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A Stochastic Framework for Incremental Average Reserve Models

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Reserves in a Stochastic World

- At a point in time (valuation date) there is a range of possible outcomes for a book of (insurance) liabilities. Some possible outcomes may be more likely than others
- Range of possible outcomes along with their corresponding probabilities are the distribution of outcomes for the book of liabilities – i.e. reserves are a distribution
- The distribution of outcomes may be complex and not completely understood
- Uncertainty in predicting outcomes comes from
  - Process (pure randomness)
  - Parameters (model parameters uncertain)
  - Model (selected model is not perfectly correct)

Stochastic Models

- In the actuarial context a stochastic could be considered as a mathematical simplification of an underlying loss process with an explicit statement of underlying probabilities
- Two main features
  - Simplified Statement
  - Explicit probabilistic statement
- In terms of sources of uncertainty two of three sources may be addressed
  - Process
  - Parameter
- Within a single model, the third source (model uncertainty) usually not explicitly addressed

Basic Traditional Actuarial Methods

- Traditional actuarial methods are simplifications of reality
  - Chain ladder
  - Bornhuetter-Ferguson
  - Berquist-Sherman Incremental Average
  - Others
- Usually quite simple thereby “easy” to explain
- Traditional reserve approaches rely on a number of methods
- Practitioner “selects” an “estimate” based on results of several traditional methods
- No explicit probabilistic component

Traditional Chain Ladder

- If C_{ij} denotes incremental amount (payment) for exposure year i at development age j
- Deterministic chain ladder

$$C_{ij+1} = f_j \sum_{k=1}^{j} C_{ik}$$

- Parameters f_j usually estimated from historical data, looking at link ratios (cumulative paid at one age divided by amount at prior age)
- Forecast for an exposure year completely dependent on amount to date for that year so notoriously volatile for least mature exposure period

Traditional Bornhuetter-Ferguson

- Attempts to overcome volatility by considering an additive model
- Deterministic Bornhuetter-Ferguson

$$C_{ij} = f_j e_i$$

- Parameters f_j usually estimated from historical data, looking at link ratios
- Parameters e_i, expected losses, usually determined externally from development data but “Cape Cod” (Stanard/Buhlmann) variant estimates these from data
- Exposure year amount not completely dependent on to-date number

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Traditional Berquist-Sherman Incremental

- Attempts to overcome volatility by considering an additive model
- Deterministic Berquist-Sherman incremental severity

$$C_{ij} = E_i \alpha_j \tau_j^i$$

- Parameters E_i exposure measure, often forecast ultimate claims or earned exposures
- Parameters α_j and τ_j usually estimated from historical data, looking at incremental averages
- Berquist & Sherman has several means to derive those estimates
- Often simplified to have

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A Stochastic Incremental Model

- Instead of incremental paid, consider incremental average $A_{ij} = C_{ij}/E_i$
- First step translating to stochastic, have expected values agree with simplified Berquist-Sherman incremental average

$$\mathrm{E}\left(A_{ij}\right) = \alpha_j \tau^i$$

- Observation – the amounts are averages of a (large?) sample, assumed from the same population
- Central Limit Theorem would imply, if variance is finite, that distribution of the average is asymptotically normal
- Thus assume the averages have Gaussian distributions (next step in stochastic framework)

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A Stochastic Incremental Model – Cont.

- Now that we have an assumption about the distribution (Gaussian) and expected value all needed to specify the model is the variance in each cell
- In stochastic chain ladder frameworks the variance is assumed to be a fixed (known) power of the mean

$$\mathrm{Var}\left(C_{ij}\right)=\sigma \mathrm{E}\left(C_{ij}\right)^{k}$$

- We will follow this general structure, however allowing the averages to be negative and the power to be a parameter fit from the data, with $e_i = \ln(E_i)$

$$\mathrm{Var}\left(A_{ij}\right)=e^{\kappa-e_i}\left(\alpha_j\tau^i\right)^{2p}$$

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Parameter Estimation

- Number of approaches possible
- If we have an a-priori estimate of the distribution of the parameters we could use Bayes Theorem to refine that estimates given the data
- Maximum likelihood is another approach
- In this case the negative log likelihood function of the observations given a set of parameters is given by

$$\mathrm{l}\left(A_{11},A_{12},\ldots,A_{n1};\alpha_1,\alpha_2,\ldots,\alpha_n,\tau,\kappa,p\right)=$$
$$\sum\frac{\kappa+\ln\left(2\pi\left(\alpha_j\tau^i\right)^{2p}\right)}{2}+\frac{\left(A_{ij}-\alpha_j\tau^i\right)^2}{2e^{\kappa-e_i}\left(\alpha_j\tau^i\right)^{2p}}$$

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Distribution of Outcomes Under Model

- Since we assume incremental averages are independent once we have the parameter estimates we have estimate of the distribution of outcomes given the parameters

$$R_i \sim \mathrm{N}\left(\sum_{j=n-i+2}^{n}\hat{\alpha}_j\hat{\tau}^i,\ \sum_{j=n-i+2}^{n}e^{\hat{\kappa}-e_i}\left(\hat{\alpha}_j\hat{\tau}^i\right)^{2\hat{p}}\right)$$

- This is the estimate for the average future forecast payment per unit of exposure, multiplying by exposures and adding by exposure year gives a distribution of aggregate future payments
- This assumes parameter estimates are correct – does not account for parameter uncertainty

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Parameter Uncertainty

- Some properties of maximum likelihood estimators
  - Asymptotically unbiased
  - Asymptotically efficient
  - Asymptotically normal
- We implicitly used the first property in the distribution of future payments under the model
- Define the Fisher information matrix as the expected value of the Hessian matrix (matrix of second partial derivatives) of the negative log-likelihood function
- The variance-covariance matrix of the limiting Gaussian distribution is the inverse of the Fisher information matrix typically evaluated at the parameter estimates

Incorporating Parameter Uncertainty

- If we assume
  - The parameters have a multi-variate Gaussian distribution with mean equal to the maximum likelihood estimators and variance-covariance matrix equal to the inverse of the Fisher information matrix
  - For a fixed parameters the losses have a Gaussian distribution with the mean and variance the given functions of the parameters
- The posterior distribution of outcomes is rather complex
- Can be easily simulated:
  - First randomly select parameters from a multi-variate Gaussian Distribution
  - For these parameters simulate losses from the appropriate Gaussian distributions

Berquist-Sherman Average Paid Data

Accident	Months of Development								Forecast
Year	12	24	36	48	60	72	84	96	Counts
1969	178.73	361.03	283.69	264.00	137.94	61.49	15.47	8.82	7,822
1970	196.56	393.24	314.62	266.89	132.46	49.57	33.66		8,674
1971	194.77	425.13	342.91	269.45	131.66	66.73			9,950
1972	226.11	509.39	403.20	289.89	158.93				9,690
1973	263.09	559.85	422.42	347.76					9,590
1974	286.81	633.67	586.68						7,810
1975	329.96	804.75							8,092
1976	368.84								7,594

Estimates

	A_1	α_2	α_3	α_4	α_5	α_6	α_7	α_8
Parameter	143.78	316.77	251.78	197.68	102.53	46.23	21.36	7.36
Std. Error	6.20	11.54	9.16	7.62	5.25	3.75	3.07	2.41

	κ	τ	p
Parameter	8.5871	1.1265	0.5782
Std. Error	0.2321	0.0077	0.0303

Forecast Average Expected Values

Accident Year	Months of Development 24	36	48	60	72	84	96	Total
1969								
1970							9.34	9.34
1971						30.54	10.52	41.06
1972					74.43	34.40	11.85	120.68
1973				185.96	83.84	38.75	13.34	321.90
1974			403.89	209.48	94.45	43.65	15.03	766.50
1975		579.48	454.96	235.97	106.39	49.17	16.93	1,442.91
1976	821.26	652.77	512.50	265.81	119.84	55.39	19.07	2,446.64

Forecast Average Variances

Accident Year	Months of Development 24	36	48	60	72	84	96	Total
1969								
1970							8.19	8.19
1971						28.10	8.19	36.29
1972					80.84	33.11	9.65	123.60
1973				235.51	93.74	38.40	11.19	378.84
1974			709.12	331.88	132.10	54.11	15.77	1,242.97
1975		1,039.02	785.45	367.61	146.32	59.93	17.47	2,415.80
1976	1,657.07	1,270.62	960.54	449.55	178.93	73.29	21.36	4,611.37

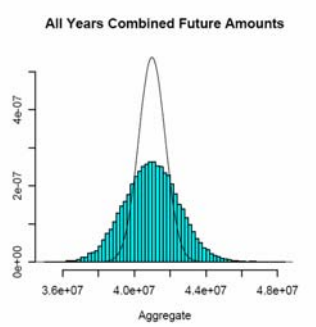
Example Accident Year Results

	Process Only		Including Parameter Uncertainty			
Accident Year	Mean	Standard Deviation	Mean	Standard Deviation	Percentile 5%	95%
1969	0	0	0	0	0	0
1970	80,981	26,503	80,551	36,442	24,148	144,035
1971	408,500	63,754	407,019	82,070	274,928	545,616
1972	1,169,365	106,448	1,169,765	137,850	945,662	1,399,015
1973	3,087,023	172,060	3,086,394	233,709	2,702,457	3,476,160
1974	5,986,335	216,225	5,984,922	344,212	5,425,005	6,551,203
1975	11,676,044	307,380	11,671,230	549,685	10,783,705	12,583,860
1976	18,579,788	375,626	18,581,701	808,465	17,258,898	19,916,569
Total	40,988,036	572,742	40,981,581	1,513,557	38,528,696	43,485,373

Example Next Year Results

	Process Only		Including Parameter Uncertainty			
Accident		Standard		Standard	Percentile	
Year	Mean	Deviation	Mean	Deviation	5%	95%
1969	0	0	0	0	0	0
1970	80,981	24,817	80,551	36,442	24,148	144,035
1971	303,859	52,742	302,553	68,934	192,431	418,164
1972	721,230	87,122	721,793	105,826	551,032	898,662
1973	1,783,372	147,171	1,783,236	172,967	1,502,286	2,075,631
1974	3,154,365	207,974	3,154,597	240,834	2,764,684	3,559,245
1975	4,689,180	260,836	4,686,348	309,909	4,179,644	5,204,351
1976	6,236,615	309,130	6,236,267	372,667	5,629,261	6,854,599
Total	16,969,602	489,384	16,965,345	652,968	15,893,889	18,045,385

Distribution of Outcomes from Model



Some Observations

- The data imply that the variance for payments in a cell are roughly proportional to the square root of the mean in this case, much lower than the powers of 1 and 2 usually used in stochastic chain ladder models
- The variance implied by the estimators for the aggregate future payment forecast is 573K
- Incorporating parameter risk gives a total variance of outcomes <u>within this model</u> is 1,513K
- Obviously process uncertainty is much less important than parameter
- <u>MODEL UNCERTAINTY IS NOT ADDRESSED HERE AT ALL</u>

More Observations

- We chose a relatively simple model for the expected value
- Nothing in this approach makes special use of the structure of the model
- Model does not need to be linear nor does it need to be transformed to linear by a function with particular properties
- Variance structure is selected to parallel stochastic chain ladder approaches (overdispersed Poisson, etc.) and allow the data to select the power
- The general approach is also applicable to a wide range of models
- This allows us to consider a richer collection of models than simply those that are linear or linearizable

Some Cautions

- MODEL UNCERATINTY IS NOT CONSIDERED thus distributions are distributions of outcomes under a specific model and must not be confused with the actual distribution of outcomes for the loss process
- An evolutionary Bayesian approach can help address model uncertainty
  - Apply a collection of models and judgmentally weight (a subjective prior)
  - Observe results for next year and reweight using Bayes Theorem
- We are using asymptotic properties, no guarantee we are far enough in the limit to assure these are close enough
- Actuarial “experiments” not repeatable so frequentist approach (MLE) may not be appropriate