

SOME STOCHASTIC CONTROL PROBLEMS IN
INSURANCE

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Some stochastic control problems in insurance

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1 Introduction and Summary

Stochastic control has been used in insurance for some time for peculiar problems (see Martin-Löf [7], or Brockett[2]). In this paper we shall consider continuous time problems which lead to Hamilton-Jacobi-Bellman equations. These are problems of optimal choice of new business, and of optimal proportional reinsurance. Our objective function will be infinite time ruin probability (which is chosen for simplicity and for the purpose of illustration). Other objective functions (based on utility functions or on expected discounted dividend) are possible and can be treated with essentially the same methods. See e.g. Asmussen and Taksar [1] or Hoejgaard and Taksar [6].

The corresponding problems with optimal excess of loss reinsurance and with optimal investment are dealt with in Hipp and Taksar [4].

We consider the classical Lundberg process for insurance business

$$dR_0(t) = cdt - dS(t), \quad R_0(0) = s,$$

where $S(t)$ is a compound Poisson process. So, $S(t) = X_1 + \dots + X_{N(t)}$, where $N(t)$ is a Poisson process - with constant intensity λ - which is independent of the independent and identically distributed claim sizes X_1, X_2, \dots

The number c is a fixed premium intensity. The classical infinite time ruin probability

$$\psi_0(s) = \mathbb{P}\{R_0(t) < 0 \text{ for some } t \geq 0\}$$

is based on the assumption that the insurer does not change his or her risk management strategy, such as the reinsurance program, investment strategy, or writing new business. We shall consider ruin probability under the assumption that an adjustment of risk management decisions is possible in continuous time, and that this adjustment is chosen such that the ruin probability is minimized. This leads to stochastic control problems which can be solved using Hamilton-Jacobi-Bellman equations. The usual steps for the solution are

- transform the given problem into a Hamilton-Jacobi-Bellman (HJB) equation;
- solve the HJB equation numerically;
- for a smooth solution of the HJB equation, prove a verification theorem which states that the solution of the equation is a solution of the given problem.

The first step in this process is purely heuristic, it is based on smoothness assumptions which are not justifiable. The second step is the hardest one, while the third has two components: the dynamics and the initial condition. The argument concerning the dynamics is quite general: the HJB equation yields that the processes under consideration are (local) martingales and submartingales, respectively. The initial condition is needed to compare the two processes, and for this step we need arguments which are specific for each given problem. Roughly speaking, we want to compare two functions f_1 and f_2 . We first show that $f_1'(t) \leq f_2'(t)$, and with an initial condition like

$$f_1(t_0) = f_2(t_0)$$

we arrive at the desired result $f_1(t) \leq f_2(t)$ for all t .

1.1 HJB equation for optimal new business

To be more precise we consider a first example. This is the optimization of new business in the presence of a non traded fixed insurance portfolio which is modelled by a classical Lundberg process. Assume that at each time point $t \geq 0$ a proportion $b(t)$ of new business can be written from an independent risk process which is again a classical Lundberg process: if $b(t)$

is the proportion of new business held at time t , then the dynamics of the process including new business equals

$$dR(t) = dR_0(t) + b(t)c_1 dt - dS(t, b(t)), \quad R(0) = s,$$

where c_1 is the premium rate for the new business, and $S(t, b)$ is the compound Poisson process with claim sizes Y_1, Y_2, \dots and intensity $\lambda_1 b$. So the written proportion has influence on the premium rate and the claims intensity, but not on the claim sizes. Now the proportion is adjusted continuously in order to make the ruin probability as small as possible:

$$\psi(s) = \mathbb{P}(R(t) < 0 \text{ for some } t \geq 0) = \min!$$

For notational convenience we now switch to survival probabilities $\delta(s) = 1 - \psi(s)$.

The proportion chosen at time t may depend on the history of the process $R(s)$ up to time t ; if a claim occurs at time t , then $b(t)$ may not depend on the size of this claim. This is necessary for predictability of $b(t)$.

For $\delta_0(s)$ the ruin probability without new business we have

$$0 = \lambda E[\delta_0(s - X) - \delta_0(s)] + c\delta'_0(s). \quad (1)$$

This follows by considering the two distinct cases

- there is exactly one claim in the interval $[0, dt]$ which happens with probability λdt , and after this claim of size X we are left with a surplus $s - X$; or
- there is no claim in the interval $[0, dt]$ which happens with probability $1 - \lambda dt$, and we are left with a surplus of size $s + cdt$.

Averaging over all possible claim sizes we arrive at the equation

$$\delta_0(s) = \lambda dt E[\delta_0(s - X)] + (1 - \lambda dt)\delta_0(s + cdt).$$

Assuming that $\delta_0(s)$ has a (right) derivative $\delta'_0(s)$ we obtain equation (1).

For the ruin probability with new business we obtain in exactly the same way - with $b = b(0)$ given - the equation

$$0 = \lambda E[\delta(s - X) - \delta(s)] + \lambda_1 b E[\delta(s - Y) - \delta(s)] + (c + c_1 b)\delta'_0(s),$$

or, more explicitly

$$\delta(s) = \delta(s) + dt\{\lambda E[\delta(s-X) - \delta(s)] + \lambda_1 b E[\delta(s-Y) - \delta(s)] + (c + bc_1)\delta'_0(s)\}.$$

An optimal choice for b (which might be seen as the proportion written in the interval $[0, dt]$) is obtained by minimizing the bracket:

$$0 = \sup_b \{\lambda E[\delta(s-X) - \delta(s)] + \lambda b E[\delta(s-Y) - \delta(s)] + (c + bc_1)\delta'_0(s)\}. \quad (2)$$

An optimal strategy for all time points is obtained by solving this Hamilton-Jacobi-Bellman equation for all state variables s , i.e. derived from a solution $(\delta(s), B(s))$ of (2). This solution - if it exists - has the following properties:

$$\lambda E[\delta(s-X) - \delta(s)] + \lambda B(s) E[\delta(s-Y) - \delta(s)] + (c + B(s)c_1)\delta'_0(s) = 0, \quad (3)$$

and for arbitrary functions $\hat{B}(s)$ we have

$$\lambda E[\delta(s-X) - \delta(s)] + \lambda \hat{B}(s) E[\delta(s-Y) - \delta(s)] + (c + \hat{B}(s)c_1)\delta'_0(s) \leq 0. \quad (4)$$

The optimal proportion written at time t is

$$b(t) = B(R(t-)).$$

Later in section 2 we shall see that the optimal strategy has to be restricted (since otherwise the optimal parameter b would be $-\infty$ or $+\infty$, respectively), and that it is bang-bang, i.e. takes extremal values only. Furthermore, the optimal strategy is unrealistic for application since even non profitable business will be written and sold.

1.2 HJB equation for optimal proportional reinsurance

As a second example we consider the optimization of expensive proportional reinsurance. For earlier different approaches to this problem see Hoejgaard and Taksar [5] or Dayananda [3]. At each time point we can reinsure a proportion $a(t)$ of our insurance portfolio, the premium for this reinsurance in an interval of length dt is $a(t)c_1dt$, and this insurance is called expensive when

$$c_1 > c.$$

This assumption is needed to exclude the strategy $a(t) = 1$. Now our risk process has the dynamics

$$dR(t) = (c - a(t)c_1)dt - dS(t, a(t)), \quad R(0) = s,$$

where for fixed number a the process $S(t, a)$ is compound Poisson with intensity λ and claim sizes

$$Y_i = aX_i.$$

In this case the HJB equation is

$$0 = \sup_a \{ \lambda E[\delta(s - (1-a)X) - \delta(s)] + (c - ac_1)\delta'(s) \}. \quad (5)$$

Here, the dependence on a is more complicated. According to real life constraints the range of a must be restricted to $0 \leq a \leq 1$.

2 Computation of optimal strategies

2.1 Optimal new business

Since the expression in brackets in the HJB equation (2) is linear in b , the infimum will be attained only if the set of admissible b 's is restricted. We shall introduce the constraints $0 \leq b \leq 1$. Then the supremum is attained by

$$b(t) = \begin{cases} 1 & \text{if } \lambda E[\delta(s - Y) - \delta(s)] + c_1\delta'_0(s) < 0 \\ 0 & \text{if } \lambda E[\delta(s - Y) - \delta(s)] + c_1\delta'_0(s) \geq 0 \end{cases}.$$

This means that one of the following equation must hold:

$$0 = \lambda E[\delta(s - X) - \delta(s)] + \lambda_1 E[\delta(s - Y) - \delta(s)] + (c + c_1)\delta'(s) \text{ or } (6)$$

$$0 = \lambda E[\delta(s - X) - \delta(s)] + c\delta'(s) \quad (7)$$

For the numerical solution of HJB we choose $\delta(0)$ arbitrarily and solve for $\delta'(0)$ in (6) or (7). We choose the value which is smallest. We discretize at $\Delta, 2\Delta, 3\Delta, \dots$. Our updating and recursion for $\delta(\Delta), \delta(2\Delta), \dots$ and $\delta'(\Delta), \delta'(2\Delta), \dots$ is

$$\delta(\Delta) = \delta(0) + \Delta\delta'(0),$$

and $\delta'(\Delta) =$ from (6) or (7), whichever makes $\delta'(\Delta)$ smallest, and so on. At the end (i.e. when a sufficiently large multiple $k\Delta$ of Δ is reached), a norming is performed to adjust the result to the restriction $\delta(\infty) = 1$:

$$\delta^{norm}(j\Delta) := \delta(j\Delta)/\delta(k\Delta), \quad j = 0, \dots, k.$$

The qualitative behavior of the optimal strategy $b(t)$ is best visible at the point $s = 0$, i.e. when the initial surplus is zero and the insurer is very close to ruin. The choice $b(0) = 0$ or $b(0) = 1$ depends on the size of $\delta'(0)$ computed with the two defining equations

$$\begin{aligned} 0 &= \lambda[1 - \delta(0)] + \lambda_1[1 - \delta(0)] - (c + c_1)\delta'_0(0) \text{ or} \\ 0 &= \lambda[1 - \delta(0)] - c\delta'_0(0), \end{aligned}$$

i.e.

$$\begin{aligned} \delta'_0(0) &= (\lambda[1 - \delta(0)] + \lambda_1[1 - \delta(0)]) / (c + c_1) \text{ or} \\ \delta'_0(0) &= \lambda[1 - \delta(0)] / c \end{aligned}$$

We have $b(0) = 1$ iff the first equation leads to a smaller $\delta'(0)$, i.e. iff

$$\lambda[1 - \delta(0)] / c > (\lambda[1 - \delta(0)] + \lambda_1[1 - \delta(0)]) / (c + c_1) \text{ or}$$

$$\lambda / c > \lambda_1 / c_1.$$

This means that close to ruin new business is written irrespectively of the mean claim size of new business. Even non profitable business will be written in order to collect premia, and this money will be used to pay the next claim. If the company survives, then at some large surplus s the (possibly non profitable) new business will be sold ($b(t) = 0$), and this will not be possible in real life.

2.2 Optimal new business without selling

For a more realistic setup we now consider constraint optimization: written business cannot be sold later. This means that we restrict the proportion functions $b(t)$ to the set of functions which are nondecreasing and bounded by 0 and 1. To solve this new problem we add a new state variable B which is the current proportion written. Our value function $\delta(s, B)$ is defined via the HJB equation

$$\begin{aligned} 0 &= \sup_{B \leq \beta \leq 1} \{ \lambda E[\delta(s - X, \beta) - \delta(s, \beta)] \\ &\quad + \lambda b E[\delta(s - Y, \beta) - \delta(s, \beta)] + (c + bc_1)\delta'(s, B) \} \end{aligned} \quad (8)$$

The initial conditions are $\delta(s, 1)$ which is a classical ruin probability, and arbitrary values for $\delta(0, B)$ which must be adjusted with the norming procedure. The optimal strategy is

$$b(t) = \beta(R(t), B(t)),$$

where $\beta(s, B)$ is the point at which the supremum in (8) is attained, and $B(t)$ is the proportion written at time t . For the numerical computation we have to discretize the state space: s and B are integer multiples of the step size Δ (one could of course take two different step sizes for the s -values and the β -values). We first compute $\delta(s, 1)$ for $s = j\Delta$, $j = 0, \dots, k$, using

$$0 = \lambda E[\delta(s - X, 1) - \delta(s, 1)] + \lambda b E[\delta(s - Y, 1) - \delta(s, 1)] + (c + bc_1)\delta'(s, 1).$$

For $B = 1 - \Delta$ we have only two possible values for β : $\beta = 1$ and $\beta = B$. We start with an arbitrary value for $\delta(0, B)$. Next, if $\beta = 1$ then $\delta'(0, B) = \delta'(0, 1)$, and if $\beta = B$ then $\delta'(0, B)$ is taken from

$$0 = \lambda E[\delta(s - X, B) - \delta(s, B)] + \lambda b E[\delta(s - Y, B) - \delta(s, B)] + (c + bc_1)\delta'(s, B)$$

for $s = 0$. Notice that the last equation is the integro-differential equation for the classical ruin probability $\delta_0(s, B)$ with a fixed proportion B in new business. The decision for $\beta = 1$ or $\beta = B$ is made according to the rule: minimize the value for $\delta'(0, B)$. Hence

$$\delta'(s, B) = \min(\delta'(s, 1), \delta'_0(s, B)), \quad s \geq 0.$$

The norming is done according to $\delta(k\Delta, B) = 1$, and the following updating is used:

$$\delta(s - \Delta, B) = \delta(s, B) - \Delta \delta'(s - \Delta, B).$$

For the next steps $B = 1 - j\Delta$, $j > 1$, we similarly obtain

$$\delta'(s, B) = \min(\delta'(s, B + \Delta), \delta'_0(s, B)), \quad s \geq 0,$$

and we use the same norming and updating procedure.

Alternatively to the above approach, one could allow for selling of written new business via expensive proportional reinsurance. This would lead to a joint optimization of writing new business and writing reinsurance.

2.3 Optimal proportional reinsurance

For a solution of the HJB equation (5) we have to compute the expectations

$$g(s, a) := E\delta(s - (1 - a)X)$$

for all possible values of a . For simplicity we assume that the claim size distribution is an exponential distribution with mean $1/\theta$. Then the functions $g(s, a)$ satisfy the differential equations

$$g_s(s, a) = \frac{\theta}{1 - a}(\delta(s) - g(s, a)).$$

The value function $\delta(s)$ has derivative

$$\delta'(s) = \min_a (\lambda(\delta(s) - g(s, a)) / (c - ac_1)). \quad (9)$$

Again, the optimal strategy is given by $a(R(t))$, where $a(s)$ is the point at which the minimum in (9) is attained.

3 Verification Theorem

We shall consider only the case of new business with selling, i.e. without the restriction to non decreasing strategies $b(t)$. Assume that the HJB equation admits a smooth solution $(\delta(s), B(s))$, i.e. $\delta(s)$ is twice continuously differentiable, $B(s)$ is the value at which the supremum is attained, for which

$$0 \leq \delta(s) \leq 1,$$

and

$$\lim_{s \rightarrow \infty} \delta(s) = 1.$$

Then the two relations (3) for $B(s)$ and (4) for arbitrary functions $\hat{B}(s)$ hold. Assume that $\hat{\delta}(s)$ is the survival probability using the strategy $\hat{B}(\hat{R}(t-))$, and $\delta^*(s)$ is the survival probability with strategy $B(R^*(t-))$. Here $\hat{R}(t)$ and $R^*(t)$ are the surplus processes resulting from strategies \hat{B} and B . Let τ^* and $\hat{\tau}$ be the ruin times for the processes $R^*(t)$ and $\hat{R}(t)$, respectively. Consider the stochastic processes

$$V^*(t) = \delta(R^*(t \wedge \tau^*)), t \geq 0,$$

and

$$\widehat{V}(t) = \delta(\widehat{R}(t \wedge \widehat{\tau})), t \geq 0.$$

The dynamics of the two processes $R^*(t)$ and $\widehat{R}(t)$ are as follows:

- the process $R^*(t)$ has no jump in the interval $[t, t+dt]$ with probability $1-\lambda dt + \lambda_1 B(R^*(t-))$, its increment in this interval is $(c+B(R^*(t-))c_1)dt$.
- the process $R^*(t)$ has a jump of size X with probability λdt ; and
- the process $R^*(t)$ has a jump of size Y with probability $\lambda_1 B(R^*(t-))dt$.

For $\widehat{R}(t)$ we just have to replace $*$ -objects by \sim -objects.

This implies that by (3) the process $V^*(t)$ is a martingale, while $\widehat{V}(t)$ is a supermartingale according to (4). We have to consider the stopped processes since the two relations (3) and (4) hold for $s \geq 0$ only. Hence

$$E\widehat{V}(t) \geq \widehat{V}(0) = \delta(s) = V^*(0) = EV(t), t \geq 0. \quad (10)$$

On $\{\tau^* = \infty\}$ we have $R^*(t) \rightarrow \infty$, and therefor

$$V^*(t) \rightarrow 1_{\{\tau^* = \infty\}}.$$

By dominated convergence this implies

$$\lim_{t \rightarrow \infty} EV^*(t) = \mathbb{P}\{\tau^* = \infty\} = \delta^*(s).$$

Similarly,

$$\widehat{\delta}(s) = \mathbb{P}\{\widehat{\tau} = \infty\} = \lim_{t \rightarrow \infty} E\widehat{V}(t)$$

With (10) we obtain

$$\widehat{\delta}(s) \geq \delta^*(s),$$

and this optimal survival probability is attained using strategy $B(R^*(t-))$.

Notice that for most HJB equation the value function has to be convex and smooth. This is not obvious for value functions which are ruin probabilities: the classical ruin probabilities with discrete claim size distributions are neither convex nor smooth, they are not differentiable.

4 Numerical examples

4.1 New business

We consider an exponential claim size distribution for both, new and old business. We take $\lambda = \lambda_1 = 1$, $c = 2$, $c_1 = 3$, and the means of X and Y are 1 and 0.8, respectively. Figure 4.1 shows the survival probabilities with and without new business in the unrestricted case, together with the optimal strategy which is 1 up to $s = 4.285$, and 0 for larger values of the surplus. We obtain $\delta(0) = 0.5512$. The corresponding values for the restricted case do not differ much, they are equal up to possible discretization error. So it might be that we obtain the same results for both, the restricted and the unrestricted case.

4.2 Proportional reinsurance

Again we consider claim sizes having an exponential distribution with mean 1. We let $c = 2, c_1 = 2.1$, and $\lambda = 1$. As possible values for a we take 0 and 0.5 in a first, and 0, 0.25, 0.5, 0.75 in a second attempt. The resulting survival probabilities and optimal strategies are given in Figure 4.2. The optimal strategy in the second attempt takes the values 0 and 0.75 only.

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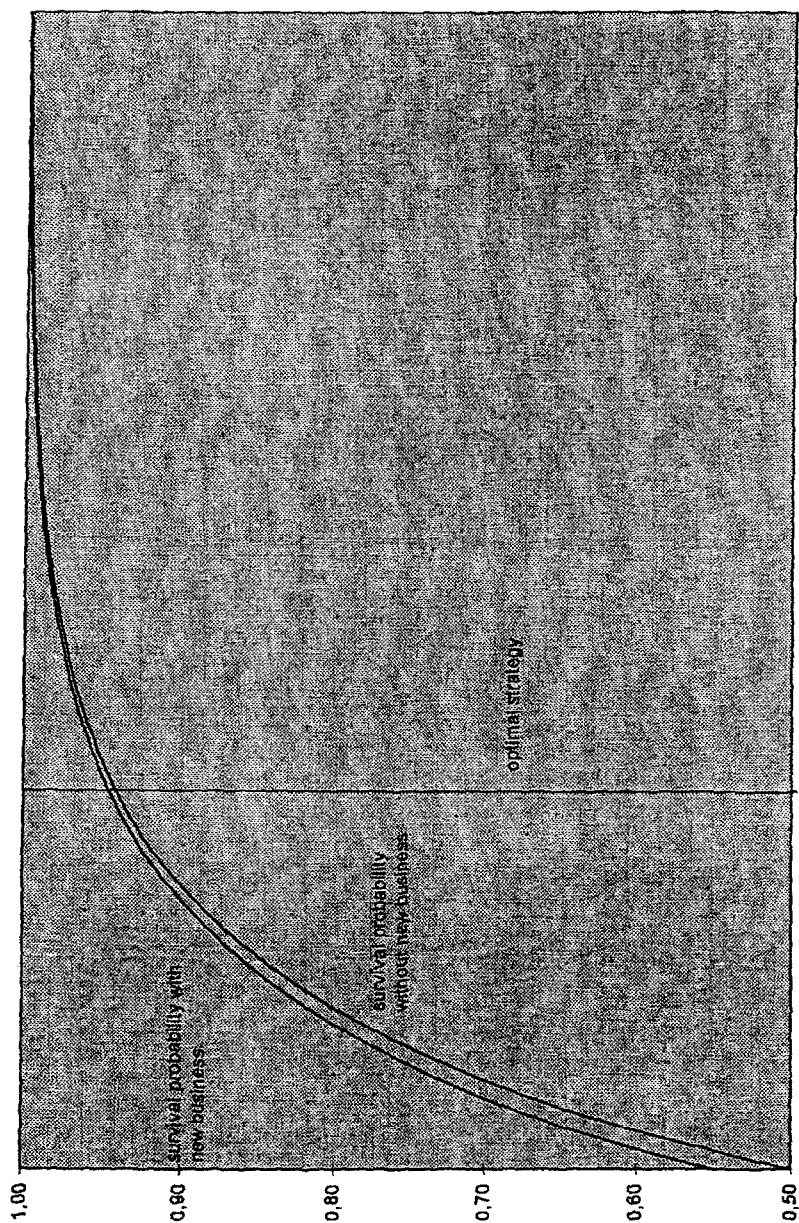


Figure 4.1

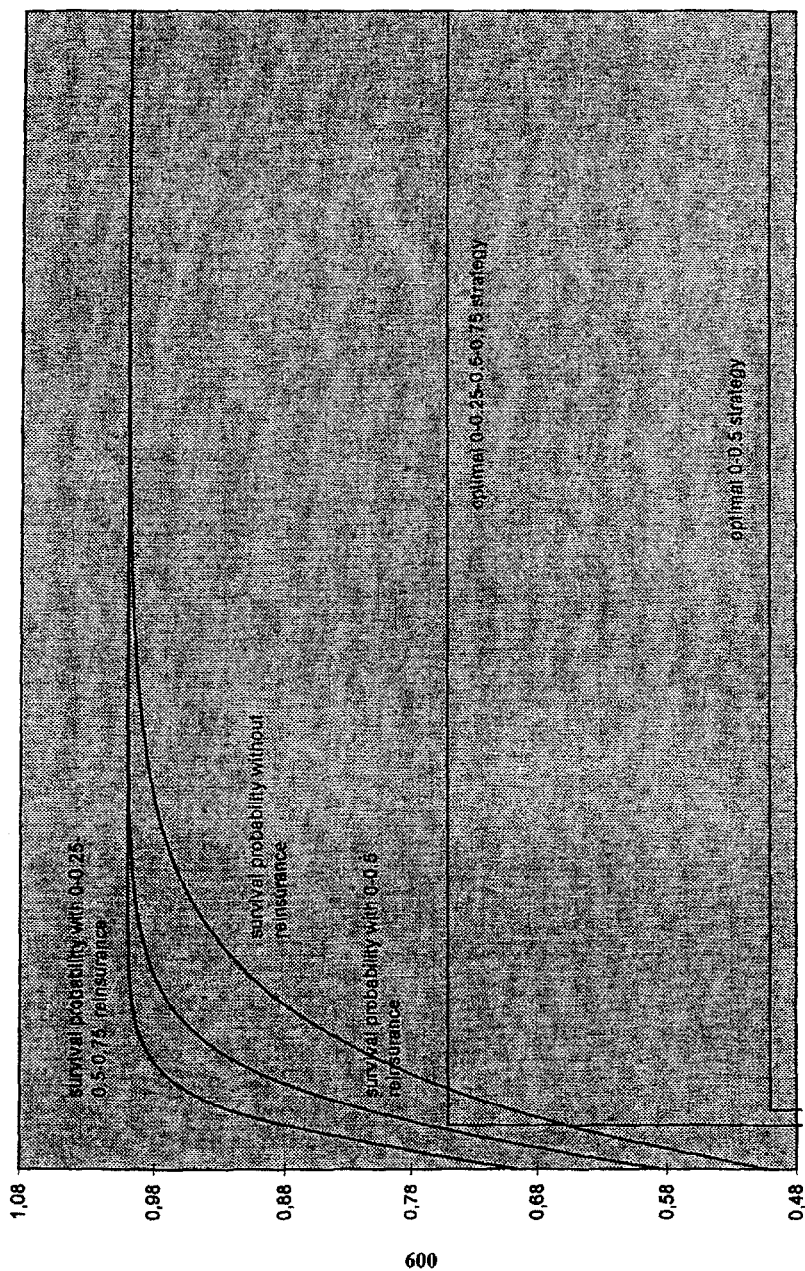


Figure 4.2