$s(0) = 1\,; \tag{1.1}$$

s is strictly monotonic decreasing ; (1.2)

$$\lim_{x \to \infty} s(x) = 0. \tag{1.3}$$

All functions in this paper will be assumed to be continuous with domain the non-negative real numbers.

In terms of a given survival function s two associated functions

q^s, the annual death rate, and m^s, the central death rate, are defined by

$$q^s_x = \frac{s(x) - s(x+1)}{s(x)}, \tag{1.4}$$

and

$$m^s_x = \frac{s(x) - s(x+1)}{\int_x^{x+1} s(t)dt}. \tag{1.5}$$

Further, if s is differentiable, the associated force of mortality μ^s is defined by

$$\mu^s_x = \frac{-1}{s(x)} . s'(x). \tag{1.6}$$

As is well known, it follows from this last definition that

$$s(x) = \exp\left\{-\int_0^x \mu^s_t dt\right\} \tag{1.7}$$

In particular we note that s is determined by μ^s. On the other hand q^s does *not* determine s. We give below an example to show that distinct survival functions s_1 and s_2 may be such that $q^{s_1}_x = q^{s_2}_x$ for all values of x.

In section 2 we consider the corresponding problem when the central death rate is known. We prove that s is determined by m^s and we give a formula (corresponding to 1.7 above) to express $s(x)$ in terms of the values of m^s.

Finally, in section 3 we give some examples which may perhaps be of interest.

2. Throughout this section we let f be a given positive function.

Firstly we remark that there is a unique differentiable function s such that $s(0) = 1$ and

$$\frac{-1}{s(x)} . s'(x) = f(x).$$

This function is defined by

$$s(x) = \exp\left\{-\int_0^x f(t)dt\right\},$$

for which conditions 1.1 and 1.2 above are clearly satisfied. The condition 1.3 is satisfied if and only if

$$\int_0^\infty f(t)dt = \infty. \tag{2.1}$$

This last condition is thus both necessary and sufficient for the existence of a survival function s such that $\mu^s = f$.

Secondly we may ask whether or not there is a survival function s such that $q^s = f$. It is easy to give examples where f is such that no survival function exists with the required property. On the other hand there may be more than one such survival function, as the following example shows.

Suppose that $\delta > 0$ and let $f(x) = 1 - e^{-\delta}$, independent of x. Let

$$s(x) = (1 + \beta \sin 2\pi x) . e^{-\delta x},$$

where β is to be determined. Then clearly s satisfies conditions 1.1 and 1.3 above. Since

$$s'(x) = \{\beta(2\pi \cos 2\pi x - \delta \sin 2\pi x) - \delta\} . e^{-\delta x},$$

the condition

$$0 \leqslant \beta < \frac{\delta}{2\pi + \delta} \tag{2.2}$$

implies that $s'(x) < 0$. Hence, for each β satisfying condition 2.2, s is a survival function. Moreover $q^s = f$ and so, in this case, there are infinitely many survival functions having f as the associated annual death rate.

Thus, as we have already remarked, in general s is not determined by q^s. This is largely of theoretical interest only since, if q^s is known, $s(x)$ is determined for integer values of x and for practical purposes this is usually sufficient.

In the remainder of this section we consider whether or not there exists a survival function for which f is the associated central death rate. We give necessary and sufficient conditions (to be satisfied by f) for the existence of such a survival function and we show that if these conditions are satisfied the survival function is determined uniquely. Our principal result is the following.

Theorem. Let f be a given positive function. Then there exists a survival function s such that $m^s = f$ if and only if the following conditions are satisfied:

$$\int_0^\infty f(t)dt = \infty\ ; \tag{2.3}$$

the series $\sum_{r=0}^{\infty} f(x+r) . \exp\left\{-\int_0^{x+r} f(t)dt\right\}$ converges for all values of x; (2.4)

the function defined by the above series is strictly monotonic decreasing. (2.5)

Moreover, if these conditions are satisfied, then

$$s(x)=\frac{\sum_{r=0}^{\infty} f(x+r).\exp\left\{-\int_0^{x+r} f(t)dt\right\}}{\sum_{r=0}^{\infty} f(r).\exp\left\{-\int_0^{r} f(t)dt\right\}} \tag{2.6}$$

Our proof of the theorem requires the following preliminary lemma.

Lemma. The following conditions are equivalent:

$$m^s=f; \tag{2.7}$$

$$s(x+1)=s(x)-\left\{\int_0^1 s(t)dt\right\}.f(x).\exp\left\{-\int_0^x f(t)dt\right\} \text{ for all } x\geqslant 0. \tag{2.8}$$

Proof of lemma. We note that

$$s(x+1)-s(x)=\frac{d}{dx}\left\{\int_x^{x+1} s(t)dt\right\}$$

and so

$$m_x^s=\frac{d}{dx}\left\{-\log\int_x^{x+1} s(t)dt\right\}.$$

Firstly we assume that condition 2.7 above is satisfied. By our remark above this condition may be written as

$$\frac{d}{dx}\left\{-\log\int_x^{x+1} s(t)dt\right\}=f(x).$$

Integrating, we obtain

$$\int_x^{x+1} s(t)dt=\left\{\int_0^1 s(t)dt\right\}.\exp\left\{-\int_0^x f(t)dt\right\}$$

Differentiating this last equation, we see that

$$s(x+1)-s(x)=-\left\{\int_0^1 s(t)dt\right\}.f(x).\exp\left\{-\int_0^x f(t)dt\right\}, \tag{2.9}$$

which shows that condition 2.8 is satisfied.

Secondly, suppose instead that condition 2.8 is satisfied. Then equation 2.9 is valid and so

$$\frac{d}{dx}\left\{\int_x^{x+1} s(t)dt\right\}=\frac{d}{dx}\left[\left\{\int_0^1 s(t)dt\right\}.\exp\left\{-\int_0^x f(t)dt\right\}\right]$$

Integrating this last equation, we obtain

$$\int_x^{x+1} s(t)dt = \left\{\int_0^1 s(t)dt\right\}.\exp\left\{-\int_0^x f(t)dt\right\} \tag{2.10}$$

It follows from equations 2.9 and 2.10 (by division) that condition 2.7 above is satisfied, which completes the proof of the lemma.

Proof of Theorem. Suppose firstly that there exists a survival function s such that $m^s = f$. We show that conditions 2.3, 2.4 and 2.5 are satisfied and we derive formula 2.6.

Now

$$\begin{aligned}\int_0^x f(t)dt &= \int_0^x m_t^s dt \\ &= \int_0^x \left[\frac{d}{dt}\left\{-\log\int_t^{t+1} s(r)dr\right\}\right]dt \\ &= \log\frac{\int_0^1 s(r)dr}{\int_x^{x+1} s(r)dr} \end{aligned} \tag{2.11}$$

As, by hypothesis, s is a survival function, $\lim\limits_{x\to\infty} s(x)=0$ and so $\lim\limits_{x\to\infty}\int_x^{x+1} s(r)dr=0$. It therefore follows from equation 2.11 above that condition 2.3 is satisfied.

By the lemma condition 2.8 is satisfied. It follows by induction that, for each integer $n \geqslant 1$,

$$s(x+n) = s(x) - \left\{\int_0^1 s(t)dt\right\}.\sum_{r=0}^{n-1} f(x+r).\exp\left\{-\int_0^{x+r} f(t)dt\right\}. \tag{2.12}$$

Since $\lim\limits_{x\to\infty} s(x)=0$, for each x we have

$$\lim_{n\to\infty} s(x+n) = 0. \tag{2.13}$$

Combining equations 2.12 and 2.13 we see that condition 2.4 above is satisfied and

$$\sum_{r=0}^{\infty} f(x+r).\exp\left\{-\int_0^{x+r} f(t)dt\right\} = \frac{s(x)}{\int_0^1 s(t)dt}$$

This last equation establishes condition 2.5 and, since $s(0)=1$, formula 2.6 follows directly.

Secondly we assume that conditions 2.3, 2.4 and 2.5 are satisfied. We prove the existence of a survival function with the required property.

In this case we may define $s(x)$ by formula 2.6. Then $s(0)=1$ and conditions 1.2 and 1.3 follow from condition 2.5. Hence s is a survival function and it remains only to prove that $m^s=f$.

Condition 2.5 implies that for each $b>0$ the convergence of the series defining $s(x)$ is uniform on the interval $[0, b]$. Thus term by term integration is permissible and, letting

$$\lambda=\sum_{r=0}^{\infty} f(r).\exp\left\{-\int_0^r f(t)dt\right\},$$

for each $x\geqslant 0$ we have

$$\begin{aligned}\int_x^{x+1} s(u)du &= \frac{1}{\lambda}\lim_{n\to\infty}\left[\int_x^{x+1}\left[\sum_{r=0}^{n} f(u+r).\exp\left\{-\int_0^{u+r} f(t)dt\right\}\right]du\right]\\ &= \frac{1}{\lambda}\lim_{n\to\infty}\sum_{r=0}^{n}\left[\int_x^{x+1} f(u+r).\exp\left\{-\int_0^{u+r} f(t)dt\right\}du\right]\\ &= \frac{1}{\lambda}\lim_{n\to\infty}\left[\sum_{r=0}^{n}\left[\exp\left\{-\int_0^{x+r} f(t)dt\right\}\right.\right.\\ &\qquad\qquad\left.\left.-\exp\left\{-\int_0^{x+r+1} f(t)dt\right\}\right]\right]\\ &= \frac{1}{\lambda}\lim_{n\to\infty}\left[\exp\left\{-\int_0^{x} f(t)dt\right\}-\exp\left\{-\int_0^{x+n+1} f(t)dt\right\}\right]\\ &= \frac{1}{\lambda}\exp\left\{-\int_0^{x} f(t)dt\right\} \quad \text{(by 2.3 above).}\end{aligned} \tag{2.14}$$

Our definition of s implies that

$$s(x)-s(x+1)=\frac{1}{\lambda}f(x).\exp\left\{-\int_0^x f(t)dt\right\}. \tag{2.15}$$

Dividing 2.15 by 2.14 we obtain $m_x^s=f(x)$ as required.

We remark that equation 2.6 may be written in the form (analogous to 1.7 above)

$$s(x)=\frac{\sum_{r=0}^{\infty} m_{x+r}^s.\exp\left\{-\int_0^{x+r} m_t^s dt\right\}}{\sum_{r=o}^{\infty} m_r^s.\exp\left\{-\int_0^{r} m_t^s dt\right\}}. \tag{2.16}$$

3. *Some examples*

It is useful to derive a formula which can be used when m_x^s is known only for $x \geqslant \xi$

Since $\exp\left\{-\int_0^{x+r} m_t^s dt\right\} = \exp\left\{-\int_0^{\xi} m_t^s dt\right\} \cdot \exp\left\{-\int_{\xi}^{x+r} m_t^s dt\right\}$

and $\exp\left\{-\int_0^{\xi} m_t^s dt\right\}$ is a constant, it follows from equation 2.16 above that

$$\frac{s(x)}{s(\xi)} = \frac{\sum_{r=0}^{\infty} m_{x+r}^s \, . \exp\left\{-\int_{\xi}^{x+r} m_t^s dt\right\}}{\sum_{r=0}^{\infty} m_{\xi+r}^s \, . \exp\left\{-\int_{\xi}^{\xi+r} m_t^s dt\right\}}. \tag{3.1}$$

In graduating the English Life Tables No. 12 a formula of the type

$$m_x^s = a + b(1+e^{-\alpha(x-x_1)})^{-1} + c \, . e^{-\beta(x-x_2)^2} \tag{3.2}$$

was adopted for $x \geqslant 27$ (males) and for $x \geqslant 20$ (females). The values of m_x^s were obtained for integer values of x and then q_x^s was calculated by the iterative formula

$$q_x^s = m_x^s \cdot \left[1 - \frac{q_{x-1}^s}{12[1-q_{x-1}^s]}\right] \cdot \left[1 + \tfrac{5}{12} m_x^s\right]^{-1}. \tag{3.3}$$

This last equation is equivalent to

$$\int_x^{x+1} s(t)dt = \tfrac{1}{12}\{5s(x+1) + 8s(x) - s(x-1)\}$$

which is true if $s(t)$ is a quadratic function of t on the interval $[x-1, x+1]$. The results above enable us to examine the accuracy of the approximation 3.3, assuming that mortality is defined at adult ages by 3.2.

Let m^s be defined by 3.2. It is readily verified that, if $\zeta \leqslant x_2$, then

$$\int_{\zeta}^{x} m_t^s dt = (a+b) \, . (x-\zeta) + \frac{b}{\alpha} \log\left\{\frac{1+e^{-\alpha(x-x_1)}}{1+e^{-\alpha(\zeta-x_1)}}\right\} + \frac{c\sqrt{\pi}}{2\sqrt{\beta}} \, \theta(x), \tag{3.4}$$

where

$$\theta(x) = \begin{cases} \text{erf}\{\sqrt{\beta}(x_2-\zeta)\} - \text{erf}\{\sqrt{\beta}(x_2-x)\} & \text{if } 0 \leqslant x < x_2 \\ \text{erf}\{\sqrt{\beta}(x_2-\zeta)\} + \text{erf}\{\sqrt{\beta}(x-x_2)\} & \text{if } x \geqslant x_2 \end{cases},$$

and

$$\text{erf}\,(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt$$

is the standard " error function " defined for $x \geqslant 0$.

For the Female English Life Table (No. 12) the values of the constants used in formula 3·2 above are: $a=0{\cdot}00035$, $b=0{\cdot}7574$, $c=0{\cdot}00155$, $\alpha=0{\cdot}1232$, $\beta=0{\cdot}0033$, $x_1=11{\cdot}8.\alpha^{-1}$, $x_2=56$. Using formula 3.4 we can now apply formula 3.1 to construct the mortality table defined by formula 3.2 with these constants. The infinite series in formula 3.1 converge rapidly and may be summed by computer to a very high degree of accuracy. We give below an abridged form of the resulting mortality table. The figure 97,336 has been chosen for the radix of our table, since this is the value of l_{20} in the Female English Life Table, from which the corresponding figures are also shown for purposes of comparison.

Central death rate defined by 3.2 above (constants as indicated). Mortality table obtained by 3.1.				English Life Table Number 12 (Females)		
x	l_x (See note 1)	p_x (See note 2)	m_x (See note 2)	x	l_x	p_x
20	97,336	0·99956	0·00044	20	97,336	0·99956
30	96,811	0·99925	0·00075	30	96,811	0·99925
40	95,723	0·99820	0·00180	40	95,724	0·99820
50	93,082	0·99560	0·00441	50	93,080	0·99561
60	86,966	0·98912	0·01093	60	86,967	0·98912
70	72,481	0·96897	0·03152	70	72,483	0·96896
80	41,893	0·90892	0·09542	80	41,890	0·90892
90	8,783	0·77871	0·24969	90	8,782	0·77872
100	263	0·62113	0·47535	100	264	0·62212
109	2	0·53035	0·63354	109	2	0·53421

Notes

1 The values shown for l_x are accurate to the nearest integer, given that $l(20)=97{,}336$.

2 The values shown for p_x and m_x are accurate to five decimal places.

A comparison of the two sets of figures shows the very high accuracy of the approximation 3.3 in this case. The only significant differences occur in the values of p_x at advanced ages and for practical purposes these are of no importance.

We wish to comment upon an important difference between the natures of formulae 1.7 and 2.16. It follows from formula 1.7 that, if s_1 and s_2 are survival functions such that $\mu_x^{s_1}=\mu_x^{s_2}$ for $x\leqslant x_0$ (where x_0 is some fixed positive real number), then also $s_1(x)=s_2(x)$ for $x\leqslant x_0$. In other words, an alteration in the force of mortality at ages greater than x_0 does not affect the survival function at ages less than x_0. However equation 2.16 above shows that this is not true of the central death rate. If s_3 and s_4 are survival functions such that $m_x^{s_3}=m_x^{s_4}$ for $x\leqslant x_0$, in general it does *not* follow that $s_3(x)=s_4(x)$ for $x\leqslant x_0$.

For example, let $N \geqslant 1$ be some positive integer and suppose that $\delta > 0$ and $0 < \beta < \delta$. Let f_3 and f_4 be defined by

$$f_3(x) = \delta \qquad \text{(for all } x\text{)}$$

and

$$f_4(x) = \begin{cases} \delta & \text{if } x \leqslant N \\ (1+\beta[x-N])\delta & \text{if } N < x < N+1 \\ (1+\beta)\delta & \text{if } x \geqslant N+1. \end{cases}$$

Then conditions 2.3, 2.4, and 2.5 are readily seen to be satisfied by f_3. Moreover, it can be shown that, since $0 < \beta < \delta$, these conditions are also satisfied by f_4. Hence, by the theorem, there exist survival functions s_3 and s_4 such that $m^{s_3} = f_3$ and $m^{s_4} = f_4$. Using equation 2.6 above we find that

$$s_3(x) = e^{-\delta x}.$$

Similar calculations to obtain s_4 are straightforward but lengthy and accordingly we omit the details. However, they show that

$$q_n^{s_4} > 1 - e^{-\delta} = q_n^{s_3}$$

for *every* integer $n \geqslant 0$. Thus, in this case, increasing the central death rate at ages greater than N has the effect also of increasing the annual death rates throughout the mortality table. As an extreme case, for example, even if N is very large, we still have $q_0^{s_4} > q_0^{s_3}$.

Finally we remark that an increasing function f clearly satisfies condition 2.1 and may thus be considered as a force of mortality. If also $f(x) < 1$ for all x, it is easy to show that there are infinitely many survival functions for which f is the associated annual death rate. However, it does not necessarily follow that there exists a survival function s such that $m^s = f$. It is possible for an increasing function not to satisfy condition 2.5 above, in which case there is no survival function with the required property.

REFERENCE

The Registrar General's Decennial Supplement; England and Wales 1961 Life Tables (London, H.M.S.O., 1968).