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Workshop on Stochastic Asset Models

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Summary.

Asset return scenarios are a vital ingredient to any model of an insurance enterprise. The workshop will present and contrast various approaches which have been tried for generating stochastic economic scenarios. We consider models within the following classes:

Type of Model

Random Walk

Autoregressive

Fractal

Chaotic

Equilibrium

Important Examples

Binomial, Brownian motion

Wilkie, Tilley

Mandelbrot

Logistic return model

Cox-Ingersoll-Ross, Gamma

The objective is to provide working examples of the main approaches, rather than statistical investigations. We will then outline suitability criteria for the various models and see how they square up.

Introduction and Limitations

The applications of asset models in general insurance include profit testing, solvency testing, asset-liability studies and portfolio selection. This workshop is deliberately short on applications, and instead we concentrate on the models themselves. The intention is raise the level of understanding of the models so that we can approach real applications from an intelligent standpoint.

Probably more than any other field of actuarial work, investment models are shrouded in mystery. This is because much of the published work is highly technical and not easily accessible to the layman. There is a wealth of complex mathematics to engage those who delight in such material, but I have found that the best way to understand models is to try them out, and this is the spirit of the workshop.

The paper is arranged in three sections. Section 1 outlines approaches to models which have been tried. Section 2 then suggests some criteria for evaluating these models for actuarial use. The last, and by far the longest section, is the appendices which contain technical specifications of the models discussed and the code which I used to build the examples. Computer readable copies will be available at the workshop - please bring a blank diskette.

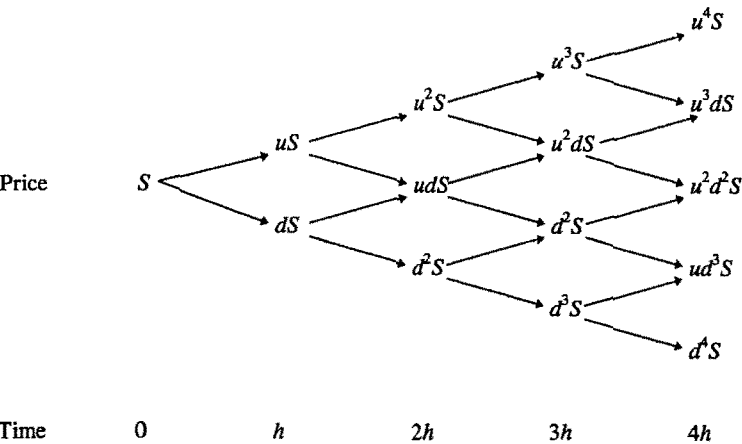
This is a working draft, not a finished paper. If you find any bugs, please let me know. Some of the material is deliberately provocative to generate discussion at the workshop. Please don't quote it without asking me. Needless to say, any views expressed are my own and not necessarily those of my employer or anybody else I have fraternised with. I have tried to use reliable information, but I don't guarantee that all the examples are correct, or that the computer programs will work on your machine.

1. Summary of Model Philosophies

Detailed implementation of the models outlined here are contained in appendices I-V. At this point we will only sketch the main features of the different approaches.

1.1 Random Walk Models

This is the simplest class of asset models. Over a time interval h , the value of an asset can either move up by a factor u or down by a factor d . The same movements are possible over the next time interval and so on, so that the possible paths can be described on a lattice, as follows:



This model is often used in association with a model for cash which earns a constant rate of interest over each time period. Such models are widely used to price options. Interestingly, for option pricing purposes, it is not necessary to know the probabilities associated with a step up or down.

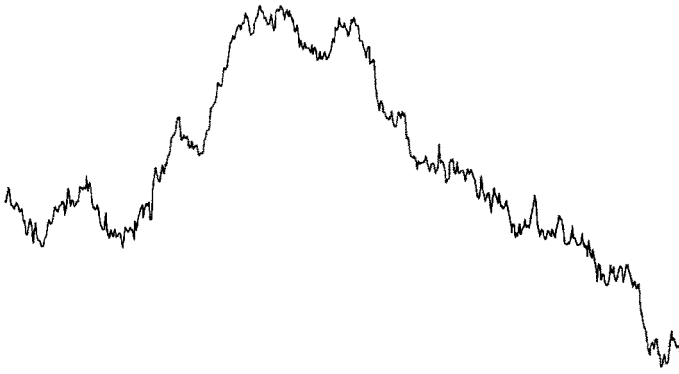
On the other hand, for simulation purposes it is essential to specify the probabilities of an upward and a downward movement. According to one version of the efficient market hypothesis, past price movements are no guide to the direction of future movements, and if we are to believe this hypothesis then the probabilities of the next step being up or down will be constant across the lattice, depending neither on the current position nor on how we got there. In such a situation, the number of up steps in the price over several time steps will have a binomial distribution.

The random walk model is restrictive in only allowing a finite range of outcomes. Instead, one might prefer to use a continuous distribution. The most commonly used distribution is a normal distribution applied to log prices, so that denoting an asset price at time t by S_t we have

$$S_{t+h} = S_t \exp(Z)$$

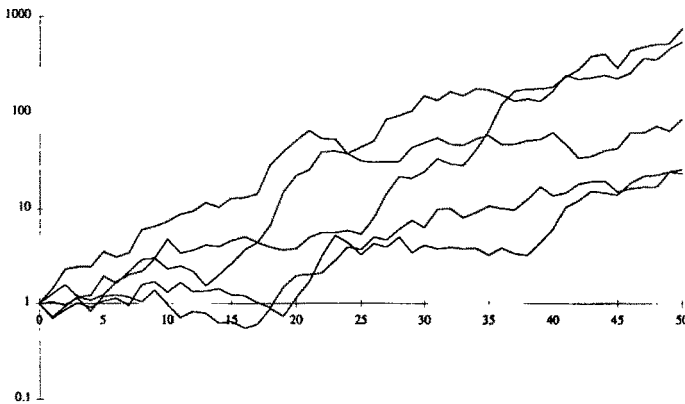
where Z has a normal distribution with mean μh and variance $\sigma^2 h$. It is usually assumed that Z is independent of all prior movements in S . The parameter μ is often called the *drift* while σ is the *volatility*. This is actually a limiting case of the discrete random walk outlined above for small time and price steps. Since the normal distribution has thin tails, models based on it do not tend to produce large jumps, and indeed, in the continuous time limit, asset prices are continuous. Such models are sometimes called *diffusion models*. The alternatives are *jump models* where asset prices are allowed to vary discontinuously with time. The underlying random walk process is sometimes called *Brownian Motion*. An example simulation appears as follows:

Random Walk

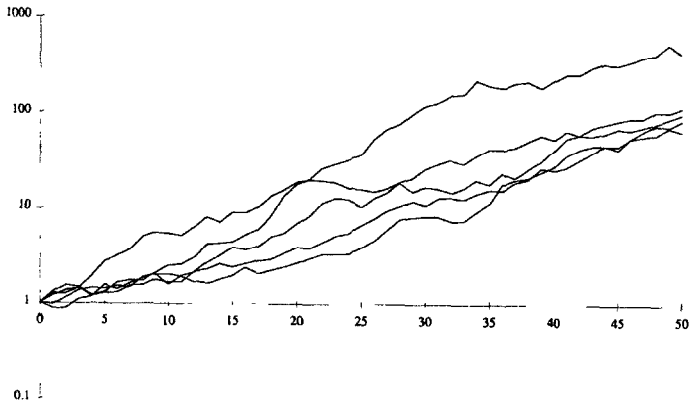


My colleague, Malcolm Kemp, has developed a model based on the random walk for use in asset - liability studies. He has generously given permission for me to reproduce some of his work for this workshop. The following charts illustrate five total return indices under the Kemp model, plotted at annual intervals for fifty years.

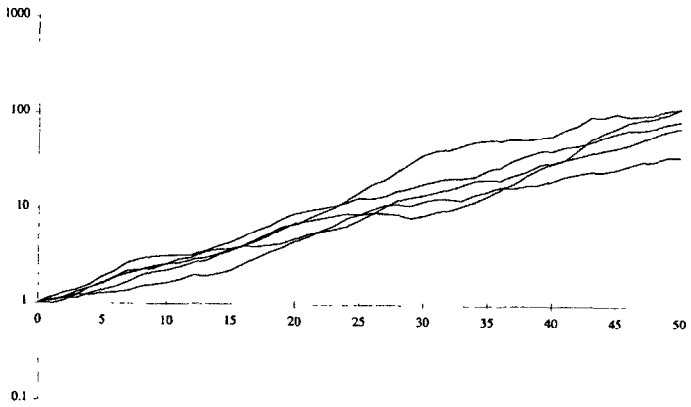
Equity total return: Kemp Model



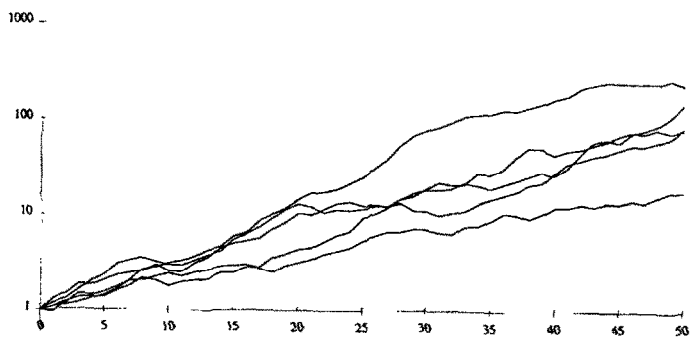
Conventional Gilts total return: Kemp Model



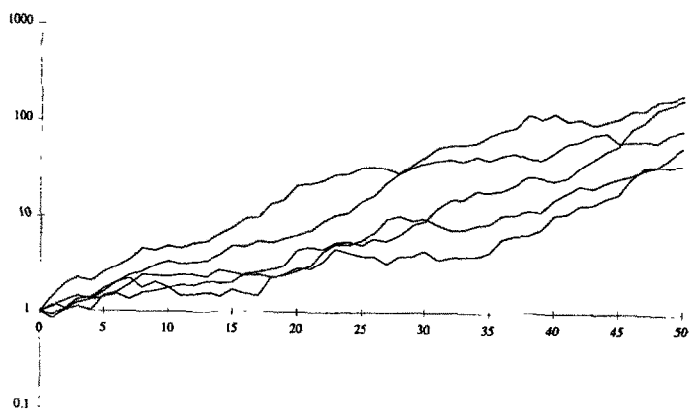
Cash total return: Kemp Model



Index Linked Gilts total return: Kemp Model



Property total return: Kemp Model



1.2 Auto-Regressive Models

It is reasonable to suppose that some economic quantities, such as rates of inflation, interest rates or dividend yields do not perform a pure random walk but instead oscillate around a long term mean. In addition, the process has some memory of where it has come from, so that the expected value at some time in the future is an average of where it is now and the longer term mean. The further into the future one looks, the more this average is biased towards the long term mean.

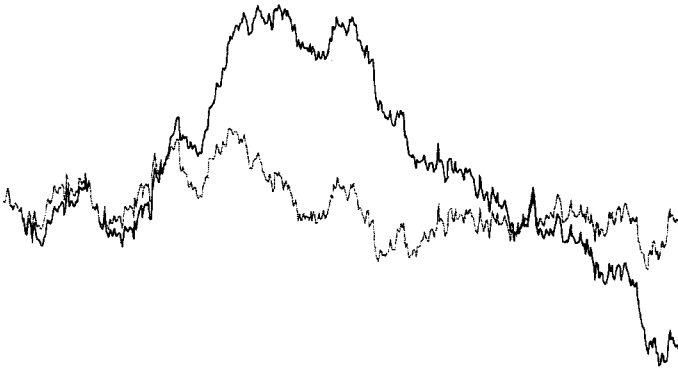
One way of constructing such models is to write

$$X_{t+h} = A^h X_t + (1-A^h)\mu + \varepsilon$$

where X is the economic quantity under scrutiny, μ is the long term mean, A is a measure of the autoregression and ε is an error term. If A is close to zero or h is large then current values of X have little influence on the future, and in the limit, the values of X are independent. On the other hand, if A is close to 1 or h is small then there is very substantial historic dependence, and in the limit we have a random walk. It is common to assume that ε has a normal distribution with mean zero and variance $\frac{1-A^{2h}}{1-A^2}\sigma^2$. This means that the predictive variance starts small

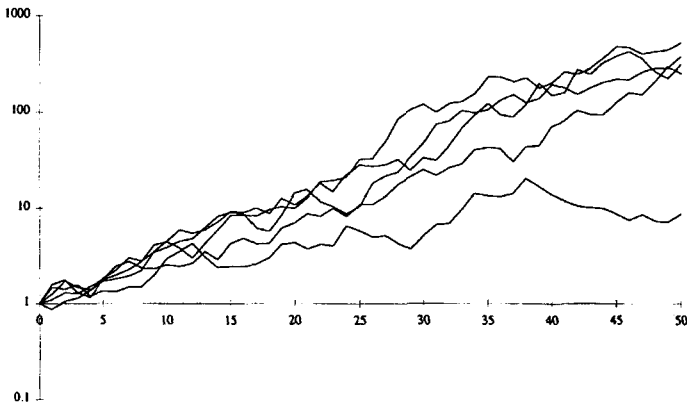
for small h and gradually increases to an upper limit of $\frac{\sigma^2}{1-A^2}$. This expression has of course been rigged to give σ^2 when $h=1$. It is interest to compare an autoregressive process to a random walk, using the same normal residuals. The tendency of the autoregressive process to revert to a long term mean then becomes clear. A chart follows:

Random Walk and Autoregressive Process

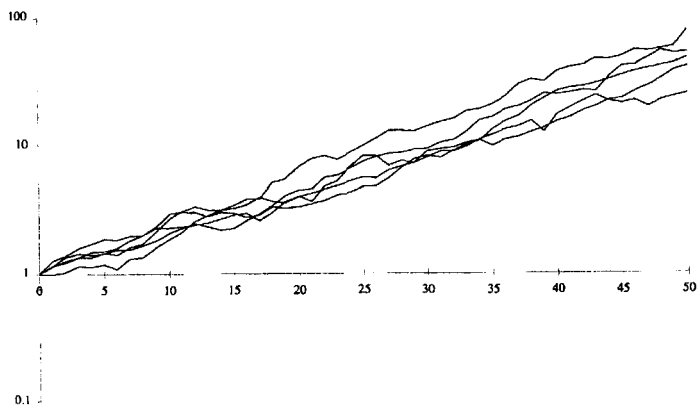


Probably the best known model using autoregressive processes is the Wilkie model. A model by Tilley, describing bonds of various terms, has also seen some applications. Five total return indices from the Wilkie model follow:

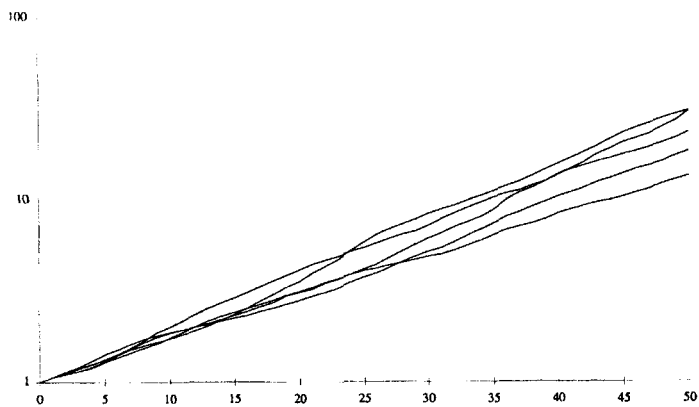
Equity total return: Wilkie Model



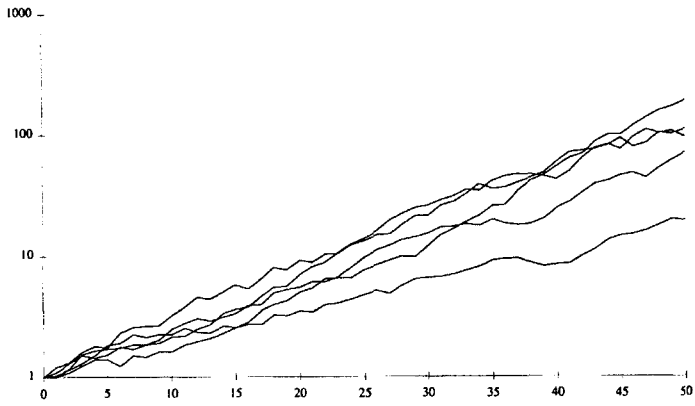
Conventional Gilts total return: Wilkie Model



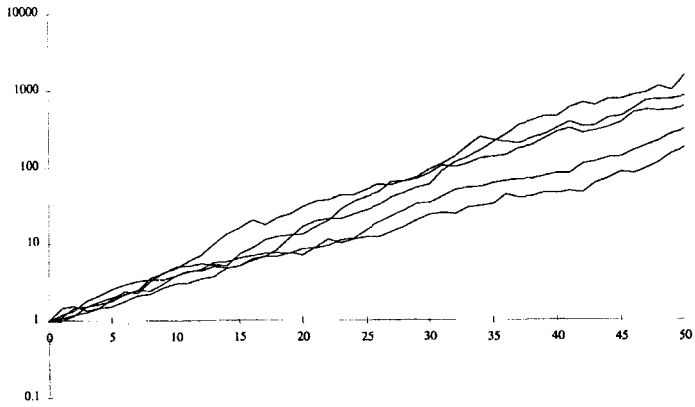
Cash total return: Wilkie Model



Index Linked Gilts total return: Wilkie Model



Property total return: Wilkie Model



1.3 Fractal Models

Suppose that you have in front of you a chart of an asset or index price plotted against time. Suppose further that the printer has bungled, and has forgotten to label both axes. Can you work out by some statistical means what the time and price scales are?

Some authors have suggested that this is impossible, even in principle. In other words, a graph of minute by minute prices looks very similar to a graph of daily prices, which in turn looks like a chart of annual prices. Analogies can be drawn with the study of *fractal sets*, which are self-similar in the sense that when a small part of the set is expanded, it looks like another larger part of the same set. Of course, for asset models this is true only in a probability sense, that is, any particular price chart will not be self similar but the underlying probability law is. If this is true, then it places some important constraints on the kind of probability laws which might be considered for asset returns - the distributions which arise are sometimes called *stable distributions*. Such models were first proposed by Mandelbrot, who is well known outside the financial world for his work on fractals.

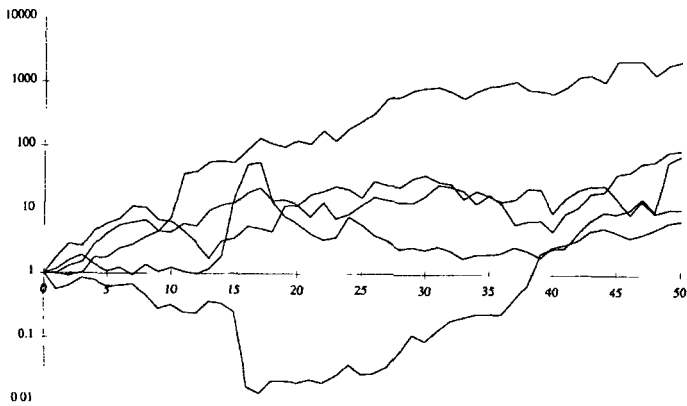
This is a family of distributions which includes the normal distribution, the Cauchy distribution and the Brownian exit distribution. With the exception of the normal distribution, all of these models can produce jumps in the underlying asset values. With the exception of the normal distribution, future asset values do not have finite means or variances, so that other measures have to be employed when analysing risk and return. A sample path from a fractal model is shown below:

Fractal Random Walk

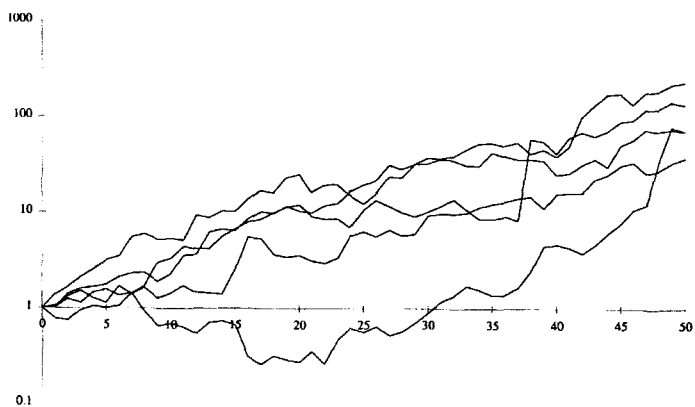


Applying this approach to asset modelling, we might obtain the following total return indices:

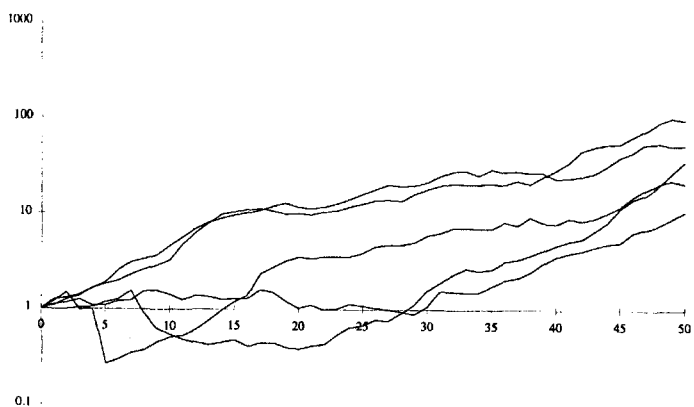
Equity total return: Mandelbrot Model



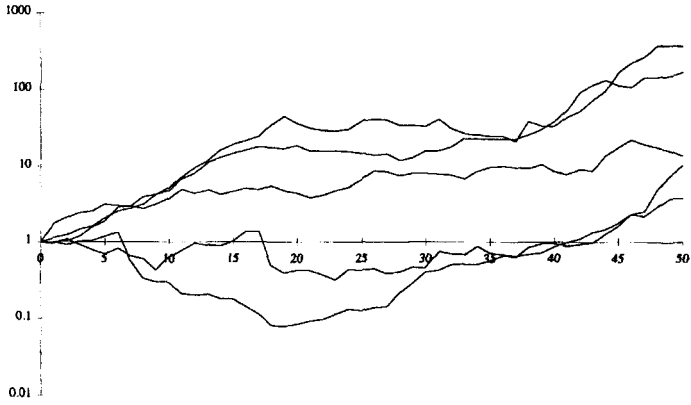
Conventional Gilts total return: Mandelbrot Model



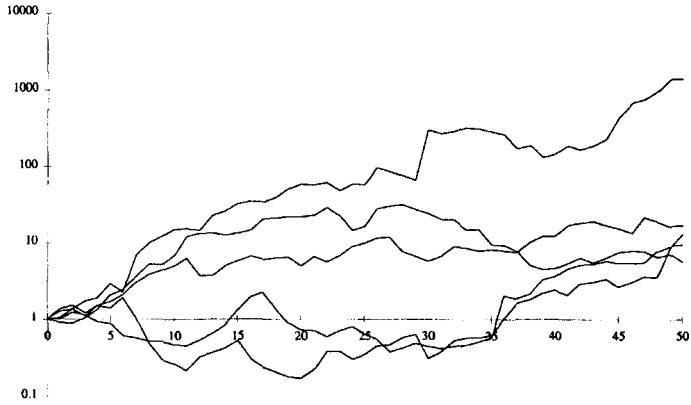
Cash total return: Mandelbrot Model



Index Linked Gilts total return: Mandelbrot Model



Property total return: Mandelbrot Model



1.4 Chaotic Models

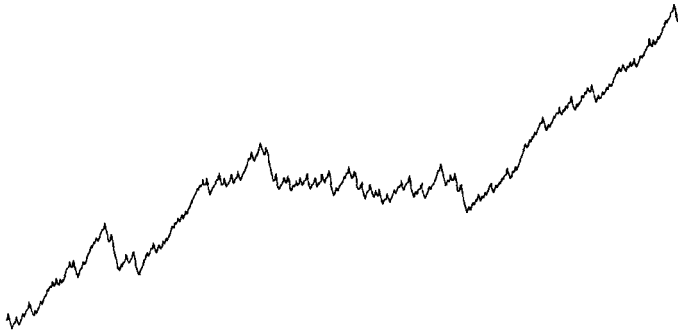
There is an alternative to stochastic modelling. It has been suggested that the state of the world at time $t+1$ is a very complicated function of the world at time t . This function is so complicated that we can't discover it easily, and so the world appears to be random.

From a philosophical point of view this may be appealing, particularly if it is believed that everything in the world, including asset values, is ultimately driven by a set of fundamental deterministic physical laws. Crucially, the chaotic approach implies that in principle fundamental research may be amply rewarded in achieved return.

Can chaos theory be used to generate asset models? In other words, can we find a model which is simple enough to build and program but complicated enough to generate apparently random effects? The answer is again yes. The key is to make the state at time $t+1$ a rather simple (but not linear) function of the state at time t , so that after applying the rule n times, the state at $t+n$ is rather a complicated function of the state at time t .

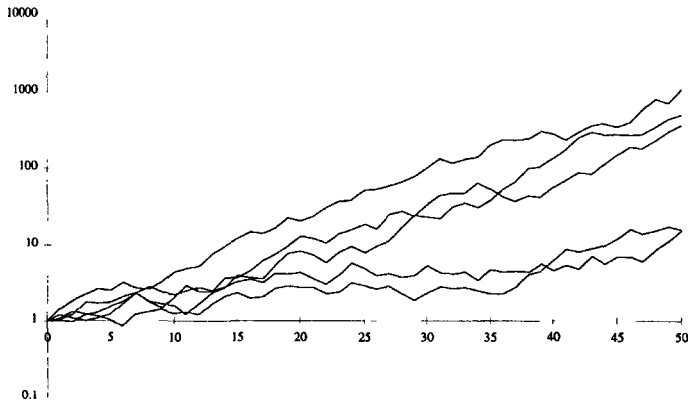
One feature of this class models is that if the world is in the same state at two different points in time, then the projections into the future from those points in time will be identical. Such a property is undesirable for investment series - we expect a total return index to rise over time, not get stuck in endless cycles. However, it is quite plausible that returns could follow a cyclical pattern - indeed, spotting repeated patterns in returns is the basis of technical analysis. Quite reasonable chaotic models can then be built with return as the driving state variable. A sample cumulative plot of chaotic variables is shown below:

Chaotic Random Walk

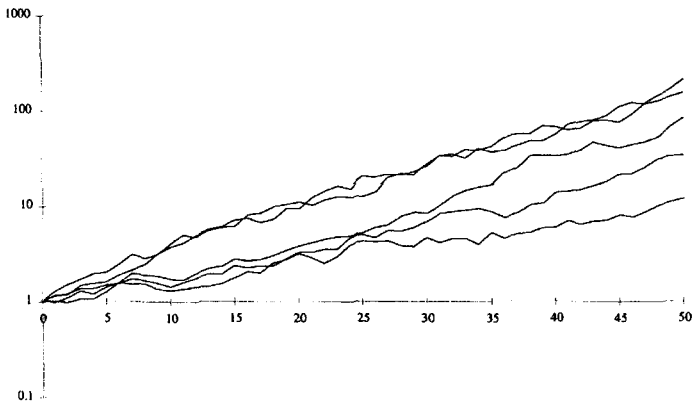


I have chosen a chaotic map made famous by Professor May of Oxford University. I have adapted his model, originally applied to population biology, to produce financial scenarios. The usual set of indices follows:

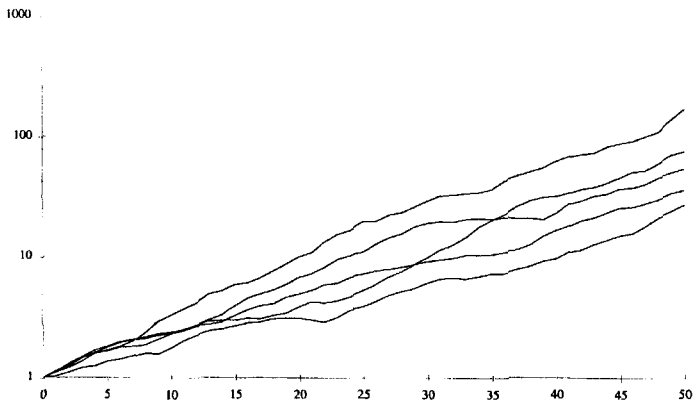
Equity total return: May Model



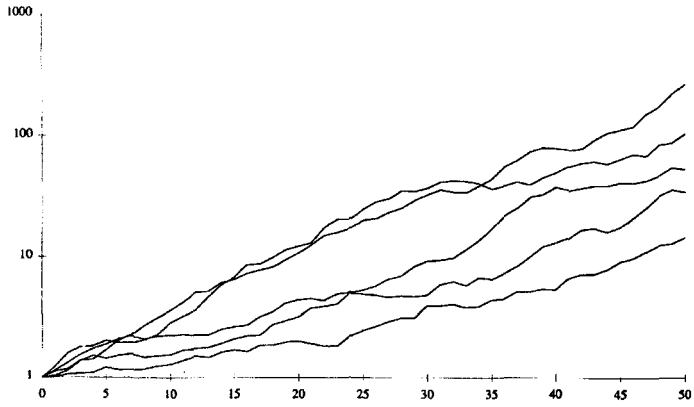
Conventional Gilts total return: May Model



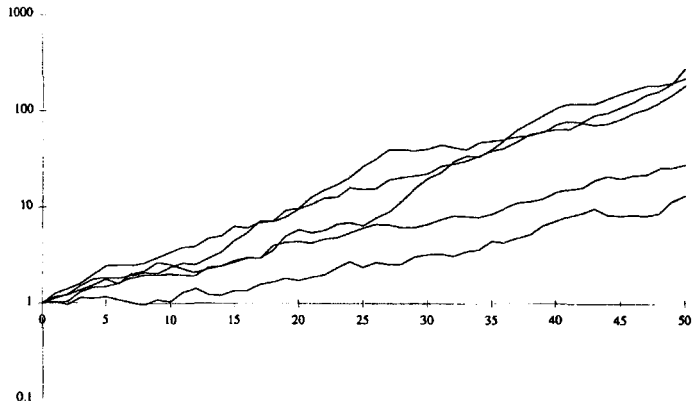
Cash total return: May Model



Index Linked Gilts total return: May Model



Property total return: May Model



1.5 Equilibrium Models

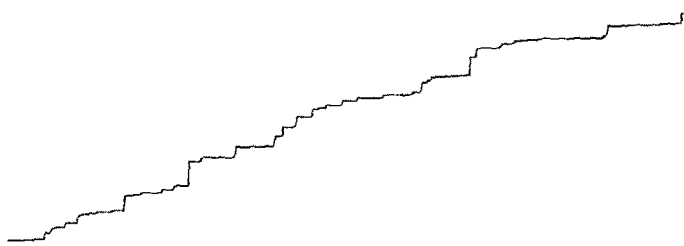
The concept behind equilibrium models is that the market might as well be made up of a number of identical simple investors. In other words, although each investor may in reality have a rather complicated agenda, in sum total all these wrinkles even out. It follows that the investment policy of the average simple investor will involve holding each of the available investments in proportion to the market capitalisation, since in total they must account for the market supply. If this is the case, it follows mathematically that certain relationships must hold between risk and return. The simplest example of such a relationship is the Capital Asset Pricing Model.

One of the first applications of this principle was the Cox-Ingersoll-Ross interest rate model. This was derived from a model describing the returns available from economic production. It is then argued that all debt instruments must have a borrower and a lender, so that in net terms there are no debt instruments, only shares in production. Supposing that the average investor has a logarithmic utility function, the authors wrote down the condition that he will wish to invest only in production, desiring neither to borrow nor to lend. This will only happen if the prices of the various debt securities satisfy certain constraints.

The intention behind the use of equilibrium models is to piggy-back off everybody else's research. In other words, if we know the portfolio decisions which investors are actually making, and we can guess at the objectives they are trying to achieve, we can infer what they must believe about asset returns. This process relies on the market following the right model, on our supposed objectives being accurate, and on investors doing their sums correctly.

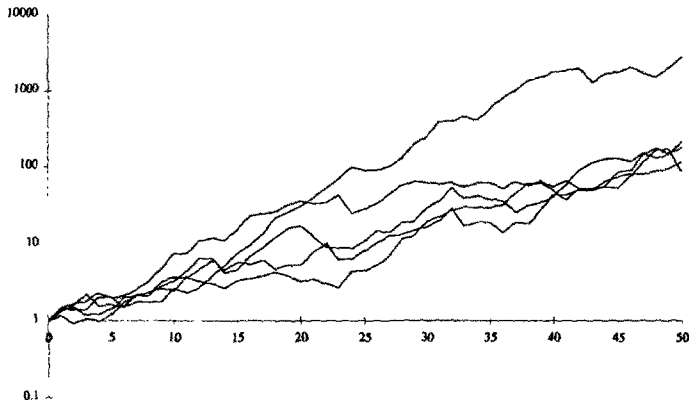
I rather like Mandelbrot's idea of jumps. I have borrowed the concept for my own equilibrium model. However, rather than using fractal processes which are analytically complex, I have used Γ processes which are rather more straightforward to handle. A Γ process has only positive jumps, so in order to get two-way jumps, one must subtract two Γ processes. A sample Γ process is as follows:

Gamma Process

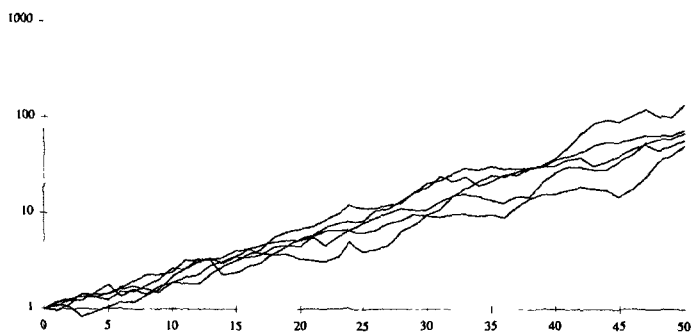


The usual five total return indices now follow:

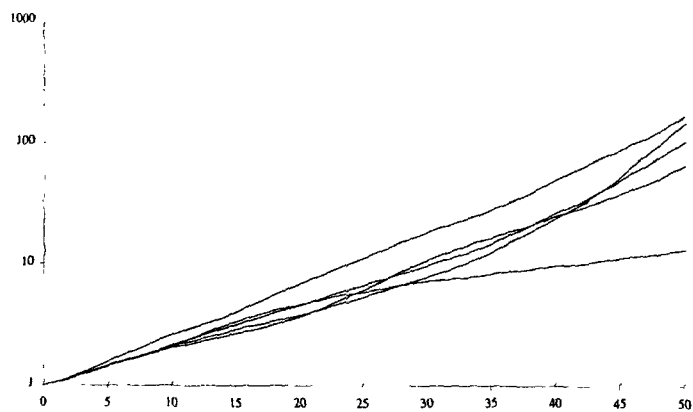
Equity total return: Smith Model



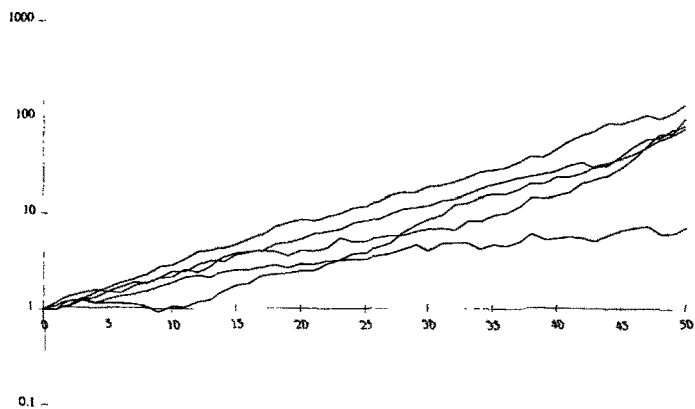
Conventional Gilts total return: Smith Model



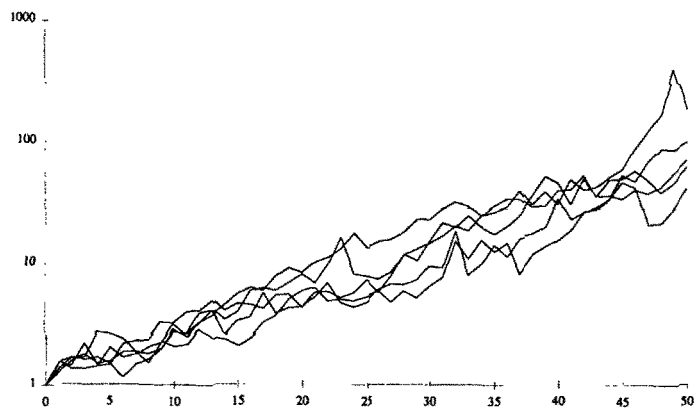
Cash total return: Smith Model



Index Linked Gilts total return: Smith Model



Property total return: Smith Model



2. Criteria for Actuarial Models

We have seen lots of approaches to constructing models. How can we decide which one is best for an insurance application? Any decision is necessarily a compromise between various conflicting objectives. Some of the points to bear in mind are as follows:

2.1 Fit to Historic Data

To judge by the actuarial literature, one might imagine that this is the only intellectually respectable criterion which might be applied to discriminate between models. If a model does not fit the facts, it should be thrown out and a better one constructed. The best model is the one which best fits the facts. This leads to models calibrated to historic data by least squares or other similar goodness of fit measures.

There is a voluminous actuarial literature explaining on statistical grounds why certain economic models should be thrown out. Empirically, the problem is that in the economic field, statistical tests virtually always seem to reject any simplification of the model. In other words, for most widely used models one can postulate a relationship not exhibited by the model, fit the relationship and find that the postulated relationship is statistically significant. Thus guided by classical statistics, ever more complex models can be constructed, and with each new parameter introduced, the standard errors of the parameters already fitted increase. In theory, this process should stop at some point when the model has been fully specified, but in my experience it doesn't usually turn out that way!

Does this mean that we have no hope of building useful models via the statistical route? No, it doesn't. It seems that there are some features of models, for example, the distributions of short term residuals, which can readily be estimated by classical statistics. Other features, such as long term means or the frequency of catastrophes, are much harder to calibrate. It turns out that the exact distribution of short term residuals often has only a limited impact on a long term model, while the long term parameters are much more important for many actuarial investigations. This suggests that it is worth using a small amount of statistics getting a description of short term residuals which is about right, and then applying educated guesswork to the longer term parameters.

Many commentators believe that we will never have a correct model for the economy. The key to economic modelling is to understand the relationship between our imperfect models and the real world. As a result, for a fixed significance level and given sufficient data, any proposed model will ultimately be refuted. We should expect this and not be worried. So what if a model is wrong ? - so is everybody else's.

More to the point, many models which can be statistically disproved still find widespread use. For example, the Black-Scholes model of option pricing implies a relationship between the prices of options with different strike prices. This relationship is not always observed to hold between market prices. Does this mean that nobody uses the Black-Scholes model? No - indeed most option traders still use this model, with suitable adjustments.

So how do our five models stack up against historic data? Both the Kemp model and the Wilkie model fall down on the non-normality of returns or other appropriate residuals. Empirically, returns tend to be *leptokurtotic*, which means

that they have fatter tails than a normal distribution. Fractal models have a much better statistical fit, allowing for the fat tails. The chaotic model is of course a hopeless historical fit because the historic data does not exactly fit the proposed deterministic model. Finally, my own model falls down because it assumes a functional relationship between the prices of bonds of different durations, and this functional relationship has not held historically. My model also does not exclude negative interest rates. No doubt further problems will become evident over time as actuaries use these models.

2.2 Economic Theory

There are many economic theories of how the market should behave. There has been some controversy in the actuarial profession concerning models which describe efficient markets. Much of this debate has centred around whether markets are or are not efficient, and the jury still seems to be out on this matter.

I have found that there are often good reasons for assuming an efficient market within an insurance model whether or not the real world is efficient. This may sound surprising - many would say that an efficient market model is altogether too cosy, and we should be prepared for the kind of nasty surprises which only occur when the market is inefficient. I disagree with this view, and claim that an efficient market should be regarded as a conservative assumption. If the market is inefficient then there are profitable speculative trading opportunities to exploit. However, it seems imprudent to suppose that your investment managers will find these opportunities to the exclusion of the rest of the market. The worst case is that there are no such opportunities, and this is what happens in an efficient market.

One area where economic theory is essential is in describing new products. When index linked gilts were first issued in 1981, traders had to build models to trade them without the benefit of historic return data. The same is true for many derivative products. *In the property market, the shift from fixed rents to upward-only reviews made a step change in how property investments behave.* Now, the upward only review seems to be out of fashion, and more flexible short term leases are becoming more widespread. A comprehensive economic theory will describe the framework within which the market will value arbitrary cash flows, and this is what is required to make intelligent assessment of new financial products.

How then do the models outlined here fit in with existing economic theory? The Kemp Model, the Mandelbrot model and my model are all consistent with efficient markets. The May model is totally inconsistent with efficient markets, in the sense that all returns can be forecast exactly. The Wilkie model lies somewhere in between. There is a considerable amount of information in past data which can be used for profitable trading, but this does not totally guarantee trading profits because of the impact of the random noise terms.

2.3 Inputs and Outputs

Several actuarial models have been proposed to describe the long term behaviour of financial markets. On the other hand, for a model of an insurance enterprise, one would wish to start from the current economic situation, and today's prices.

For example, an autoregressive model might explain how interest rates will revert to a long term mean from their current level. This prediction may be inconsistent with the interest forecasts implicit in bond prices. Out of the models described here, my model is the only one which explicitly takes the yield curve as input in

order to infer future expected price moves. By the same token, option prices may contain information regarding return distributions which differ from the distributions assumed under a specific model.

For many purposes it is important for models to produce a rich set of outputs, including for example bonds yields for all terms, forward prices for various assets, or option prices for various strikes. However, such a rich model will necessarily be difficult to calibrate to current prices, firstly because of the volume of data involved, and secondly because an exact fit requires many parameters. My own model exploits the analytical properties of the Γ distribution in order to make this calibration straightforward.

More impoverished models which lack some of the outputs may be easier to calibrate. However, this does not mean that the problem has gone away; rather, any discrepancies between the model and the market may be harder to detect if fewer price checks can be carried out. There is substantial information contained in today's market prices and ideally such information ought to be taken into account when formulating a model. Furthermore, absence of some outputs, particularly a term structure of interest rates, can seriously limit the applicability of a model.

One infuriating feature of published models is that they have all been built for different purposes, and hence tend to have different inputs and outputs. This confounds comparison between models. For the models described here, I have taken what is, in effect, a lowest common denominator in order to show a consistent set of outputs. The outputs I have chosen are:

An inflation index

Total return indices for:-

Equities

Gilts

Cash

Index-Linked Gilts

Property

Several of the models described are capable of producing more. For example, both the Wilkie model and my own model can separate capital from income in the calculation of total returns. My model also produces a full term structure of interest rates and real interest rates.

2.4 Results

It may seem self-evident that a model should only be accepted if the results are reasonable. From a philosophical point of view, if each step of the construction is properly tested then the model obtained from the combination must be unassailable. Sadly, practice does not bear this out. There are plenty of examples of models whose construction is entirely respectable but where the results are hopeless.

It is therefore imperative that any proposed model must produce plausible answers to simple investment problems. This means not only that selected strategies must produce output which can be explained, but also that the optimal behaviour recommended by the model really does make sense.

One of the tests which I always try is to ask how an investor would behave with a logarithmic utility function. Such an investor will try to maximise $E[\log(P_T)]$ where P is a total return index and T is a time horizon. I allow the investor to switch between asset classes every year, at market value. This is then an example of *dynamic optimisation*, that is, any information which becomes available over time can be used in future investment decisions.

Of course, I am not suggesting that this would be a sensible strategy for an insurer investing to meet liabilities. The logarithmic optimisation should instead be viewed as a standard test, fairly simple to execute, which is useful for comparing models. If the solution to this simple test does not look reasonable, it becomes very hard to argue that the model will be appropriate for the much more complex optimisation involved in running an insurance company.

A common feature of dynamic optimisation problems is that they are hard to solve. This is because a choice made today may restrict the choices available at some time in the future. To optimise a given utility function at some future time horizon cannot usually be achieved by a *myopic strategy* which focuses only on prospective returns one year at a time.

However, in the case of the logarithmic utility function, a myopic strategy is optimal, and this is true mathematically whatever stochastic model is employed. Furthermore, the best myopic strategy is to maximise the same utility function, that is, the log, over a single year. This means that logarithmic investors will follow the same optimal strategy whatever their time horizon.

What are the properties of the optimal portfolio? We denote the total return on the optimal portfolio by P_t and suppose that Q_t is another total return portfolio. The optimality condition of P is that, for any Q and for any $s < t$ we have

$$E_s \left(\frac{Q_t}{P_t} \right) \leq \frac{Q_s}{P_s}$$

where E_s denotes the conditional expectation given information at time s . Let us suppose that the long run returns on P and Q both exist, so we can define returns:

$$R_P = \lim_{t \rightarrow \infty} t \sqrt{\frac{P_t}{P_0}} - 1; R_Q = \lim_{t \rightarrow \infty} t \sqrt{\frac{Q_t}{Q_0}} - 1$$

Then, by the supermartingale convergence theorem, $R_Q \leq R_P$ with probability 1. In other words, the logarithmic investor maximises the long term return. Given the volume of actuarial breath expended on the concept of long term return, the optimal behaviour of the logarithmic investor should be of great interest to actuaries. The fact that the optimal behaviour follows a myopic strategy implies that short term market value movements do in fact matter, contradicting conventional actuarial wisdom on long term returns.

A simple thought experiment can give substantial insight into the usefulness of statistical fitting. Let us first consider a pair of assets with the same lognormal return distribution. Let us suppose further that the annual outperformance of one asset over the other has a standard deviation of 15%. These two assets are observed over fifty years, with a view to calibrating a model and selecting an asset allocation to maximise the long run return.

We can see by symmetry that the theoretically best portfolio is 50% in each asset class. However, the returns observed over the 50 years will have some degree of random noise associated with them. The actuary will therefore form the opinion that one asset is likely to outperform the other. In some circumstances, this out-

performance will be so extreme as to justify 100% investment in a single asset class. Presumably this should only happen in exceptional circumstances?

Actually, no. This problem arises 60% of the time. In other words, when the true optimum is a 50/50 split between asset classes, the statistical fitted model will recommend a 100/0 split 30% of the time, a 0/100 split 30% of the time and something in the middle 40% of the time. It seems to me that this observation completely undermines any suggestion that asset models fitted to historic data can be useful for portfolio selection. The statistical method is the best that can be done, but the error of parameter estimates is impossibly large in relation to the sensitivity of the answer to those parameters.

Some other input must be found to make the results more stable. If we are restricted to historic price information, we may as well all pack up and go home! One direction from which we can get some help is to use economic theories. However, if we expect these theories to be testable using price histories, we will be disappointed, because if there is enough data to test the theory, then there is enough data to calibrate the model and so we don't need the theory. This means that stochastic modelling necessarily involves untested hypotheses and professional judgement. To the uninitiated this looks like witchcraft, but to those in the know, it is educated guesswork plus the ability to sell a story.

How then do our five models perform under optimisation?

Both the Kemp model and my model recommend 100% equity for the logarithmic investor. I have deliberately calibrated the models so that this happens. This forces the models to produce reasonable answers, at least under this simple optimisation. In some of the more complex optimisations I have tried, both these models seem to produce answers which are not totally unreasonable.

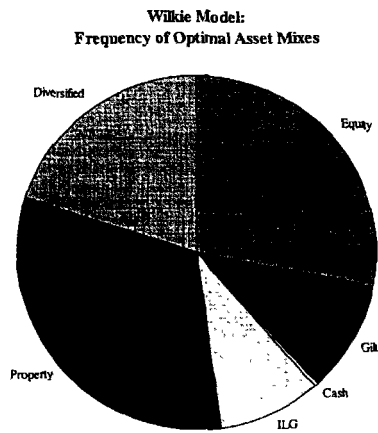
Under the fractal model, there is some rather complicated algebra involved in finding the optimal portfolio for the logarithmic investor. However some general principles can be distilled. Firstly, the composition of the optimal portfolio (i.e. the proportion by market value invested in each sector) does not change from time to time, so that, even allowing for the facility of dynamic trading, it turns out to be best not to use it. Secondly, the optimal portfolio can never be 100% in a single asset class. This arises because the fractal processes allow for very significant probabilities of total collapse for a particular market. The model thus ascribes great benefit to diversification.

The May model has some undesirable properties with regard to optimisation. Since the whole framework is deterministic, one can forecast prospective returns with absolute accuracy. In particular, one can arrange to be 100% invested in the best performing sector each year, switching year by year into the class which will do best for the following year.

Finally, let us consider the Wilkie model. I have analysed the optimal strategies for a logarithmic investor. In particular, I have considered how frequently different strategies would be optimal over a period of 1000 years. Rather surprisingly, a 100% investment in a single class is a good idea most of the time, if switching is permitted on an annual basis. We can see that the Wilkie model is coming rather closer to the chaotic model than an efficient market model.

This means that in order to use the Wilkie model for investment decisions, one must somehow constrain the feasible investment strategies so that such extreme allocations do not arise. One way of doing this is to assume a constant asset mix. However, this may rule out genuinely beneficial dynamic strategies, such as taking a greater equity exposure when a healthy solvency position permits, but retreating into gilts when solvency is tighter.

The relative frequency with which each class is optimal for the Wilkie model is shown in the pie chart below.



Such a widely fluctuating asset allocation does not make much sense to me. The problem does not seem to be caused by the particular parameter values I have used, but rather reflects a structural feature of the model. Part of the problem is that, in the Wilkie model, a high yield is usually a good predictor of price rises, so that frequent switching on a yield based rule can be extremely profitable. Another problem seems to be the use of the normal distribution, which more or less excludes catastrophic events. The benefits of diversification may therefore be understated by the Wilkie model.

2.5 Budgets and Deadlines

There are two major budget busters in stochastic modelling. The first is the model building stage, where there are always unforeseen complications which mean that a model takes longer to build than you think. Indeed, even biasing the budget to allow for this feature, the budget will still overrun.

The second budget buster is at the interpretation stage. Here, the problem is that some models are much too interesting. Indeed, the models have so many interesting features that an asset-liability study tells you more about the model than about the assets and liabilities.

One way of reducing development time is to build models for different asset classes according to a standard template. This forces assets into a particular mould which may or may not be strictly appropriate, but does at least allow a model to be bolted together from its constituent parts with ease. The random walk models exploit this fully, while my own model does distinguish between cash-type assets and bond-type assets. The Wilkie model differentiates further, with essentially different models for each asset class, although there are common underlying philosophies which result in a family likeness.

One temptation when building a model is to add *ad hoc* fixes to cover over unrealistic aspects of the model. The danger here is that an effect is included without the corresponding cause. For example, one might wish to consider adjusting a gilt model to describe corporate debentures. However, if the yield is adjusted upwards without an appropriate default risk being introduced, then debentures will appear spuriously attractive relative to other investment classes. By the same token, my model allows the possibility of negative interest rates. This is sometimes inconvenient, although their consequences are usually easy to guess.

It would be easy to fix the model so that if interest rates went below, say, 1% then they were reset to 1%. The problem with this approach is that the entire bond model has been constructed to be consistent with interest rate forecasts which could go negative. An ill considered fiddle factor applied only to the short rate would generate all sorts of subtle anomalies in the bond markets, with knock-on effects on the results which might be hard to predict. I prefer to live with the occasional negative interest rate rather than play around with the guts of the model.

The key to a speedy interpretation is to avoid interesting models. For example historically, share dividend yields have been able to predict price changes, which suggests a switching rule between equities and gilts. This is an interesting observation which is reflected in the Wilkie model. When conducting an asset liability study, provided that suitable dynamic optimisation is implemented, a yield based switching strategy will often be recommended. It would be easy to assume that this strategy had something to do with the particular asset - liability matching problem of the insurer, but such a conclusion would be wrong. Instead, this is a feature of the underlying economic model.

Producing an uninteresting model usually means a model where the optimal strategy for a simple investor is a simple passive strategy, which in practice means an equilibrium model. Any differences between the recommended asset mix for an insurer and the simple passive strategy will then genuinely reflect differences in business objectives. Using uninteresting models has the best chance of delivering a job on time and to budget.

The problem is that an interesting feature of a model may just turn out to be a genuine reflection of the real world. In that case, using a less interesting equilibrium model, an actuary can miss significant profit opportunities.

My approach to this problem is to take an equilibrium model as a bench mark, and to adjust parameters to reflect perceived inefficiencies. Such an adjustment should only be applied to the parameters for a few years, until the inefficiency has worked itself out of the system. In other words, if pushed, I can be persuaded to include specified mispricings as temporary adjustments to a base model. The recommended asset mix is then biased towards those assets which are identified as cheap.

At the end of such an exercise, a proposed asset mix can be compared with the market capitalisation of each asset class. Any differences can be ascribed either to:

Different investment objectives from the rest of the market

or

Differing views on the attractiveness of particular asset classes

It is essential to carry out such an analysis as a quality control measure. As well as being the output of a model, the result must be justified on intuitive grounds.

It is remarkable that many asset-liability studies still assume constant asset mixes over time. Plainly, such strategies would not normally be capable of exploiting short term market anomalies. It seems fundamentally silly to spend a lot of time researching anomalies and adjusting models if the trading benefits of the anomalies are excluded by the modelling approach.

Acknowledgements

I would like to record my gratitude to my colleagues Malcolm Kemp and Steve Nagle for many stimulating discussions and bright ideas. However, all errors and omissions remain my sole responsibility.

Appendix A: Model Descriptions

A.1 General Matters

For the purpose of comparability, the models in this paper have been built to describe the same economic variables, sampled on an annual basis. The variables are:

An inflation index

Total return indices for:-

Equities

Gilts

Cash

Index-Linked Gilts

Property

That is, six indices in all. Some of these models could also produce other output, but these have not been implemented here.

A.2 Random Walk Models

Under a pure random walk model, the expected returns on any asset class would be constant. In reality, we know that in many cases, investors will require higher returns at times of high inflation, and lower returns when inflation is more subdued. In order to obtain a useful model, it really is necessary to allow some cyclical inflation behaviour, and this implies a step away from a pure random walk. The model described here is due to Malcolm Kemp, which draws on the inflation model originally built by David Wilkie.

The model is driven by a series of random noise terms. At each time t , six noise variables are generated randomly. We denote these noise variables by $Z_i(t)$ for $i=1,2,3,4,5,6$. The $Z_i(t)$ are independent for distinct i and distinct t . In the example, these have a normal distribution with mean zero and variance 1.

The Wilkie model for inflation describes a retail prices index $Q(t)$ and a force of inflation $I(t)$ related by

$$Q(t) = Q(t-1) \exp\{I(t)\}$$

and $I(t)$ follows an autoregressive process of the form:

$$I(t) = QA * I(t-1) + [1 - QA] * QMU + QSD * Z_0(t)$$

We can see that this is of the form in 1.2. We can interpret QA as the speed of mean reversion, QMU as the long run mean and QSD as the amount of noise in the system.

We denote the total return indices by $R_i(t)$. The proposed model describes the total real returns as a geometric random walk with drift. The noise terms in the random walk are taken to be linear combinations of the noise variables, so that

$$\frac{R_i(t)}{Q(t)} = \frac{R_i(t-1)}{Q(t-1)} \exp\left(\mu_i + \sum_{j=1}^6 c_{ij} Z_j(t)\right)$$

It remains only to determine the values of μ_i and c_{ij} in order to run the model.

The calibration of c_{ij} is broadly obtained from historic data. We can consider the vector:

$$\mathbf{X}(t) = \begin{pmatrix} I(t) - QA * I(t-1) - [1 - QA] * QMU \\ \log\left(\frac{R_1(t)}{Q(t)}\right) - \log\left(\frac{R_1(t-1)}{Q(t-1)}\right) - \mu_1 \\ \log\left(\frac{R_2(t)}{Q(t)}\right) - \log\left(\frac{R_2(t-1)}{Q(t-1)}\right) - \mu_2 \\ \log\left(\frac{R_3(t)}{Q(t)}\right) - \log\left(\frac{R_3(t-1)}{Q(t-1)}\right) - \mu_3 \\ \log\left(\frac{R_4(t)}{Q(t)}\right) - \log\left(\frac{R_4(t-1)}{Q(t-1)}\right) - \mu_4 \\ \log\left(\frac{R_5(t)}{Q(t)}\right) - \log\left(\frac{R_5(t-1)}{Q(t-1)}\right) - \mu_5 \end{pmatrix}$$

By hypothesis, $\mathbf{X}(t)$ has zero mean. Furthermore, we have the identity:

$$\mathbf{X}(t) = \mathbf{CZ}(t)$$

where the matrix \mathbf{C} is given by

$$\begin{pmatrix} QSD & 0 & 0 & 0 & 0 & 0 \\ c_{10} & c_{11} & c_{12} & c_{13} & c_{14} & c_{15} \\ c_{20} & c_{21} & c_{22} & c_{23} & c_{24} & c_{25} \\ c_{30} & c_{31} & c_{32} & c_{33} & c_{34} & c_{35} \\ c_{40} & c_{41} & c_{42} & c_{43} & c_{44} & c_{45} \\ c_{50} & c_{51} & c_{52} & c_{53} & c_{54} & c_{55} \end{pmatrix}$$

and \mathbf{Z} is the matrix of noise terms. The variance-covariance matrix \mathbf{V} of \mathbf{X} is then given by $\mathbf{C}\mathbf{C}^T$ where $()^T$ denotes a matrix transpose. This matrix can be estimated empirically from historic data, producing an estimate of \mathbf{V} . In these worked examples I have used the following variance covariance matrix:

$V =$

Inflation	0.0041	-0.0037	-0.0046	-0.0030	-0.0003	-0.0016
Equities	-0.0037	0.0514	0.0198	0.0038	0.0043	0.0111
Gilts	-0.0046	0.0198	0.0201	0.0050	0.0046	0.0038
Cash	-0.0030	0.0038	0.0050	0.0038	0.0004	0.0011
ILG's	-0.0003	0.0043	0.0046	0.0004	0.0048	0.0003
Property	-0.0016	0.0111	0.0038	0.0011	0.0003	0.0133

The determination of C is not unique. However, there is one particularly useful algorithm for determining a possible C using Gram-Schmidt Orthonormalisation (sometimes also called Cholesky decomposition). This results in a matrix C which is in *lower triangular form*. In the example, the value of C I have used is

Inflation	0.0641	0.0000	0.0000	0.0000	0.0000	0.0000
Equities	-0.0577	0.2193	0.0000	0.0000	0.0000	0.0000
Gilts	-0.0724	0.0714	0.0986	0.0000	0.0000	0.0000
Cash	-0.0474	0.0047	0.0123	0.0377	0.0000	0.0000
ILG's	-0.0042	0.0183	0.0299	-0.0072	0.0592	0.0000
Property	-0.0243	0.0443	-0.0109	-0.0035	-0.0060	0.1029

It can be shown that different matrices C with the same value V of CC^T generate essentially the same model, that is, our particular choice of C is without loss of generality.

I have taken the mean inflation QMU directly from Wilkie's 1995 paper describing his model.

There are a number of ways in which the μ_i could be determined. I have used an equilibrium argument, where it is assumed that a long term investor would be ideally in 100% equity, whose total return I take as R_1 . The optimality of equity

implies that for each i , $E_{t-1} \left(\frac{R_i(t)}{R_1(t)} \right) = \frac{R_i(t-1)}{R_1(t-1)}$, or in parametric terms,

$$\mu_i = \mu_1 - \frac{1}{2} (v_{1i} - 2v_{ii} + v_{ii})$$

This gives the relative values of the means. The absolute value is fixed by considering a particular asset class where the expected real return may be forecast with some confidence. I take index linked gilts, with an assumed expected real return of 4%. This gives the mean vector:

Equities	6.39%
Gilts	4.80%
Cash	4.00%
ILG's	4.00%
Property	4.26%

A.2 Auto-Regressive Models

Probably the best known autoregressive model is the Wilkie model, most recently updated in 1995. Readers wanting more detail should consult this excellent paper, which provides much of the motivation behind the model. In this note, I have restricted myself to the mechanics. The implementation here omits the wage inflation and foreign assets also described by the model, for comparability with the other models also presented. The key mechanics of the model revolves around a number of state variables, each of which performs an autoregressive process. The state variables for this implementation of the Wilkie model are:

I , the force of inflation

YN , the element of the dividend yield not explained by retail prices

CN , the element of long term interest rates not explained by retail prices

BD , the relationship between long and short term rates

$\ln Z$, the log of the property yield

$\ln R$, the log of the real yield on index linked stocks

All the state variables have a mean value around which they vary; these means are defined by constants ending in MU . The distance from the mean at time t is a fraction distance at time $t-1$ plus a noise term. The fraction ends in A , and lies between 0 and 1, while the noise terms are of the form $SD * Z$, where SD is a standard deviation and Z are a series of iid $N(0,1)$ random variables (sometimes called *white noise*).

The state variables evolve stochastically according to the equations:

$$\begin{aligned}
 I(t) &= QMU + QA * [I(t-1) - QMU] + QSD * QZ(t) \\
 WN(t) &= WMU + WSD * WZ(t) \\
 YN(t) &= \ln YMU + YA * [YN(t-1) - \ln YMU] + YSD * YZ(t) \\
 CN(t) &= CA1 * CN(t-1) + CY * YSD * YZ(t) + CSD * CZ(t) \\
 BD(t) &= BMU + BA * [BD(t-1) - BMU] + BSD * BZ(t) \\
 \ln Z(t) &= \ln ZMU + ZA * [\ln Z(t-1) - \ln ZMU] + ZSD * ZZ(t) \\
 \ln R(t) &= \ln RMU + RA * [\ln R(t-1) - \ln RMU] + RBC * CSD * CZ(t) \\
 &\quad + RSD * RZ(t)
 \end{aligned}$$

There are also some moving averages of inflation which must be calculated. These are *DM*, *CM* and *EM*, which are defined inductively as follows:

$$\begin{aligned}
 DM(t) &= DD * I(t) + (1-DD) * DM(t-1) \\
 CM(t) &= CD * I(t) + (1-CD) * CM(t-1) \\
 EM(t) &= ED * I(t) + (1-ED) * EM(t-1)
 \end{aligned}$$

We have seen how the state variables perform autoregressive processes. From these state variables are constructed the actual economic variables described by the model. These variables are:

Y, the dividend yield

C, the consols yield

B, the one year interest rate

The formulae for these variables are:

$$Y(t) = \exp\{ YW * I(t) + YN(t) \}$$

$$C(t) = CM(t) + CMU * \exp\{ CN(t) \}$$

$$B(t) = C(t) * \exp\{ -BD(t) \}$$

In many situations, we want to start the simulations from the current market yields and interest rates. However, for investigations of a more general nature, one might wish to start the simulations from a *neutral* position, that is, a starting point which is not distorted by special factors at a particular point in time. There are several ways of defining this, of which one is to consider what would happen in the long run if all the noise terms were zero. Then all the autoregressive processes would be at their long term means. That would imply a starting position

$$I(0) = QMU$$

$$YN(0) = \ln YMU$$

$$CN(0) = 0$$

$$BD(0) = BMU$$

$$\ln Z(0) = \ln ZMU$$

$$\ln R(0) = \ln RMU$$

which implies the following derived quantities:

$$Y(0) = \exp\{ YW * QMU + \ln YMU \}$$

$$C(0) = QMU + CMU$$

$$B(0) = \exp(-BMU) * [CW * QMU + CMU]$$

The moving averages *DM*, *CM* and *EM* are also initialised to *QMU*.

There are ostensibly 3 indices in the model. These are

Q , the retail price index

D , the share dividend index

E , the property income index

Other quantities can be calculated from these; for example an equity price index is defined as $P(t) = D(t)/Y(t)$. In the same way, a property price index $A(t)$ is defined as $E(t)/Z(t)$.

The updating equations for the explicit indices are as follows:

$$Q(t) = Q(t-1) \exp \{I(t)\}$$

$$D(t) = D(t-1) \exp \{ DW * DM(t) + (1-DW) * I(t) + DMU + DY * YSD * YZ(t-1) \\ + DB * DSD * DZ(t-1) + DSD * DZ(t) \}$$

$$E(t) = E(t-1) \exp \{ EW * EM(t) + (1-EW) * I(t) + EMU + EBZ * ZSD * ZZ(t) \\ + ESD * EZ(t) \}$$

The simulation results also include total return indices for each investment category. The total return indices are

PR , the total return on shares

CR , the total return on consols

BR , the total return on cash

RR , the total return on index linked gilts

AR , the total return on property

The roll up formulae are simple, being all effectively of the form:

$$\text{total return}(t) = \text{total return}(t-1) * \left(\frac{\text{price}(t) + \text{income}(t)}{\text{price}(t-1)} \right)$$

Thus, the gross return indices satisfy the formulae

$$\frac{PR(t)}{PR(t-1)} = \frac{P(t) + D(t)}{P(t-1)} = Y(t-1) * \frac{D(t)}{D(t-1)} \left(\frac{1}{Y(t)} + 1 \right)$$

$$\frac{CR(t)}{CR(t-1)} = \frac{1 / C(t) + 1}{1 / C(t-1)} = C(t-1) * \left(\frac{1}{C(t)} + 1 \right)$$

$$\frac{BR(t)}{BR(t-1)} = 1 + B(t-1)$$

$$\frac{RR(t)}{RR(t-1)} = \frac{Q(t) / R(t) + Q(t)}{Q(t-1) / R(t-1)} = R(t-1) * \frac{Q(t)}{Q(t-1)} \left(\frac{1}{R(t)} + 1 \right)$$

$$\frac{AR(t)}{AR(t-1)} = \frac{A(t) + E(t)}{A(t-1)} = Z(t-1) * \frac{E(t)}{E(t-1)} \left(\frac{1}{Z(t)} + 1 \right)$$

We aim to reproduce the total returns between $t-1$ and t according to the Wilkie model. The initial information available is:

The state variables I , WN , YN , CN , BD , $\ln Z$ and $\ln R$ at time $t-1$

The indices Q , W at time $t-1$

The rolled up indices PR , CR , BR , RR and AR at time $t-1$

We want to avoid storing a huge array of returns. Instead, we overwrite each year's index with the next year's and so on. This means that there is some care required in making sure that intermediate quantities are sampled at the right time in the process. We can divide the updating into three separate steps:

Step 1 (corresponding to the subroutine *preindex*)

Multiply each of the indices where appropriate by the elements which relate to state variables at time $t-1$. This involves the assignments

$$\begin{aligned}
 W &\leftarrow W * \exp\{ WW2 * I \} \\
 PR &\leftarrow PR * \exp\{ YW * I + YN + DY * YSD * YZ + DB * DSD * DZ \} \\
 CR &\leftarrow CR * [CM + CMU * \exp(CN)] \\
 BR &\leftarrow BR * \{ 1 + [CM + CMU * \exp(CN)] * \exp[-BD] \} \\
 RR &\leftarrow RR * \exp\{ \ln R \} \\
 AR &\leftarrow AR * \exp\{ \ln Z \}
 \end{aligned}$$

Step 2 (corresponding to the subroutine *advance*)

Move all the state variables from time $t-1$ to t , according to the updating formulae. This involves the assignments in respect of the autoregressive variables:

$$\begin{aligned}
 I &\leftarrow QMU + QA * [I - QMU] + QSD * QZ \\
 YN &\leftarrow \ln YMU + YA * [YN - \ln YMU] + YSD * YZ \\
 CN &\leftarrow CA1 * CN + CY * YSD * YZ + CSD * CZ \\
 BD &\leftarrow BMU + BA * [BD - BMU] + BSD * BZ \\
 \ln Z &\leftarrow \ln ZMU + ZA * [\ln Z - \ln ZMU] + ZSD * ZZ \\
 \ln R &\leftarrow \ln RMU + RA * [\ln R - \ln RMU] + RBC * CSD * CZ + RSD * RZ
 \end{aligned}$$

and in respect of the moving averages:

$$DM \leftarrow DD * I + (1-DD) * DM$$

$$CM \leftarrow CD * I + (1-CD) * CM$$

$$EM \leftarrow ED * I + (1-ED) * EM$$

Step 3 (corresponding to the subroutine *postindex*)

Multiply each of the indices by the elements which involve state variables at time t . This involves the assignments

$$Q \leftarrow Q * \exp\{I\}$$

$$W \leftarrow W * \exp\{WW1 * I + WMU + WSD * WZ\}$$

$$PR \leftarrow PR * \exp\{DW * DM + (1-DW) * I + DMU + DSD * DZ\} * (\exp\{-YW * I - YN\} + 1)$$

$$CR \leftarrow CR * (1 / [CM + CMU * \exp(CN)] + 1)$$

$$RR \leftarrow RR * \exp\{I\} * (\exp\{-\ln R\} + 1)$$

$$AR \leftarrow AR * \exp\{EW * EM + (1-EW) * I + EMU + EBZ * ZSD * ZZ + ESD * EZ * (\exp\{-\ln Z\} + 1)\}$$

This completes the recursive updates of the indices as required.

We notice that the derived state variables Y , C and B are not explicitly required for calculating the indices, so we save effort by not storing them.

We can also see that it is not necessary to store the dividend index D or the property income index E for the purposes of calculating the total returns. Neither is it necessary to store the price indices P and A .

We can see that different error terms are required at different steps of the induction algorithm. There are eight in all, and the matrix is as follows:

Error term	Step 1	Step 2	Step 3
QZ		✓	
YZ	✓	✓	
DZ	✓		✓
CZ		✓	
BZ		✓	
ZZ		✓	✓
EZ			✓
RZ		✓	

The only complication is that in order to move forward from $t-1$ to t we need to remember $YZ(t-1)$ and $DZ(t-1)$ in step 1. The easiest way to do this is to include them as other state variables. In fact, to be consistent with Wilkie, we do not store YZ and DZ but quantities YE and DE defined as $YSD * YZ$ and $DSD * DZ$ respectively.

A.3 Fractal Models

The construction of fractal models is very similar to the construction of a compound claims process in collective risk theory. We consider the log of a total return index, and describe the movement as a series of “claims”, or jumps. Unlike a conventional claims process, we must allow for downward jumps as well as upward jumps.

We define the fractal property as follows. A jump process is *stable* if for some α , the combined operation of stretching the process values by a factor of k and stretching the time axis by a factor k^α has no effect on the underlying probability law. For most financial applications, a value of α between 1 and 2 is appropriate.

Let us suppose that we have a stable jump process for some parameter α . We generate jumps as a Poisson process. Let us first consider only those jumps which have absolute value greater than 1. Let the frequency of upward jumps exceeding 1 be denoted by u and the frequency of downward jumps exceeding 1 be d .

We can use the fractal property to determine the frequency of other jump sizes. If we expand the process by a space factor k and a time factor k^α , jumps which were previously greater than 1 now have size greater than k . For the unscaled process, such jumps have frequency u (upwards) and d (downwards). For the scaled process, the frequencies are therefore $uk^{-\alpha}$ and $dk^{-\alpha}$ respectively. But, by hypothesis, the scaled process has the same probability law as the original. This means that the frequencies of jumps of all sizes are as follows (for $k > 0$):

Jumps $\geq k$ with frequency $uk^{-\alpha}$; jumps $\leq -k$ with frequency $dk^{-\alpha}$.

We can see that the smaller the jumps, the more frequent they are. The stable process is unlike the claims process in the sense that we have infinitely many small claims. However, as most of the claims are small, the total is finite and so we can construct a meaningful process.

Let us denote this process by X_t . Then for any particular value of t we can write

$$S = \left(\frac{2\Gamma(\alpha) \sin\left(\frac{\pi\alpha}{2}\right)}{(u+d)\pi t} \right)^{1/\alpha} X_t$$

By the fractal assumption, the distribution of S does not depend on the choice of t . The rather odd looking scaling factor is, unfortunately, an established convention. We say that S has a *Lévy stable distribution* with parameters α , β . The parameter β is conventionally defined as $\frac{d-u}{d+u}$. We notice that a positive value of β means a larger proportion of downward jumps. Walter (1989) fitted such a model to the French market based on daily data, and obtained (for the CAC 40 equity index) $\alpha = 1.65$ and $\beta = -0.11$. This is an odd result, suggesting that, after adjusting for the drift, the upward jumps in the equity index are more frequent than downward ones. Experience suggests that over the longer term, the reverse is the case, but perhaps we remember the bad times more clearly than the good ones!

In the example, I have built a fractal model by adjusting the error structures of the random walk model. Instead of using $N(0,1)$ random noise, I have replaced this with stable (1.65, -0.11) in each case, that is, the increments of a stable process. This does make sense in principle, in that these quantities do have zero mean, and while the variance is infinite, the upper and lower quartiles are of the same order of magnitude as a $N(0,1)$. Plainly, other parameters could have been chosen.

A.4 Chaotic Models

The essential ingredient to a chaotic model is a simple non linear map which we can iterate to obtain complex patterns. My favourite map is the quadratic one:

$$Z_{t+1} = \sqrt{2}[1 - Z_t^2]$$

starting with Z_0 in $(-\sqrt{2}, \sqrt{2})$. The consecutive values from iterating this map are then fed in to the error terms of the random walk model. The starting values are generated randomly with a uniform distribution.

One of the nice features of this series is that the terms behave in some sense like $N(0,1)$ random variables. Given virtually any initial distribution for Z_0 , the value of Z_t for large t has mean zero and variance 1, and furthermore, distinct Z_t are uncorrelated. This explains why the paths of a chaotic model may not easily be distinguished from a random walk, and indeed, the graphs we saw earlier reinforce this point.

A.5: Equilibrium Models

We have derived the expected returns for the random walk model based on what is, essentially, an equilibrium argument. However, the random walk model does not exhibit jumps, and when we allow jumps, the derivation of the expected returns falls over. This note outlines a new stochastic model, previously unpublished which exhibits jumps, but for which expected returns and the yield curve are determined by equilibrium constraints. The proposed stochastic model is based on a set of four assets, denoted by P_i for $i=1,2,3,4$. These represent

Sterling

Inflation

Equity

Property

We do not count bonds as a separate class, since (modulo credit risk) these are simply deferred currency payments. Bonds are described by the model of the asset in which they are denominated, either sterling or inflation. The model naturally extends to other asset classes, such as foreign investments.

For each asset class, we can define income and capital components. For the traded examples above, the relevant definitions are relatively straightforward:

For currencies, the capital is the currency itself, and the income is the interest generated on short deposits at market rates

For equities, the capital is the share or basket of shares, and the income is the dividends paid. We model all dividends gross of tax on income.

For properties, the capital is the property itself while the income is the potential rent on a short lease, less the cost of maintaining the property to the extent that this is borne by the lessee.

The income on a basket of commodities is slightly harder to define. We define this as the cost of short term lending income, that is, the fee which receivable from a third party who wishes to borrow the commodities for the short term. Under such lending arrangements, the borrower has to return equivalent commodities; not necessarily the physical mass lent, especially if the commodity concerned is perishable! Broadly speaking, this is equivalent to the *convenience yield* implied in futures prices. The most significant commodity basket for actuarial purposes is the basket which defines the RPI or the equivalent in other countries.

It is important to appreciate that an asset does not have a uniquely defined numerical value of itself. Instead, the price depends on the selected accounting unit, usually a currency. Thus, if P_1 is sterling, and P_3 is a suitable basket of 100 equities, then the FTSE 100 index at time t is the price of P_3 in units of P_1 which we denote at time t by $\frac{P_3(t)}{P_1(t)}$. We could equally well measure the price of the shares in some other units, such as Yen or even barrels of oil; the choice of sterling is simply a convention, which is convenient since many holders of UK equities will often account in sterling.

We say that the assets P_i have *asset dimension*, while the price of one asset in terms of another is dimensionless, and consequently has a uniquely defined numerical value. This means that there is some arbitrariness in the construction of models where the P_i are modelled as stochastic processes, since the multiplication

of each P_i by a common positive process will have no effect on the economic impact of the model.

We notice in passing that currency exchange rates can also be represented as the price of one asset in terms of another, and also that under our construction such currency rates automatically satisfy relevant consistency conditions, such as:

$$\frac{P_3(t)}{P_1(t)} = \frac{P_3(t)}{P_2(t)} \times \frac{P_2(t)}{P_1(t)}$$

Under the proposed model each asset has a *term structure*. This means that we can value an entitlement to receive the asset at a known point of time in the future. In the case of currencies this enables us to value bonds and annuities, while for a commodity price index the term structure can give us the value of index linked bonds. For property, this gives us the value of a reversion, so that the appropriate fixed rent for a known period is essentially the current value of the property minus the reversion, all divided by the annuity price for the appropriate term. This does depend on the revenue being secure, and for some tenants the appropriate rent would have to be adjusted upwards to allow for credit risk.

The term structure information allows us to value future income streams, for example, one might receive the interest on a currency starting in two years' time and continuing for one year. This is the value of the use of one currency unit for a year starting in two years time, that is, the difference between a two year bond and a three year bond. This kind of argument is useful for pricing derivatives such as swaps.

The model produces price indices P_i for each asset class, so that the price of one asset in terms of another is obtained by taking ratios. These prices P_i are functions of time t , and are defined for all $t \geq 0$. We denote the relative price of asset i at

time t by $P_i(t)$. We model this numerically, even though the price is only defined up to multiplication by a function of time.

The model also produces term structures for each of the asset classes. We define a quantity $V_i(s,t)$ to be:

The value at time s in units of $P_i(s)$ of a bond maturing at time t ,
delivering one unit of $P_i(t)$.

Since the units are specified, $V_i(s,t)$ is numeric that is, a dimensionless quantity. It is the price of a zero coupon bond, or alternatively, the appropriate term dependent factor at time s for discounting quantities due at time t . Naturally, this discount factor depends on the currency (or more generally, the asset) in question - hence the subscript i .

In order to start the model off, we need to know initial prices and term structures, that is, we need $P_i(0)$ for each i , and $V_i(0,t)$ for each i and $t \geq 0$.

Various other indices can be constructed from prices and term structures. For example, it is often helpful to consider a portfolio of bonds which is continually rebalanced so as to maintain a constant maturity profile, say with a fixed outstanding term τ . This is in contrast to a portfolio of bonds held to maturity, whose outstanding terms will gradually shorten over time.

We drive the model is driven by a series of random generating processes $G_j(t)$ for $j=1,2,..,n$, each starting at 0. These process are increasing processes, which capture the cumulative effect of jumps or other changes in the system. We can generate processes with both upward and downward jumps as the difference

between two such processes. The G_j are supposed to represent underlying fundamental variables in the economy, upon which the values of all investments ultimately depend.

We also define integrated processes $H_j(t)$ by

$$H_j(t) = \int_0^t G_j(s) ds$$

Since G_j is positive and increasing, it follows that $H_j(t)$ is positive, increasing and convex. It is also, of course, continuous.

The price model is given by

$$P_i(t) = V_i(0,t)P_i(0) \prod_{j=1}^n \left[\frac{(1 - \beta_{ij} + \gamma_{ij}t)^{-\beta_{ij} + \gamma_{ij}t}}{(1 - \beta_{ij})^{1-\beta_{ij}}} \right]^{\alpha_j/\gamma_{ij}} \exp \left[\sum_{j=1}^m -\alpha_j t + \beta_{ij} G_j(t) - \gamma_{ij} H_j(t) \right]$$

where α_j , β_{ij} and γ_{ij} are real constants, satisfying $\alpha_j > 0$, $\beta_{ij} < 1$ and $\gamma_{ij} \geq 0$. We can see that β_{ij} is the sensitivity to G_j which generate jumps in the asset price. The term involving γ_{ij} is the sensitivity to the cumulative effect of G_j and is effectively a momentum term. When G_j is higher than previously expected, H_j increases at a higher rate, and so the price has a downward trend. By the same token, if G_j has been low relative to expectations, then any downward trend in asset prices will be less noticeable. Of course, the trend is not meaningful in an absolute sense, since asset prices are only defined relative to another asset. The price appreciation or depreciation of an asset will be determined by the trend of the asset itself relative to the trend of the chosen accounting unit. It will be clear that this momentum term in some sense contradicts conventional random walk models. The constant term in the product may seem hard to interpret - we will see later exactly why the expression makes sense. This is certainly not the kind of formula which immediately jumps to mind for statistical fitting!

We now describe the evolution of the term structure variables $V_i(s, t)$. In this case, we suggest a model of the form:

$$V_i(s, t) = \frac{V_i(0, t)}{V_i(0, s)} \prod_{j=1}^n \left[\frac{[1 - \beta_{ij}]^{-\beta_{ij}} [1 - \beta_{ij} + \gamma_{ij} t]^{-\beta_{ij} + \gamma_{ij} t}}{[1 - \beta_{ij} + \gamma_{ij} s]^{-\beta_{ij} + \gamma_{ij} s} [1 - \beta_{ij} + \gamma_{ij} (t - s)]^{-\beta_{ij} + \gamma_{ij} (t - s)}} \right]^{\alpha_j / \gamma_{ij}} \\ \times \exp \left[-(t - s) \sum_{j=1}^n \gamma_{ij} G_j(s) \right]$$

We notice that all the stochastic terms are in the second line. Furthermore, the argument of the stochastic terms is $t-s$ which means that all possible yield curves at any point in time are parallel. Thus, we can expect that the traditional actuarial measures of duration and convexity would still be relevant to this rather fancy model.

Furthermore, we can see that the highest yields are associated with high values of the G_j which in turn are related to negative momentum in the underlying asset - even the coefficients γ_{ij} are the same! Thus, when an asset is depreciating, investors discount future receivables more deeply. Again, this feature is intuitively consistent with existing theory.

We can now set about calculating the value of constant maturity indices as described previously. After some manipulation, the result is:

$$RP_i(t; \tau) \approx RP_i(0, \tau) \prod_{j=1}^n (1 - \beta_{ij} + \gamma_{ij} \tau)^{\alpha_j / \tau} \exp \left(\sum_{j=1}^n (\beta_{ij} - \gamma_{ij} \tau) G_j(t) \right)$$

where $RP_i(t; \tau)$ is the total return constant maturity index for bonds of duration τ denominated in asset i . It may be surprising at first sight that such a simple

expression should result from such a complex model. Of course, anyone who knows me will appreciate that the formulae were rigged from the start so that this kind of thing would work nicely.

We now consider probability laws which might generate the jump processes $G_j(t)$. One way of constructing these processes is to use Γ distributions. We therefore suggest that

$$G_j(t) - G_j(s) \sim \Gamma[\alpha_j(t-s)]$$

independent of the other processes and of all history prior to time s . It is easily shown that the sum of two independent Γ distributions is another Γ distribution, and the parameter of the sum is the sum of the parameters of the constituents. This means that in principle we can construct a consistent process which satisfies the above distribution for all $s < t$. The effect of changing α is to scale the time axis; larger values of α mean that the process is speeded up.

This generates price behaviour which is qualitatively different from the behaviour under diffusion models, that is, models using the normal distribution. Several commentators have suggested that risk should be subdivided into *volatility risk*, that is, the risk of small movements each day, and *catastrophe risk*, that is, the risk of a sudden jump, as a result of some unforeseen disaster. The Γ process takes this further by allowing a continuum of jump sizes both small and large. By contrast, diffusion models have continuous paths, so only allow infinitesimal jumps.

Under this Γ model, it can be shown that

$$E_s[P(t)] = V(s,t)P(s)$$

so that the value of a bond is simply its expected value, and furthermore, that this holds simultaneously for all assets. Thus, within the proposed model, the market values all cash flows according to their expected values. We refer to this as the *martingale property*. From the tower law of conditional expectation, we can produce the more general result: $E_s[P(t)V(t,T)] = V(s,T)P(s)$ which says that the value of any bond portfolio (on a market value total return basis) is a martingale. A fundamental result of financial theory then implies that the value of any total return bond index is a martingale, even if the index is actively traded. This observation has a number of economic consequences, one of which is the absence of arbitrage. In other words, it is impossible to construct a portfolio whose current value is ≤ 0 , but has a value ≥ 0 at some future point, unless that portfolio is identically zero.

A consistent way to price other derivatives would be to extend this principle, that is to say, the value of an option is simply the conditional expectation of its payoff. This will guarantee that derivative prices are also consistent with the absence of arbitrage. When computing the payoff of options, it is essential to consider explicitly the currency in which the option is denominated. For example, if P_1 is sterling and P_2 is a suitable basket of 100 equities then the payoff of a call option with strike k is $\max\{P_2(t)-kP_1(t), 0\}$, not $\max\{P_2(t)-k, 0\}$. A useful check is that, most of the time, expressions to be valued turn out to be first order homogeneous functions of the underlying asset prices. This means that derivatives still have asset dimension. If an expression is not homogeneous, it is likely that an accounting

currency has inadvertently been *hard coded*, and of course this may give the wrong answer.

We now turn to the calibration of the model. Firstly, we examine the variance and skewness structure of the log relative total returns. Let us suppose that we are examining the total return of bonds with duration τ_i for asset i relative to the total return of bonds with duration τ_j for asset j . The log relative return is

$$\log\left(\frac{RP_j(t, \tau_j)}{RP_i(t, \tau_i)}\right) = \sum_{k=1}^n \alpha_k t \log\left(\frac{1 - \beta_{jk} + \gamma_{jk}\tau_j}{1 - \beta_{ik} + \gamma_{ik}\tau_i}\right) + (\beta_{jk} - \beta_{ik} - \gamma_{jk}\tau_j + \gamma_{ik}\tau_i)G_j(t)$$

From this, we can calculate any moments that are required. We do not use this to get the mean, because that is hopelessly subject to estimation error. However, we do examine variances, covariances and skewnesses. We first note that

$$\text{Var}[G_k(t)] = \alpha_k t$$

and the third centralised moment is given by

$$\text{Skew}[G_k(t)] = 2\alpha_k t^2$$

This implies that

$$\begin{aligned} \text{Cov}\left(\log\frac{RP_j(t, \tau_j)}{RP_i(t, \tau_i)}, \log\frac{RP_k(t, \tau_j)}{RP_i(t, \tau_i)}\right) \\ = t \sum_{m=1}^n \alpha_m (\beta_{jm} - \beta_{im} - \gamma_{jm}\tau_j + \gamma_{im}\tau_i)(\beta_{km} - \beta_{im} - \gamma_{km}\tau_k + \gamma_{im}\tau_i) \\ \text{Skew}\log\left(\frac{RP_j(t, \tau_j)}{RP_i(t, \tau_i)}\right) = 2t \sum_{m=1}^n \alpha_m (\beta_{jm} - \beta_{im} - \gamma_{jm}\tau_j + \gamma_{im}\tau_i)^3 \end{aligned}$$

In general, given these values, it might still be quite hard to solve the equations for the underlying parameters. However, we can simplify matters considerably by assuming a lower triangular form for the asset classes. In effect, we write the total return indices in a hierarchical pyramid, such that shocks to higher elements can also hit lower elements, but not the other way around. The proposed hierarchy is then:

Gilt total return / cash total return

Index Linked total return / cash total return

Equity total return / cash total return

Property total return / cash total return

A historically based variance covariance matrix for these is:

Gilts	0.013939	0.003483	0.014945	0.003004
ILG's	0.003483	0.004345	0.003321	0.001037
Equity	0.014945	0.003321	0.047754	0.012464
Property	0.003004	0.001037	0.012464	0.013426

The Gram-Schmidt square root of this decomposes the volatility of each series in terms of the underlying processes. Together with some appropriate skewness assumptions, we can use this to derive the α parameters for each of the four generating processes, as follows:

α	16	9.874861	1.432999	0.927582
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We now assume that the inflation index does not have any jumps, but that its slope can jump. We also assume that the optimal strategy for a logarithmic investor is 100% equity. Together with some suitable estimates of bond durations for the historic data, this enables us to identify the remaining parameters.

Cash	β	0.031646	-0.00223	0.148691	0
Inflation	β	0.031646	-0.00223	0.148691	0
Equity	β	0	0	0	0
Property	β	0.025284	-0.00377	0.105155	-0.10403
Cash	γ	0.001968	0	0	0
Inflation	γ	0.000737	0.001876	0	0
Equity	γ	0	0	0	0
Property	γ	0	0	0	0

Appendix B Implementations

I have implemented all the above models in Microsoft Excel 5 for PC's. I have made extensive use of the Visual Basic macro facility supplied with Excel. The reason for using this is that Excel is a popular and inexpensive package, so that there is a reasonable chance that others will be able to use the code straight away. Visual Basic produces code which is fairly easy to read and understand. The code displayed here is not particularly efficient from a computational point of view. This is because I have deliberately avoided some short cuts in order to make the program logic clearer.

I have split the code up into six modules to make it easier to follow. These are as follows:

Distributions: This model contains the code to generate random variables from the normal, Lévy stable and Γ distributions.

Generic: This module contains all the boring housekeeping for simulations, taking and interpreting input parameters, calling the appropriate model routine and poking the answers back into a suitable array.

RandomWalk: This module implements the Kemp, Mandelbrot and May models.

Wilkie: This implements the Wilkie model

Optimise: This performs the optimisation of a logarithmic investor under the Wilkie model.

Smith: This contains the code for my own model.

Now it's up to you. Try the models out. Bring your results to the workshop!

Distributions

Option Explicit

```

Function normgen() As Double
'Return sample from N(0,1) distribution
Static got_one As Boolean, stored As Double
Dim x1 As Double, x2 As Double, r As Double
If got_one Then
    normgen = stored
    got_one = False
Else
    'generate a point in the unit square
    Do
        x1 = 2 * Rnd - 1
        x2 = 2 * Rnd - 1
        r = x1 ^ 2 + x2 ^ 2
        'reject unless inside unit circle
    Loop While r >= 1 Or r = 0
    r = Sqr(-2 * Log(r) / r)
    stored = x1 * r
    normgen = x2 * r
    got_one = True
End If
End Function

Function gammgen(alpha) As Double
'Return sample from Gamma(alpha) distribution
Dim beta As Double, gamma As Double, k As Double
Dim p1 As Double, p2 As Double, y As Double
beta = Sqr(alpha + 0.25) - 0.5
gamma = beta + 1
k = (alpha + gamma) ^ ((alpha + gamma) / (beta + gamma)) / _
    (alpha - beta) ^ ((alpha - beta) / (beta + gamma)) * Exp(-1)
p1 = beta * k ^ (-beta) * (alpha - beta) ^ (beta - alpha) * _
    Exp(alpha - beta)
p2 = gamma * k ^ gamma * (alpha + gamma) ^ (-gamma - alpha) * _
    Exp(alpha + gamma)
p1 = p2 / (p1 + p2)
Do
    If Rnd() < p1 Then
        'use beta distribution
        y = k * Rnd() ^ (1 / beta)
        p2 = (y / (alpha - beta)) ^ (alpha - beta) * _
            Exp(alpha - beta - y)
    Else
        'use pareto distribution
        y = k / Rnd() ^ (1 / gamma)
        p2 = (y / (alpha + gamma)) ^ (alpha + gamma) * _
            Exp(alpha + gamma - y)
    End If
Loop Until Rnd() < p2
gammgen = y
End Function

Function Levy(alpha, beta) As Double
'return sample from Levy stable (alpha, beta) distribution
Dim phi0 As Double, phi As Double, W As Double
phi0 = (alpha - 2) / 2 / alpha * 3.1415927 * beta
phi = 3.1415927 * (Rnd() - 0.5)
W = -Log(Rnd())
Levy = Sin(alpha * (phi - phi0)) / Cos(phi) ^ (1 / alpha) * _

```

Distributions

```

      (Cos(phi - alpha * (phi - phi0)) / W) ^ ((1 - alpha) / alpha)
End Function

Function IID(nosims, seed, distribution, _
    Optional alpha, Optional beta)
ReDim outvec(1 To nosims) As Double
Dim dummy As Double, t As Integer
dummy = Rnd(-1.414 - 1.723 * Abs(seed) ^ 2.718)
' Now iterate round a few times so senstive to initial conditions
dummy = Rnd()
dummy = Rnd()
For t = 1 To nosims
    Select Case distribution
    Case "normal"
        outvec(t) = normgen
    Case "gamma"
        outvec(t) = gammgen(alpha)
    Case "Levy"
        outvec(t) = Levy(alpha, beta)
    End Select
Next t
IID = outvec
End Function

```

Generic

Option Explicit

Type Return_Index

'These are all total return indices
'Including income gross of tax
'with the exception of RPI, which has no income
Year As Integer
Retail_Price As Double
Equity As Double
Fixed_Interest As Double
Cash As Double
Index_Linked As Double
Property As Double

End Type

Sub Unit(S As Return_Index)

With S
 .Year = 0
 .Retail_Price = 1
 .Equity = 1
 .Fixed_Interest = 1
 .Cash = 1
 .Index_Linked = 1
 .Property = 1

End With

End Sub

Sub Fill(index_array As Variant, S As Return_Index)

With S
 index_array(.Year, 1) = .Year
 index_array(.Year, 2) = .Retail_Price
 index_array(.Year, 3) = .Equity
 index_array(.Year, 4) = .Fixed_Interest
 index_array(.Year, 5) = .Cash
 index_array(.Year, 6) = .Index_Linked
 index_array(.Year, 7) = .Property

End With

End Sub

Sub Update(S As Return_Index, Model As String)

Select Case Model
 Case "Kemp", "Mandelbrot", "May"
 RandomWalk_Update S, Model
 Case "Wilkie"
 Wilkie_Update S
 Case "Smith"
 Smith_Update S

End Select

S.Year = S.Year + 1

End Sub

Function Scenario(Model As String, horizon As Integer, _
 seed As Double)

'Skeleton function for putting model output into an array
'Use seed for random number generator,
'with arbitrary non-round coefficients
Dim dummy As Double
dummy = Rnd(-1.414 - 1.723 * Abs(seed) ^ 2.718)
'Now iterate round a few times so sensitive to initial conditions

Generic

```
dummy = Rnd()  
dummy = Rnd()  
'Set out range for output  
ReDim outscen(0 To horizon, 1 To 7) As Double  
Dim S As Return_Index  
  
Unit S  
Fill outscen, S  
Do  
    Update S, Model  
    Fill outscen, S  
Loop Until S.Year = horizon  
Scenario = outscen  
End Function
```

RandomWalk

```

Option Explicit

Const QMU = 0.047
Const QA = 0.58

Type Six_Error
    Z(0 To 5) As Double
End Type

Sub Kemp_gen(Z As Six_Error)
    Dim i As Integer
    For i = 0 To 5
        Z.Z(i) = normgen()
    Next i
End Sub

Sub Mandelbrot_gen(Z As Six_Error)
    Dim i As Integer
    For i = 0 To 5
        Z.Z(i) = Levy(1.65, -0.11)
    Next i
End Sub

Sub May_gen(Z As Six_Error)
    Static initialised As Boolean
    Dim i As Integer
    If initialised Then
        For i = 0 To 5
            Z.Z(i) = 1.414 * (1 - Z.Z(i) ^ 2)
        Next i
    Else
        For i = 0 To 5
            Z.Z(i) = 2.818 * Rnd() - 1.414
        Next i
    End If
End Sub

Sub RandomWalk_Update(S As Return_Index, Model As String)
    'Make the only state variable static
    Static inflation_rate As Double, mu(0 To 5)
    Static C(0 To 5, 0 To 5) As Double, Z As Six_Error
    Dim i As Integer, j As Integer, E As Six_Error
    If S.Year = 0 Then
        inflation_rate = QMU
        'read in array for C
        For i = 0 To 5
            If i = 0 Then
                mu(i) = 0
            Else
                mu(i) = Range("Muvector").Cells(i, 1).Value
            End If
            For j = 0 To i
                C(i, j) = Range("Cmatrix").Cells(i + 1, j + 1).Value
            Next j
        Next i
    End If
    Select Case Model
    Case "Kemp"
        Kemp_gen Z
    Case "Mandelbrot"

```

RandomWalk

```

Mandelbrot_gen Z
Case "May"
  May_gen Z
End Select
For i = 0 To 5
  E.Z(i) = 0
  For j = 0 To i
    E.Z(i) = E.Z(i) + C(i, j) * Z.Z(j)
  Next j
Next i
inflation_rate = QA * inflation_rate + (1 - QA) * QMU + E.Z(0)
With S
  .Retail_Price = .Retail_Price * Exp(inflation_rate)
  .Equity = .Equity * Exp(mu(2) + E.Z(1) + inflation_rate)
  .Fixed_Interest = .Fixed_Interest * Exp(mu(3) +
    + E.Z(2) + inflation_rate)
  .Cash = .Cash * Exp(mu(3) + E.Z(3) + inflation_rate)
  .Index_Linked = .Index_Linked * Exp(mu(4) +
    + E.Z(4) + inflation_rate)
  .Property = .Property * Exp(mu(5) + E.Z(5) + inflation_rate)
End With
End Sub

```

Option Explicit

Type Wilkie State

'contains the stationary state variables

'starting with the AR1 variables

i As Double

YN As Double

CN As Double

BD As Double

lnZ As Double

lnR As Double

'now the moving averages of inflation

DM As Double

CM As Double

EM As Double

'and finally, the error terms carried forward

YE As Double

DE As Double

End Type

Type Eight Normal

'all the iid $N(0,1)$ random innovations

QZ As Double

YZ As Double

DZ As Double

CZ As Double

BZ As Double

ZZ As Double

EZ As Double

RZ As Double

End Type

'retail price inflation parameters page 9

Const QMU As Double = 0.047

Const QA As Double = 0.58

Const QSD As Double = 0.0425

'dividend yields, page 46

Const YW As Double = 1.8

Const YA As Double = 0.55

'Const YMU As Double = 0.0375 this is always logged,

'so store logged value

Const lnYMU As Double = -3.283414346

Const YSD As Double = 0.155

'dividends, page 68

Const DW As Double = 0.58

Const DD As Double = 0.13

Const DMU As Double = 0.016

Const DY As Double = -0.175

Const DB As Double = 0.57

Const DSD As Double = 0.07

'long term interest rates, page 86

Const CD As Double = 0.045

Const CMU As Double = 0.0305

Const CAI As Double = 0.9

Const CY As Double = 0.34

Const CSD As Double = 0.185

Wilkie

```

'short term interest rates, page 95
Const BMU As Double = 0.23
Const BA As Double = 0.74
Const BSD As Double = 0.18

'property yields, page 101
'Const ZMU As Double = 0.074 this is always logged,
' so use log value instead
Const lnZMU As Double = -2.603690186
Const ZA As Double = 0.91
Const ZSD As Double = 0.12

'property rents, page 103
Const EW As Double = 1
Const ED As Double = 0.11
Const EMU As Double = 0.003
Const EBZ As Double = 0.24
Const ESD As Double = 0.06

'index linked gilt yields, page 107
'Const RMU As Double = 0.04 always logged so store log instead
Const lnRMU As Double = -3.218875825
Const RA As Double = 0.55
Const RBC As Double = 0.22
Const RSD As Double = 0.05

'initialisation of states
Sub Neutral(W As Wilkie_State)
With W
'starting with the ARI variables
.i = QMU
.YN = lnYMU
.CN = 0
.BD = BMU
.lnZ = lnZMU
.lnR = lnRMU
'now the moving averages of inflation
.DM = QMU
.CM = QMU
.EM = QMU
'and finally, the error terms carried forward
.YE = 0
.DE = 0
End With
End Sub

Sub Preindex(S As Return_Index, W As Wilkie_State)
Dim C As Double
With W
S.Equity = S.Equity * Exp(YW * .i + .YN _
+ DY * .YE + DB * .DE)
C = .CM + CMU * Exp(.CN)
S.Fixed_Interest = S.Fixed_Interest * C
S.Cash = S.Cash * (1 + C * Exp(-.BD))
S.Index_Linked = S.Index_Linked * Exp(.lnR)
S.Property = S.Property * Exp(.lnZ)
End With
End Sub

```


Wilkie

```

Sub Advance(W As Wilkie_State, Z As Eight_Normal)
With W
.i = QMU + QA * (.i - QMU) + QSD * Z.QZ
.YE = YSD * Z.YZ
.YN = lnYMU + YA * (.YN - lnYMU) + .YE
.DM = DD * .i + (1 - DD) * .DM
.DE = DSD * Z.DZ
.CM = CD * .i + (1 - CD) * .CM
.CN = CA1 * .CN + CY * .YE + CSD * Z.CZ
.BD = BMU + BA * (.BD - BMU) + BSD * Z.BZ
.lnZ = lnZMU + ZA * (.lnZ - lnZMU) + ZSD * Z.ZZ
.EM = ED * .i + (1 - ED) * .EM
.lnR = lnRMU + RA * (.lnR - lnRMU) -
+ RBC * CSD * Z.CZ + RSD * Z.RZ
End With
End Sub

Sub Postindex(S As Return_Index, W As Wilkie_State, _
Z As Eight_Normal)
With W
S.Retail_Price = S.Retail_Price * Exp(.i)
S.Equity = S.Equity * Exp(DW * .DM + (1 - DW) * .i -
+ DMU + .DE) * (Exp(-YW * .i - .YN) + 1)
S.Fixed_Interest = S.Fixed_Interest
* (1 / (.CM + CMU * Exp(.CN)) + 1)
S.Index_Linked = S.Index_Linked * Exp(.i) * (Exp(-.lnR) + 1)
S.Property = S.Property * Exp(EW * .EM + (1 - EW) * .i
+ EMU + EBZ * ZSD * Z.ZZ + ESD * Z.EZ) * (Exp(-.lnZ) + 1)
End With
End Sub

Sub Eight_Normal_Generate(Z As Eight_Normal)
With Z
.QZ = normgen()
.YZ = normgen()
.DZ = normgen()
.CZ = normgen()
.BZ = normgen()
.ZZ = normgen()
.EZ = normgen()
.RZ = normgen()
End With
End Sub

Sub Wilkie_Update(S As Return_Index)
Static W As Wilkie_State
Dim Z As Eight_Normal
'dummy code
If S.Year = 0 Then
Neutral W
End If
Preindex S, W
Eight_Normal_Generate Z
Advance W, Z
Postindex S, W, Z
End Sub

```

Optimise

Option Explicit

```
Function optcount(horizon, seed) As Variant
'determine how many times 100% investment in one class is optimal.
Dim dummy As Double
Dim cumopt(1 To 5) As Integer, t As Integer
Dim sumquot(1 To 5, 1 To 5) As Double
Dim H(1 To 10) As Eight_Normal, Z As Eight_Normal
Dim W As Wilkie_State, Whyp As Wilkie_State, S As Return_Index
Dim i As Integer, j As Integer, isbest As Boolean
    dummy = Rnd(-1.414 - 1.723 * Abs(seed) ^ 2.718)
    dummy = Rnd()
    dummy = Rnd()
'generate test scenarios
For i = 1 To 10
    With H(i)
        .QZ = 1.414 * Sin(i * 0.6283185307)
        .YZ = 1.414 * Sin(i * 2 * 0.6283185307)
        .DZ = 1.414 * Sin(i * 3 * 0.6283185307)
        .CZ = 1.414 * Sin(i * 4 * 0.6283185307)
        .BZ = 1.414 * Cos(i * 0.6283185307)
        .ZZ = 1.414 * Cos(i * 2 * 0.6283185307)
        .EZ = 1.414 * Cos(i * 3 * 0.6283185307)
        .RZ = 1.414 * Cos(i * 4 * 0.6283185307)
    End With
Next i
Neutral W
For t = 1 To horizon
    For i = 1 To 5
        For j = 1 To 5
            sumquot(i, j) = 0
        Next j
    Next i
    For i = 1 To 10
        Unit S
        Whyp = W
        Preindex S, Whyp
        Advance Whyp, H(i)
        Postindex S, Whyp, H(i)
        With S
            sumquot(1, 2) = sumquot(1, 2) _
                + .Equity / .Fixed_Interest _
            sumquot(1, 3) = sumquot(1, 3) _
                + .Equity / .Cash _
            sumquot(1, 4) = sumquot(1, 4) _
                + .Equity / .Index_Linked _
            sumquot(1, 5) = sumquot(1, 5) _
                + .Equity / .Property _
            sumquot(2, 1) = sumquot(2, 1) _
                + .Fixed_Interest / .Equity _
            sumquot(2, 3) = sumquot(2, 3) _
                + .Fixed_Interest / .Cash _
            sumquot(2, 4) = sumquot(2, 4) _
                + .Fixed_Interest / .Index_Linked _
            sumquot(2, 5) = sumquot(2, 5) _
                + .Fixed_Interest / .Property _
            sumquot(3, 1) = sumquot(3, 1) _
                + .Cash / .Equity _
            sumquot(3, 2) = sumquot(3, 2) _
                + .Cash / .Fixed_Interest _
```

Optimise

```

sumquot(3, 4) = sumquot(3, 4) -
+ .Cash / .Index_Linked
sumquot(3, 5) = sumquot(3, 5) -
+ .Cash / .Property
sumquot(4, 1) = sumquot(4, 1) -
+ .Index_Linked / .Equity
sumquot(4, 2) = sumquot(4, 2) -
+ .Index_Linked / .Fixed_Interest
sumquot(4, 3) = sumquot(4, 3) -
+ .Index_Linked / .Cash
sumquot(4, 5) = sumquot(4, 5) -
+ .Index_Linked / .Property
sumquot(5, 1) = sumquot(5, 1) -
+ .Property / .Equity
sumquot(5, 2) = sumquot(5, 2) -
+ .Property / .Fixed_Interest
sumquot(5, 3) = sumquot(5, 3) -
+ .Property / .Cash
sumquot(5, 4) = sumquot(5, 4) -
+ .Property / .Index_Linked
End With
Next i
'Now determine if any single class is optimal
For i = 1 To 5
  isbest = True
  For j = 1 To 5
    isbest = (isbest And sumquot(j, i) < 10)
  Next j
  If isbest Then
    cumopt(i) = cumopt(i) + 1
  End If
Next i
'finally, need to advance W randomly
'but not really interested in return achieved
Eight_Normal_Generate Z
Advance W, Z
Next t
optcount = cumopt
End Function

```

Option Explicit

```

Sub Smith_Update(S As Return_Index)
Static alpha(1 To 4) As Double, beta(1 To 4, 1 To 4) As Double
Static gamma(1 To 4, 1 To 4) As Double
Static G(1 To 4) As Double, H(1 To 4) As Double
Static init_yield(1 To 4) As Double, duration(1 To 4) As Double
Dim i As Integer, j As Integer, t As Integer
Dim price(1 To 4), rollup(1 To 4), bond(1 To 4)
If S.Year = 0 Then
    initialise everything
    For i = 1 To 9
        For j = 1 To 4
            Select Case i
            Case 1
                alpha(j) = Range("EquGreeks").Cells(i, j).Value
            Case 2 To 5
                beta(i - 1, j) = Range("EquGreeks").Cells(i, j).Value
            Case 6 To 9
                gamma(i - 5, j) = Range("EquGreeks").Cells(i, j).Value
            End Select
        Next j
    Next i
    For i = 1 To 4
        G(i) = 0
        H(i) = 0
        adopt continuously compounded conventions
        init_yield(i) = Log(1 + Range("Yields").Cells(i, 1).Value)
        duration(i) = Range("Durations").Cells(i, 1).Value
    Next i
End If
For i = 1 To 4
    'update gamma processes and integrated processes
    ' using trapezium rule
    H(i) = H(i) + G(i) / 2
    G(i) = G(i) + gammgen(alpha(i))
    H(i) = H(i) + G(i) / 2
    'calculate time. Note that working to project next year
    t = S.Year + 1
    'all these quantities are stored as logs
    price(i) = -t * init_yield(i)
    rollup(i) = 0
    bond(i) = 0
    For j = 1 To 4
        If gamma(i, j) = 0 Then
            price(i) = price(i) + alpha(j)
                * t * (1 + Log(1 - beta(i, j)))
        Else
            price(i) = price(i) + alpha(j) / gamma(i, j) * _
                ((1 - beta(i, j) + gamma(i, j) * t) * _
                Log(1 - beta(i, j) + gamma(i, j) * t) -
                (1 - beta(i, j)) * Log(1 - beta(i, j)))
        End If
        price(i) = price(i) - alpha(j) * t + beta(i, j) * G(j) _
            - gamma(i, j) * H(j)
        rollup(i) = rollup(i) + alpha(j) * t * Log(1 - beta(i, j)) _
            + beta(i, j) * G(j)
        bond(i) = bond(i) + alpha(j) * t * Log(1 - beta(i, j)) _

```

Smith

```

      + duration(i) * gamma(i, j)) + (beta(i, j) - _
duration(i) * gamma(i, j)) * G(j)
    Next j
Next i
Now poke answer back into S
With S
  .Retail_Price = Exp(price(2) - price(1))
  .Equity = Exp(rollup(3) - price(1))
  .Fixed_Interest = Exp(bond(1) - price(1))
  .Cash = Exp(rollup(1) - price(1))
  .Index_Linked = Exp(bond(2) - price(1))
  .Property = Exp(rollup(4) - price(1))
End With
End Sub

```